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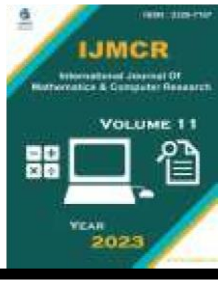

Dass and Gupta's Fixed-Point Theorem in Digital Metric Space

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ABSTRACT

This paper aims to define Dass and Gupta's contraction in the context of digital metric space and to establish a fixed point theorem for it. An example is also given in support of our proven result. Our findings widen and deepen a number of previously established findings in the literature.

KEYWORDS: Fixed point, Digital Metric Space, Digital Image, Digital Contraction.

2020 Mathematics Subject Classification: 54H25, 54E50, 47H10.

1. INTRODUCTION

The digital metric space proposed by Ege and Karaca [6] is one of the many generalizations of metric space found in the fixed point theory. Digital topology, in which we examine the topological and geometrical digital features of an image, is related to the idea of digital metric space. In computer graphic design and many other business activities involving computers, an image is treated as an object. A digital image is viewed in this kind of work as a collection of organized dots known as pixels or voxels. We examine these points and their adjacency relationship in digital topology. Rosenfeld [8] was the first to study the characteristics of almost fixed points using a digital topology. Boxer [4, 5] provides the topological concept in digital form later on. On the basis of this idea, Ege and Karaca [6] created the digital metric space in 2015 and validated a number of other fixed-point outcomes there. Then, numerous authors contributed to this field ([1], [3], [7], [9], [10]).

On the other hand, the "Banach Contraction Principle" has been widely generalized by scholars who have used it to analyze fixed points and common fixed points under various contraction conditions. By utilizing a rational type contractive condition, Dass and Gupta's [2] provided another extension of the "Banach Contraction Principle" in 1975. The digital version of Dass and Gupta's contraction was introduced in this study along with proof of the fixed point theorem. Our findings widen and deepen a number of previously established findings in the literature. The following definitions are required before we can prove our main result.

2. PRELIMINARIES

Definition 2.1. [5] "Let $F \subseteq \mathbb{Z}^n$, $n \in \mathbb{N}$ where $\mathbb{Z}^n$ is a lattice point set in the Euclidean n - dimensional space and (F, Υ) represent a digital image, with Υ -adjacency relation between the members of F and (F, Φ, Υ) represent a digital metric space, where (F, Φ) is a metric space. Let l, n be two positive integers, where $1 \le l \le n$ and g, h are two distinct points,

$$g = (g_1, g_2, \ldots\ldots g_n),\ h = (h_1, h_2, \ldots\ldots h_n) \in \mathbb{Z}^n.$$

Then the points g and h are said to be Υ_1– adjacent if there are at most l indices i such that $|g_i - h_i| = 1$ and for all other indices j, $|g_j - h_j| \neq 1$, $g_j = h_j$."

Definition 2.2. [5] "Let $\kappa \in \mathbb{Z}^n$, then the set –

$$N_\Upsilon(\kappa) = \{\ \sigma\ \mid \sigma \text{ is } \Upsilon - \text{adjacent to } \kappa\}$$

Represent the Υ – neighbourhood of κ for $n \in \{1, 2, 3\}$. Where $\Upsilon \in \{2, 4, 6, 8, 18, 26\}$."

Definition 2.3. [6] "The digital image $(F, \Upsilon) \subseteq \mathbb{Z}^n$ is called Υ –connected if and only if for every pair of different points $g, h \in F$, there is a set $\{g_0, g_1, \ldots\ldots g_s\}$ of points of digital image (F, Υ), such that $g = g_0$, $h = g_s$, and g_e and g_{e+1} are Υ -neighbours where $e = 0,1,2\ldots\ldots$ s-1."

Definition 2.4. [6] "Let $K: F \to$ K is a function and $(F, \Upsilon_0) \subset \mathbb{Z}^{n_0}$, (K, Υ_1) $\subset \mathbb{Z}^{n_1}$ are two digital images. Then –

(i). K is (Υ_0, Υ_1) - continuous if there exists Υ_0 - connected subset σ of F, for every $K(\sigma)$, Υ_1- connected subset of K.

