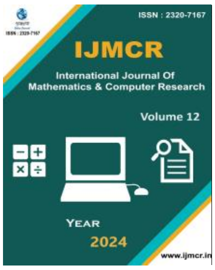
International Journal of Mathematics and Computer Research

ISSN: 2320-7167

Volume 12 Issue 01 January 2024, Page no. 3941-3948

Index Copernicus ICV: 57.55, Impact Factor: 7.362

DOI: 10.47191/ijmcr/v12i1.03



Initial Coefficient Bounds for a Class Functions Involving q-Sal˘ Agean˘ Differential Operator

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ARTICLE INFO	ABSTRACT
Published Online: 03 January 2024 Corresponding Author: **K.Amarender Reddy**	By making use of the q-analogue of famous S˘al˘agean differential operator, the authors define a new subclass of analytic functions with respect to other points. Fekete-Szeg¨o inequality and initial coefficient bounds of a certain bi-starlike functions are obtained. Further several examples, remarks and applications of our results are enumerated.

KEYWORDS: Starlike functions, Spiralike Functions, Bi-Univalent functions, Coefficient inequalities, Fekete-Szeg¨o, q-calculus, S˘al˘agean differential operator, and Symmetric function.

1. INTRODUCTION, DEFINITIONS AND PRELIMINARIES

Let A denote the class of all analytic functions $f(z)$ normalized by the condition $f(0) = f^0(0) - 1 = 0$ and is of the form

(1.1) $$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in \mathcal{U}).$$

Let S denote the class of all function in A which are univalent in U. The well-known example in this class is the Koebe function, $k(z)$ defined by

$$k(z) = \frac{z}{(1-z)^2} = z + \sum_{n=2}^{\infty} n z^n.$$

The Bieberbach conjecture about the coefficient of the univalent functions in the unit disk was formulated by Bieberbach [2] in the year 1916. The conjecture states that for every function $f \in$S, given by (1.1), we have $|a_n| \le n$ for every n. Strict inequality holds for all n unless f is the Koebe function or one of its rotation. For many years, this conjecture remained as a challenge to mathematicians. After the proof of $|a_3| \le 3$ by L¨owner in 1923, Fekete-Szeg¨o surprised the mathematicians with the complicated inequality

$$\left|a_3 - \mu a_2^2\right| \le 1 + 2\exp\left(\frac{-2\mu}{1-\mu}\right),$$

Which holds good for all values $0 \le \mu \le 1$. Note that this inequality region was thoroughly investigated by Schaefer and Spencer [16].For a class functions in A and a real (or more generally complex) number μ, the Fekete-Szego¨ problem is all about finding the best possible constant $C(\mu)$ so that $\left|a_3 - \mu a_2^2\right| \le C(\mu)$ for every function in A. Many papers have been devoted to this

Problem see [3, 4, 5, 7, 8, 10]. In this paper, we obtain the estimates of a_2, a_3 and also the Fekete-Szego¨ inequality for a subclass of spiralike functions of complex order defined using subordination.

It is well known that every function $f \in$S has an interval f^{-1}, defined by

$$f^{-1}\{f(z)\}=z;(z\in U) \text{ and } f\left\{f^{-1}(w)\right\} = w \left(|w| < r_0(f); r_0(f) \ge \frac{1}{4}\right).$$

In fact, the inverse function f^{-1} is given by

(1.2

$$f^{-1}(w) = w - a_2 w^2 + (2a_2 w^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots$$

.

A function $f \in$A is said to be biunivalent in U if both $f(z)$ and $f^{-1}(z)$ are univalent in U. Let f and g be analytic in the open unit desk U. The function f is subordinate to g written as $f \prec g$ in U, if there exist a function w analytic in U with $w(0) = 0$ and $|w(z)| < 1$; $(z \in U)$ such that $f(z) = g(w(z))$, $(z \in U)$.

Let $S^*(\alpha)$ and $C(\alpha)$ denote the well-known subclasses of the univalent function class S which are respectively defined as follows.

$$\mathcal{S}^*(\alpha) = \left\{ f \in \mathcal{A} : \ Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha; \ 0 \le \alpha < 1 \right\}$$

And

$$\mathcal{C}(\alpha) = \left\{ f \in \mathcal{A} : \ Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha; \ 0 \le \alpha < 1 \right\}$$

Using Alexander transform, it follows that $f(z) \in C(\alpha)$ if and only if $zf^0(z) \in S^*(\alpha)$.

