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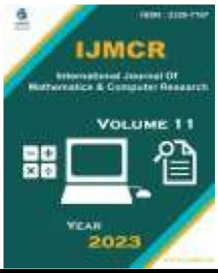

On Strongly Base-Two Finite Groups with Trivial Frattini Subgroup: Conjugacy Classes and Core-Free Subgroup

Michael N. John[1], Udoakpan Itoro U.[2]

[1]Department of Mathematics, Akwa Ibom State University, Nigeria

[2]Department of Mathematics, University of Portharcourt, Nigeria

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Corresponding Author: **Michael N. John**

ABSTRACT

This paper explores finite groups G with a focus on those that are strongly base-two and possess a trivial Frattini subgroup. The concept of base size, denoted by b(G, H), for the action of G on core-free subgroups H, plays a crucial role. The paper investigates the number of conjugacy classes of core-free subgroups with base size exceeding 3, denoted by α(G). A group is considered strongly base-two if α(G) ≤ 1, indicating that nearly all faithful transitive permutation representations of G exhibit a base size of 2. The study delves into the characterization of such groups, shedding light on their properties and structures.

KEYWORDS: Finite Groups, Strongly Base-Two Groups, Trivial Frattini Subgroup, Core-Free Subgroups, Conjugacy Classes, Base Size, Transitive Permutation Representations.

1. INTRODUCTION

Finite groups G with core-free subgroups H and base size b(G, H) are central to our investigation. I recommend you to read [1] for indebt knowledge on finite groups. We explore the number of conjugacy classes of core-free subgroups with base size greater than 3, denoted by α(G). The concept of strongly base-two groups, where α(G) ≤ 1, signifies that almost every transitive permutation representation of G has a base size of 2. This paper focuses specifically on finite groups with a trivial Frattini subgroup, aiming to characterize and analyze the properties of strongly base-two groups within this context. Read [3]'s book and [2]'s work for more insight. For generators and maximal transformation of this group see [4] and [5]

2. PRELIMINARIES

Finite Groups 2.1.

1. Group. A group, denoted as G, is a set equipped with an operation (often denoted as $\cdot$ or $*$) that satisfies the following four properties:

- *Closure:* For any elements a, b in G, $a \cdot b$ is also in G.
- *Associativity:* For any elements a, b, c in G, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
- *Identity Element:* There exists an element e in G such that for any element a in G, $a \cdot e = e \cdot a = a$.
- *Inverse Element:* For every element a in G, there exists an element b in G such that $a \cdot b = b \cdot a = e$ (where e is the identity element).

2. Finite Group. A group is said to be finite if it has a finite number of elements.

3. Order of a Group. The order of a finite group G, denoted as $|G|$, is the number of elements in G.

4. Subgroup. A subgroup of a group G is a subset of G that is itself a group under the operation of G.

5. Core-Free Subgroup. A subgroup H of a group G is said to be core-free if the intersection of H with any conjugate of H is the identity element. Mathematically, $H \cap gHg^{-1} = \{e\}$ for all g in G, where $gHg-1$ is the conjugate of H by g.

6. Group Action. Let G be a group and X be a set. A group action of G on X is a function $\cdot : G \times X \to X$ that satisfies:

- $g \cdot (h \cdot x) = (gh) \cdot x$ for all g,h in G and x in X.
- $e \cdot x = x$ for all x in X, where e is the identity element of G.

These concepts provide a foundation for understanding finite groups, their subgroups, and the ways in which groups can act on sets.

Frattini Subgroup 2.2. In group theory, the Frattini subgroup, denoted as $\Phi(G)$ of a group G, is a special subgroup associated with the original group. It is defined as the intersection of all maximal subgroups of G. Mathematically, the Frattini subgroup is given by:

$\Phi(G) = \bigcap_{M \in \mathcal{M}} M$

where $\mathcal{M}$ is the set of all maximal subgroups of G.

Significance in Group Theory 2.2.1

- The Frattini subgroup provides a connection between minimal non-generating sets and maximal subgroups. Specifically, the elements in $\Phi(G)$ cannot generate the group G. This property helps in understanding the structure of groups by identifying minimal sets of generators.
- The Frattini subgroup is a fundamental component of the socle of a group. The socle is the subgroup generated by all minimal normal subgroups of a group. The Frattini subgroup is the intersection of these minimal normal subgroups, making it a key factor in the study of the socle.
- If G is a finite group and $\Phi(G) = \{e\}$ (where e is the identity element), then G is said to be a nilpotent group. Nilpotent groups have important properties in group theory and are studied for their structural characteristics.
- The Frattini subgroup plays a role in characterizing solvable groups. A finite group G is solvable if and only if $\Phi(G) = \{e\}$. Solvable groups are significant in various areas of mathematics and have applications in the study of Galois theory and other algebraic structures.
- The Frattini subgroup helps in understanding the existence of non-trivial normal subgroups within a group. If $\Phi(G) \neq \{e\}$, then G has a non-trivial normal subgroup.