(ii). K is (Υ_0, Υ_1) - continuous if for every Υ_0 - adjacent point $\{\sigma_0, \sigma_1\}$ of F, either $K(\sigma_0) = K(\sigma_1)$ or $K(\sigma_0)$ and $K(\sigma_1)$ are Υ_1 - adjacent in K.

(iii). K is said to be (Υ_0, Υ_1) - isomorphism, if K is (Υ_0, Υ_1) - continuous bijective and K^{-1} is (Υ_0, Υ_1)- continuous, also it is denoted by $F \cong K_{(\Upsilon_0,\Upsilon_1)}$.”

Definition 2.5. [6] “Let a $(2, \Upsilon)$ continuous function K: $[0, \sigma]_z \rightarrow F$ s.t. $K(0) = \alpha$ and $K(\sigma) = \beta$. Then in the digital image (F, Υ), it is called a digital Υ - path from α to β.”

Definition 2.6. [8] **“**Let $K: (F, \Upsilon) \rightarrow (F, \Upsilon)$ be a (Υ, Υ) - continuous function on a digital image (F, Υ), then we said that the property of fixed point satisfied by the digital image (F, Υ) if for every (Υ, Υ) - continuous function $K: F \rightarrow F$ there exists $\alpha \in F$ such that $K(\alpha) = \alpha$.”

Definition 2.7. [6] “Let $\{u_n\}$ is a sequence in digital metric space (F, Φ, Υ), then the sequence $\{u_n\}$ is called-

(i). Cauchy sequence if and only if there exists $\varrho \in N$ such that, $\Phi(u_n, u_m) < \epsilon, \forall\, n, m > \varrho$.

(ii). Converge to a limit point $\ell \in F$ if for every $\epsilon > 0$, there exists $\varrho \in N$ such that for all $n > \varrho$, $\Phi(u_n, \Upsilon) < \epsilon$.”

Theorem 2.8. [6] *“A digital metric space (F, Φ, Υ) is complete.”*

Proposition 2.9. [6] *“Every digital contraction map $K: (F,\Phi, \Upsilon) \rightarrow (F, \Phi, \Upsilon)$ is digitally Υ - continuous.”*

Lemma 2.14. [1] Let $\{u_n\}$ be a sequence in complete digital metric space (F, Φ, Υ), and if there exists $\rho \in (0, 1)$, such that $\Phi(u_{n+1}, u_n) \le \Phi(u_n, u_{n-1})$ for all n then, sequence $\{u_n\}$ converges to a point in F.

3. MAIN RESULT

Theorem 3. *Let (F, Φ, Υ) be a complete digital metric space, and $K: F \rightarrow F$ be a mapping which satisfies the following rational contraction condition*

$$\Phi(Ku, Kv) \le \frac{\xi_1\Phi(v, Kv)[1+\Phi(u, Ku)]}{1+\Phi(u, v)} + \xi_2\Phi(u, v) \quad (3.1)$$

For all $u, v \in F$. Here $\xi_1, \xi_2 > 0$ s.t., $\xi_1 + \xi_2 < 1$. Then mappings K has a unique fixed point in F.

Proof. Let $u_0 \in F$ be an arbitrary point. Define a sequence $\{u_n\} \in F$, such that, $u_1 = K(u_0)$ and $u_2 = K(u_1)$, in general, we have $u_{2n+1} = K(u_{2n})$ and $u_{2n+2} = K(u_{2n+1})$.