One of the very interesting generalization of the function class S^* is the so called starlike functions of complex order b which satisfies the condition

$$1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right) \prec \phi(z), \quad (f \in \mathcal{A}),$$

Where $\varphi \in P$, the class of functions with positive real part and we denote it by $S_b(\varphi)$. Similarly, let $C_b(\varphi)$ denote the class of functions in A satisfying the condition

$$1 + \frac{1}{b}\frac{zf''(z)}{f(z)} \prec \phi(z), \quad (f \in \mathcal{A}).$$

Note that $S_b(1 + z/1 - z) = S_b$ and $C_b(1 + z/1 - z) = C_b$ are the classes considered by Nasr and Aouf in [11] and by Wiatrowski in [18].

q - calculus has been studied by various authors due to the fact that applications of basic Gaussian hyper geometric function to the diverse subjects of combinatorics, quantum theory, number theory, statistical mechanics , are constantly being uncovered. The q-difference operator denoted as $D_qf(z)$ is defined by

$$D_qf(z) = \frac{f(z) - f(qz)}{z(1-q)}, \ (f \in \mathcal{A}, \ z \in \mathcal{U} - \{0\}),$$

and $D_qf(0) = f^0(0)$, where $q \in (0,1)$. It can be easily seen that $D_qf(z) \to f^0(z)$ as $q \to 1^-$. If $f(z)$ is of the form (1.1), a simple computation yields

$$D_qf(z) = 1 + \sum_{n=2}^{\infty} \frac{1-q^n}{1-q} a_n z^{n-1}, \ (z \in \mathcal{U}). \tag{1.3}$$

The inverse function of (1.3) is given by

$$D_qg(w) = 1 - (1+q)a_2w - (1+q+q^2)a_3w^2 + 2(1+q)^2a_2^2w^2 + \cdots.$$

The q- analogue of Sa˘l˘agean differential operator ($R_q^m f(z) : \mathcal{A} \to \mathcal{A}$ see [14]) , for $m \in N$, is formed as follows.

$$\begin{aligned} R_q^0 f(z) &= f(z), \\ R_q^1 f(z) &= z(D_q f(z)), \\ \vdots \quad & \quad \vdots \quad \quad \vdots \\ R_q^m f(z) &= R_q^1 \quad R_q^{m-1} f(z)). \end{aligned}$$

Motivated by the concept introduced by Sakaguchi in [13], recently several subclasses of analytic functions with respect to k-symmetric points were introduced and studied by various authors. In this paper, we introduce a new subclass of spiralike biunivalent functions using subordination and we obtained the estimates of the $|a_2|$ and $|a_3|$ for the functions belonging to this new subclass.

DEFINITION 1.1.

Let $h(z)$ be a convex univalent function with $h(0) = 1$. A function $f \in A$ is said to be in the class $\mathcal{S}_b^{\lambda,m}(\beta, s, t, h)$ if and only if it satisfies the analytic condition,

$$e^{i\beta}\left[1 + \frac{1}{b}\left\{\frac{[(s-t)z]^{1-\lambda} R_q^{m+1} f(z)}{[R_q^m f(sz) - R_q^m f(tz)]^{1-\lambda}} - 1\right\}\right] \prec h(z)\cos\beta + i\sin\beta$$

And

$$e^{i\beta}\left[1 + \frac{1}{b}\left\{\frac{[(s-t)w]^{1-\lambda} R_q^{m+1} g(w)}{[R_q^m g(sw) - R_q^m g(tw)]^{1-\lambda}} - 1\right\}\right] \prec h(w)\cos\beta + i\sin\beta.$$

Where ($z \in \mathcal{U}$; $\lambda \geq 0$; $\frac{-\pi}{2} \lessdot \beta < \frac{\pi}{2}$; $b \in \mathbb{C} - \{0\}$). and $s,t \in \mathbb{C}$ with $s \neq t$, $|t| \leq 1$.

Remark 1.1. On specializing the parameters and the function $h(z)$, we obtain several new and well known subclasses of analytic functions. Here we list a few of them.

1. If we let $\beta = 0$ and $h(z) = 1 + \frac{\gamma-\alpha}{\pi} i \log\left(\frac{1-e^{2\pi i(1-\alpha)\backslash(\gamma-\alpha)z}}{1-z}\right)$.then the class $S_b^{\lambda,m}(\beta,s,t,h)$ reduces to the form
$$\alpha < Re\left\{1 + \frac{1}{b}\left\{\frac{[(s-t)z]^{1-\lambda} R_q^{m+1} f(z)}{[R_q^m f(sz) - R_q^m f(tz)]^{1-\lambda}} - 1\right\}\right\} < \gamma.$$
Which is analogues to the class introduced by Kuroki and Owa in [6].
2. If we let $m = 0$, $q \to 1^-$ and $b = 1$ in $S_b^{\lambda,m}(\beta,s,t,h)$. then the class reduces to the class which is analogous to the class introduced and studied by Altınkaya and Yal¸cın in [1]
3. If we set $h(z) = \frac{1+(1-2\alpha)z}{1-z}$, $0 \leq \alpha < 1$ in the class $S_b^{\lambda,0}(\beta,s,t,h)$ we have

$S_b^{\lambda,m}(\beta,s,t,\alpha)$ and defined as
$$Re\left\{e^{i\beta}\left\{\frac{[(s-t)z]^{1-\lambda} z f'(z)}{[f(sz) - f(tz)]^{1-\lambda}}\right\}\right\} > \alpha\cos\beta \quad , z \in \mathcal{U}$$
and
$$Re\left\{e^{i\beta}\left\{\frac{[(s-t)w]^{1-\lambda} z g'(w)}{[g(sw) - g(tw)]^{1-\lambda}}\right\}\right\} > \alpha\cos\beta \quad ,$$

Where $g(w) = f^{-1}(w)$, $s,t \in \mathbb{C}$ with $s \neq t$, $|t| \leq 1$ $\beta \in (\frac{-\pi}{2}, \frac{-\pi}{2})$ and $\lambda \geq 0$.