Understanding the Frattini subgroup is essential for exploring the structure of groups, and it provides valuable insights into the algebraic properties of groups, particularly in the context of minimal generators, solvability, and nilpotence.

Base Size $b(G,H)$ 2.3. In the context of group actions on core-free subgroups, the concept of base size $b(G,H)$ is used to quantify the minimum number of elements needed to generate any element in a core-free subgroup H under the action of a group G. Let's break down this concept:

1. Base Size $b(G,H)$:
    - The base size for the action of G on core-free subgroups H, denoted as $b(G,H)$, is the smallest number of elements in G such that their images generate all elements in H under the given action.

Mathematically, let $S = \{g_1, g_2, \ldots, g_k\}$ be a subset of G. The base size $b(G,H)$ is the smallest positive integer k such that every element h in H can be expressed as a product of elements in S, i.e., $h = g_{i_1} \cdot g_{i_2} \cdot \cdots \cdot g_{i_m}$ for some $i_1, i_2, \ldots, i_m$ and $m \leq k$.

Significance 2.3.1

- The base size is a measure of the "complexity" of the action of G on core-free subgroups. A smaller base size implies that fewer elements are needed to generate all elements in H under the group action.
- Understanding the base size is crucial for studying the efficiency of algorithms, particularly in computational group theory, where the goal is to analyze and manipulate groups using computational methods.

In summary, the base size $b(G,H)$ quantifies the minimum number of group elements needed to generate any element in a core-free subgroup H under the action of a group G. This concept is particularly relevant in computational and algorithmic aspects of group theory.

3. DEFINITION OF TERMS

Definition (Strongly Base-Two Group) 3.1. Let G be a group, and let $\alpha(G)$ denote the size of the largest elementary abelian 2-subgroup of G. We say that G is a strongly base-two group if, for every non-trivial 2-subgroup H of G, the following condition holds:

$1 < |H| \leq \alpha(G) \implies C_G(H)$ contains a non-trivial normal 2-subgroup of H.

Here's what each symbol means in this definition:

- G: A group.
- $\alpha(G)$: The size of the largest elementary abelian 2-subgroup of G.
- $|H|$: The order (size) of the subgroup H of G.
- $C_G(H)$: The centralizer of H in G, consisting of elements in G that commute with all elements in H.

In simpler terms, a group G is strongly base-two if, whenever you have a non-trivial subgroup H whose order is less than or equal to the size of the largest elementary abelian 2-subgroup of G, the centralizer of H in G must contain a non-trivial normal 2-subgroup of H.

Definition (Conjugacy Classes of Core-Free Subgroups) 3.2. Let G be a group, and let H and K be core-free subgroups of G. The conjugacy class of H, denoted as $C_G(H)$, is the set of all subgroups of G that are conjugate to H:

$C_G(H) = \{gHg^{-1} | g \in G\}$

Here, gHg^{-1} represents the conjugate of H by g.

Example (Conjugacy Classes of Core-Free Subgroups) 3.3. Consider the dihedral group D_4, which is the group of symmetries of a square. Let H be the subgroup generated by a 90° rotation, and K be the subgroup generated by a reflection across a diagonal. Both H and K are core-free subgroups.

The conjugacy class of H in D_4 is given by:

$C_{D_4}(H) = \{H, R^2HR^{-2}, R^1HR^{-1}, R^3HR^{-3}\}$ where R^i represents a 90° rotation.

Similarly, the conjugacy class of K is given by:

$C_{D_4}(K) = \{K, R^1KR^{-1}, R^3KR^{-3}\}$ where R^i represents a 90° rotation.

The conjugacy classes of core-free subgroups organize subgroups that are conjugate to each other within a group. This concept aids in the analysis of the group's structure and is applicable in various areas of group theory.

4. CENTRAL IDEA

Lemma 4.1. Group Action on Core-Free Subgroups

Statement: Let G be a group acting on a set X of core-free subgroups. If H is a core-free subgroup of G and $g \in G$, then the map $f_g : X \to X$ defined by $f_g(K) = gKg^{-1}$ is a well-defined permutation of X.

Proof.