Now consider,

$$\Phi(u_1, u_2) = \Phi(Ku_0, Ku_1)$$

$$\Phi(Ku_0, Ku_1) \le \frac{\xi_1\Phi(u_1, Ku_1)[1+\Phi(u_0, Ku_0)]}{1+\Phi(u_0, u_1)} + \xi_2\Phi(u_0, u_1)$$

$$\Phi(u_1, u_2) \le \frac{\xi_1\Phi(u_1, u_2)[1+\Phi(u_0, u_1)]}{1+\Phi(u_0, u_1)} + \xi_2\Phi(u_0, u_1)$$

$$\Phi(u_1, u_2) \le \xi_1\, \Phi(u_1, u_2) + \xi_2\, \Phi(u_0, u_1)$$

$$\Phi(u_1, u_2) \le \left(\frac{\xi_2}{1-\xi_1}\right) \Phi(u_0, u_1)$$

Similarly,

$$\Phi(u_2, u_3) \le \left(\frac{\xi_2}{1-\xi_1}\right) \Phi(u_1, u_2)$$

$$\Phi(u_2, u_3) \le \left(\frac{\xi_2}{1-\xi_1}\right)^2 \Phi(u_0, u_1) \quad (3.2)$$

Now, let $\eta = \left(\frac{\xi_2}{1-\xi_1}\right)$, then from (3.2) we obtain

$$\Phi(u_2, u_3) \le \eta^2\Phi(u_0, u_1)$$

In general, we can easily, get

$$\Phi(u_n, u_{n+1}) \le \eta^n\Phi(u_0, u_1)$$

$$\Phi(u_{n+1}, u_{n+2}) \le \eta^{n+1}\Phi(u_0, u_1)$$

Then by using the triangular inequality property, we get

$$\Phi(u_n, u_{n+k}) \le (\eta^n + \eta^{n+1} + \eta^{n+2} + \cdots \eta^{n+k-1})\, \Phi(u_0, u_1)$$

$$\Phi(u_n, u_{n+k}) \le \left(\frac{\eta^n}{1-\eta}\right) \Phi(u_0, u_1)$$

Now as $n \rightarrow \infty$, $\left(\frac{\eta^n}{1-\eta}\right) \Phi(u_0, u_1) \rightarrow 0$. This suggested that the sequence $\{u_n\}$ is a digital Cauchy sequence in digital metric space (F, Φ, Υ), thus there exists a point $\mu \in F$ such that $\{u_n\} \rightarrow \mu$.Therefore, subsequence $K(u_{2n}) \rightarrow \mu$ and $K(u_{2n+1}) \rightarrow \mu$. Since K is (Υ_0, Υ_1) – continuous function, so we have $K\mu = \mu$.

Now, to show the uniqueness of the fixed point, suppose that λ is another fixed point, then from (3.1) we have,

$$\Phi(\mu, \lambda) = (K\mu, K\lambda)$$

$$\Phi(\mu, \lambda) \le \frac{\xi_1\Phi(\lambda, K\lambda)[1+\Phi(\mu, K\mu)]}{1+\Phi(\mu, \lambda)} + \xi_2\Phi(\mu, \lambda)$$

$$\Phi(\mu, \lambda) \le \frac{\xi_1\Phi(\lambda, \lambda)[1+\Phi(\mu, \mu)]}{1+\Phi(\mu, \lambda)} + \xi_2\Phi(\mu, \lambda)$$

$$\Phi(\mu, \lambda) \le \xi_2\Phi(\mu, \lambda)$$

$$\Phi(\mu, \lambda) = 0 \qquad \text{as } 0 < \lambda < 1$$

Therefore, $\mu = \lambda$.

Hence, μ is a unique fixed point of mappings K.

Example 3.2. Let $F = [0, 1]$ and $\Phi(u, v) = |u - v|$ be a digital metric on F then clearly (F, Φ, Υ) represent a digital metric space. Let $K: F \rightarrow F$ be a mapping defined by $K(u) = \frac{u}{5}$ $\forall\, u \in \mathcal{F}$. It is then simple to demonstrate that all of the prerequisites and criteria stated in Theorem 3.1 apply for $\xi_1 = \frac{1}{25}$ and $\xi_2 = \frac{4}{25}$ and that there exists a unique fixed point '0' for mapping K.

4. CONCLUSION

In this paper, we introduce the notion of Dass and Gupta's contraction in the digital metric space and prove a digital fixed point theorem for rational contraction in this space. Many well-known results in the literature are expanded upon and broadened by our findings. In fixed point theory, this outcome has an application. It can be used to reduce the size of digital photographs and is useful for processing and changing how images are stored.

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