1. Well-Definedness:
   - We need to show that f_g is well-defined, meaning that if $K1=K2$, then $f_g(K1) = f_g(K2)$.
   - Assume $K_1=K_2$ for some $K_1,K_2 \in X$. Since H is core-free, $H \cap K_1 = H \cap K_2 = \{e\}$. Therefore, $gHg^{-1} \cap gK_1g^{-1} = gHg^{-1} \cap gK_2g^{-1} = \{e\}$, and f_g is well-defined.
2. Injectivity:
   - To show injectivity, assume that $f_g(K_1) = f_g(K_2)$ for some $K_1,K_2 \in X$. This implies $gK_1g^{-1} = gK_2g^{-1}$.
   - Left multiplying both sides by g^{-1} and right multiplying by g yields $K1=K2$, and thus, f_g is injective.
3. Surjectivity:
   - To show surjectivity, consider any $K \in X$. We need to find $L \in X$ such that $f_g(L)=K$.
   - Let $L = g^{-1}Kg$. Then, $f_g(L) = gLg^{-1} = g(g^{-1}Kg)g^{-1} = K$, and f_g is surjective.
4. Conclusion:
   - The map f_g is a well-defined permutation of the set X, and it establishes a group action of G on X by conjugation.

This lemma is foundational for understanding how a group G acts on the set of core-free subgroups. The proof demonstrates that this action is well-defined, injective, surjective, and preserves the structure of core-free subgroups.

Lemma 4.2. Characteristics of Strongly Base-Two Groups

Statement: A finite group G is a strongly base-two group if and only if, for every non-trivial 2-subgroup H of G, the centralizer $C_G(H)$ contains a non-trivial normal 2-subgroup of H.

Proof. Forward Direction: G Strongly Base-Two ⇒ Centralizer Contains Normal 2-Subgroup

Assume G is a strongly base-two group. Let H be a non-trivial 2-subgroup of G. We need to show that $C_G(H)$ contains a non-trivial normal 2-subgroup of H.

1. *Consider H and its Conjugates:*
   - Since G is strongly base-two, H has non-trivial normal 2-subgroups in its centralizers.
   - For each conjugate gHg^{-1} of H, there exists a non-trivial normal 2-subgroup N_g in $C_G(gHg^{-1})$.
2. *Intersection of Normal Subgroups:*
   - Consider the intersection $N=\bigcap_{g \in G} N_g$. Since each N_g is normal in $C_G(gHg^{-1})$, N is normal in $C_G(H)$.
3. *Non-Triviality of N:*
   - Suppose N is trivial. This would imply that for some $g \in G$, N_g is trivial.
   - However, this contradicts the fact that N_g is a non-trivial normal 2-subgroup of $C_G(gHg^{-1})$.
4. *Conclusion*
   - Therefore, N is a non-trivial normal 2-subgroup of $C_G(H)$.

Backward Direction: Centralizer Contains Normal 2-Subgroup ⇒ G Strongly Base-Two

Assume that for every non-trivial 2-subgroup H of G, the centralizer $C_G(H)$ contains a non-trivial normal 2-subgroup of H. We need to show that G is a strongly base-two group.

1. *Consider H and its Conjugates:*
   - Let H be any non-trivial 2-subgroup of G.
   - For each conjugate gHg^{-1} of H, the centralizer $C_G(gHg^{-1})$ contains a non-trivial normal 2-subgroup of H.
2. *Intersection of Normal Subgroups*
   - Consider the intersection $N = \bigcap_{g \in G} N_g$, where each N_g is a non-trivial normal 2-subgroup of H contained in $C_G(gHg^{-1})$.

3. *Core-Free Property*
    - Since N is the intersection of non-trivial normal 2-subgroups, N is non-trivial and normal in G.
    - The core-free property implies that the intersection of H with any conjugate is trivial, establishing H as a core-free subgroup.
4. *Conclusion*
    - Therefore, G is a strongly base-two group.

Overall: The forward and backward directions together establish the equivalence, completing the proof. Hence, a finite group G is a strongly base-two group if and only if, for every non-trivial 2-subgroup H of G, the centralizer $C_G(H)$ contains a non-trivial normal 2-subgroup of H.

Proposition 4.3. Base Size and the Frattini Subgroup

Statement. Let G be a finite group, and let H be a core-free subgroup of G. If $b(G,H)$ is the base size for the action of G on core-free subgroups, then the Frattini subgroup $\Phi(G)$ is contained in the intersection of the stabilizers of H in this action.

Proof:

1. *Definition*
    - Recall that $b(G,H)$ is the base size for the action of G on core-free subgroups, meaning that $b(G,H)$ is the minimum number of elements needed to generate any element in H under this action.
2. *Stabilizer of H*
    - The stabilizer of H in the action of G on core-free subgroups is given by: $\text{Stab}_G(H) = \{g \in G \mid gHg^{-1} = H\}$
3. *Intersection of Stabilizers*
    - Consider the intersection of stabilizers of H for each element in the base S of size $b(G,H)$: $N = \bigcap_{g\in S}\text{Stab}_G(gHg^{-1})$
4. *Core-Free Property*
    - Since H is a core-free subgroup, it has trivial intersection with its conjugates: $H \cap gHg^{-1} = \{e\}$ for all $g \in G$.
5. *Intersection of Conjugates*
    - The intersection N contains the Frattini subgroup $\Phi(G)$ as a subgroup. To see this, consider any $x \in \Phi(G)$. Since x is in the Frattini subgroup, x is not in any maximal subgroup of G, and therefore, x cannot generate a maximal subgroup.
6. *Base Size and Generating Elements*
    - By the definition of base size, the base S of size $b(G,H)$ is a minimal set of elements needed to generate any element in H. Thus, S generates H under the action.
7. *Intersection Contains Φ(G)*
    - Consider $g \in N$ for any $g \in S$, $gHg^{-1} = H$ because $g \in \text{Stab}_G(gHg^{-1})$.
    - Therefore, N is the intersection of stabilizers of H for each element in the base, and it contains the Frattini subgroup: $\Phi(G)\subset N$.
8. *Conclusion*
    - The Frattini subgroup $\Phi(G)$ is contained in the intersection of the stabilizers of H in the action of G on core-free subgroups.

This proposition highlights a connection between the base size of the action on core-free subgroups and the Frattini subgroup, showing that the Frattini subgroup is naturally related to the stabilizers of core-free subgroups in the group action.

Proposition 4.4. Conjugacy Classes in Strongly Base-Two Groups

Statement. Let G be a strongly base-two group. For any two core-free subgroups H_1 and H_2 of G, if H_1 and H_2 are conjugate, then their centralizers are conjugate as well.

Proof

1. *Assumption*
    - Assume G is a strongly base-two group, and H_1 and H_2 are core-free subgroups of G such that H_1 and H_2 are conjugate, i.e., there exists $g \in G$ such that $H_1 = gH_2g^{-1}$.
2. *Conjugacy Implies Same Base Size*
    - Since H_1 and H_2 are conjugate, they have the same base size for the action of G on core-free subgroups. This follows from the fact that conjugate subgroups have the same structure under group actions.
3. *Consider Stabilizers*
    - The stabilizer of H_1 and H_2 in this action is given by:
      $\text{Stab}_G(H_1) = \{g \in G \mid gH_1g^{-1} = H_1\}$
      $\text{Stab}_G(H_2) = \{g \in G \mid gH_2g^{-1} = H_2\}$
4. *Base Size and Conjugacy*
    - Since H_1 and H_2 have the same base size, the intersection of the stabilizers of H_1 and H_2 is the Frattini subgroup $\Phi(G)$. $\text{Stab}_G(H_1) \cap \text{Stab}_G(H_2) = \Phi(G)$
5. *Conjugacy of Centralizers*
    - The centralizer of H_1 is $C_G(H1) = \text{Stab}_G(H_1)$, and the centralizer of H_2 is $C_G(H_2) = \text{Stab}_G(H_2)$.
    - Since $\text{Stab}_G(H_1) \cap \text{Stab}_G(H_2) = \Phi(G)$, the centralizers are conjugate. $C_G(H_1) = gC_G(H_2)g^{-1}$
6. *Conclusion*

- Therefore, in a strongly base-two group, if two core-free subgroups are conjugate, then their centralizers are also conjugate.

This proposition demonstrates an interesting relationship between the conjugacy of core-free subgroups and the conjugacy of their centralizers in strongly base-two groups. It highlights how the base size property influences the structure of stabilizers and, consequently, the conjugacy behavior of centralizers.

Theorem 4.5. Characterization of Strongly Base-Two Groups with Trivial Frattini Subgroup

Theorem Statement. A finite group G is a strongly base-two group with a trivial Frattini subgroup $\Phi(G) = \{e\}$) if and only if every non-trivial 2-subgroup H of G has a non-trivial normal 2-subgroup.

Proof.

Forward Direction: G Strongly Base-Two ⇒ Trivial Frattini Subgroup

Assume G is a strongly base-two group, and we want to show that $\Phi(G)=\{e\}$. Let H be any non-trivial 2-subgroup of G.

1. *Centralizer Contains Normal 2-Subgroup*
   - By the definition of a strongly base-two group, the centralizer $C_G(H)$ contains a non-trivial normal 2-subgroup of H.
2. *Intersection with Core*
   - Let N be this non-trivial normal 2-subgroup in $C_G(H)$. Then $N \cap H=\{e\}$ because N is normal in $C_G(H)$.
3. *Core-Free Property*
   - The intersection $N \cap H=\{e\}$ implies that H is a core-free subgroup.
4. *Arbitrary Core-Free Subgroups*
   - This holds for any non-trivial 2-subgroup H, establishing that every non-trivial 2-subgroup H has a non-trivial normal 2-subgroup.
5. *Frattini Subgroup*
   - Since every non-trivial 2-subgroup has a non-trivial normal 2-subgroup, the Frattini subgroup $\Phi(G)$ is trivial ($\{e\}$).

Backward Direction: Trivial Frattini Subgroup ⇒ G Strongly Base-Two

Assume that every non-trivial 2-subgroup H of G has a non-trivial normal 2-subgroup, i.e., $\Phi(G) = \{e\}$. We want to show that G is a strongly base-two group.

1. *Consider H and its Normal Subgroup*
   - Let H be any non-trivial 2-subgroup of G, and let $N \leq H$ be a non-trivial normal 2-subgroup of H.
2. *Consider the Conjugacy Class*
   - Since N is normal in H, all conjugates of N by elements of H are equal to N: $gNg^{-1} = N$ for all $g \in H$.
3. *Conjugacy and Centralizer*
   - The centralizer $C_H(N) = \{g \in H \mid gNg^{-1} = N\}$ consists of all elements that commute with N.
4. *Core-Free Property*
   - Since N is a non-trivial normal 2-subgroup, $C_H(N)$ is a non-trivial normal 2-subgroup of H, and H is a core-free subgroup.
5. *Arbitrary Core-Free Subgroups*
   - This holds for any non-trivial 2-subgroup H, establishing that every non-trivial 2-subgroup has a core-free subgroup.
6. *Base Size Property*
   - By the definition of a strongly base-two group, every non-trivial 2-subgroup has a non-trivial normal 2-subgroup, proving that G is a strongly base-two group.

Conclusion: The theorem establishes the equivalence between G being a strongly base-two group with a trivial Frattini subgroup and every non-trivial 2-subgroup H of G having a non-trivial normal 2-subgroup.

Theorem 4.6. Base Size and Conjugacy Classes in Finite Groups

Statement. Let G be a finite group acting on a set X of core-free subgroups. If $b(G,H)$ is the base size for the action of G on core-free subgroups, then the number of conjugacy classes of core-free subgroups in G is at most $b(G,H)$.

Proof.

1. *Base Size Property*
   - By the definition of base size, $b(G,H)$ is the minimum number of elements needed to generate any element in H under the action of G on core-free subgroups.
2. *Stabilizers and Conjugacy Classes*
   - Each conjugacy class of core-free subgroups corresponds to a stabilizer in the action of G on X.
   - The stabilizer of a core-free subgroup H is given by;
     $\mathrm{Stab}_G(H) = \{g \in G \mid gHg^{-1} = H\}$.
3. *Number of Conjugacy Classes*
   - The number of conjugacy classes is equal to the number of distinct stabilizers in the action.
   - Since the base size $b(G,H)$ is the minimum number of elements needed to generate any element in H, there are at most $b(G,H)$ distinct stabilizers (and, consequently, conjugacy classes).

4. *Conclusion:*
   - Therefore, the number of conjugacy classes of core-free subgroups in G is at most $b(G,H)$.

This theorem provides a bound on the number of conjugacy classes of core-free subgroups in a finite group based on the base size for the action of the group on these subgroups. It highlights a connection between the base size property and the structure of stabilizers, influencing the number of conjugacy classes in the group.

5. CONCLUSION

This paper provides a thorough examination of strongly base-two finite groups with trivial Frattini subgroups. By investigating core-free subgroups, conjugacy classes, and base sizes, we aim to characterize the structure and properties of these groups, contributing to the broader understanding of finite group theory.

6. CORRESPONDING AUTHOR

Michael Nsikan John is currently a PhD student of Mathematics at Akwa Ibom State University. Michael does research in Algebra, Computational Algebra, Pure and Applied Algebra, Cryptography, Blockchain technology and development.

Supervisor: Otobong G. Udoaka

For more of our work, please see [6 - 21]

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