LEMMA 1.1.

[12] *Let the function* $\phi(z)$ *given by* $\phi(z) = \sum_{n=1}^{\infty} \mathcal{B}_n z^n$ *be convex in* U. *If* $h(z) \prec \varphi(z)$, $(z \in U)$, *then* $|h_n| \leq |B_1|$, $n \in \mathbb{N} = \{1,2,3,...\}$.

LEMMA 1.2.

[9]*If* $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ *is a function with positive real part in* U *and* μ *is a complex number, then*
$$|c_2 - \mu c_1^2| \leq 2\max\{1; |2\mu - 1|\}.$$
The result is sharp for the functions given by
$$p(z) = \frac{1+z^2}{1-z^2} \text{ and } p(z) = \frac{1+z}{1-z}.$$

2. MAIN RESULTS

In this section, we obtain very interesting Fekete-Szego¨ inequalities for a certain subclass of analytic functions.

Theorem 2.1.

Let $\varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots$ *with* $B_1 \neq 0$. *If* $f \in$ A *satisfies the differential inequality*
$$e^{i\beta}\left[1 + \frac{1}{b}\left\{\frac{[(s-t)z]^{1-\lambda} R_q^{m+1} f(z)}{[R_q^m f(sz) - R_q^m f(tz)]^{1-\lambda}} - 1\right\}\right] \prec \phi(z)\cos\beta + i\sin\beta \quad . \tag{2.1}$$
Then
$$|a_3 - \mu a_2^2| \leq \frac{|b|\cos\beta\,|B_1|}{|\kappa_1|}\max\left\{1, \left|\frac{B_2}{B_1} + \frac{\kappa_1\kappa_2 B_1 b\cos\beta\, e^{-i\beta}}{[(1+q)+(\lambda-1)(s+t)]^2(1+q)^{2(m-1)}}\right|\right\}, \tag{2.2}$$

Where $\kappa_1 = (1+q+q^2)^m + (\lambda-1)(s^2+st+t^2)(1+q+q^2)^{m-1}$ *and*

$$\kappa_2 = \frac{[(1+q)^m + (\lambda-1)(s+t)(1+q)^{m-1}](1-\lambda)(s+t)(1+q)^{m-1}}{\kappa_1} - \frac{\frac{\lambda(1-\lambda)}{2}(s+t)^2(1+q)^{2(m-1)}}{\kappa_1} - \mu.$$

The result is sharp.

Proof. Let $f \in A$ satisfies (2.1), then there exist Schwarz function w analytic in U with $w(0) = 0$ and $|w(z)| < 1$ in U such that

$$(2.3) \qquad e^{i\beta} \left[1 + \frac{1}{b}\left\{\frac{[(s-t)z]^{1-\lambda} R_q^{m+1} f(z)}{[R_q^m f(sz) - R_q^m f(tz)]^{1-\lambda}} - 1\right\}\right] = \phi(w(z))\cos\beta + i\sin\beta .$$

Define $p(z)$ by

$$(2.4) \qquad p(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + \cdots .$$

Since $w(z)$ is a Schwarz function, it is clear that $Re p(z) > 0$ and $p(0) = 1$. Therefore

$$\begin{aligned}
\phi(z) &= \phi\left(\frac{p(z)-1}{p(z)+1}\right) \\
(2.5) \qquad &= \phi\left(\frac{1}{2}\left[c_1 z + \left(c_2 - \frac{c_1^2}{2}\right)z^2 + \left(c_3 - c_1c_2 + \frac{c_1^3}{4}\right)z^3 + \cdots\right]\right) \\
&= 1 + \frac{B_1c_1}{2}z + \left[\frac{B_1}{2}(c_2 - \frac{c_1^2}{2}) + \frac{B_2c_1^2}{4}\right]z^2 + \cdots .
\end{aligned}$$

Now by substituting (2.5) in (2.3)

$$e^{i\beta}\left[1 + \frac{1}{b}\left\{\frac{[(s-t)z]^{1-\lambda} R_q^{m+1} f(z)}{[R_q^m f(sz) - R_q^m f(tz)]^{1-\lambda}} - 1\right\}\right] = \left(1 + \frac{B_1c_1}{2}z + [\frac{B_1}{2}(c_2 - \frac{c_1^2}{2}) + \frac{B_2c_1^2}{4}]z^2 + \cdots\right)\cos\beta + i\sin\beta .$$

From this equation, we obtain

$$e^{i\beta}\frac{1}{b}\left[(\lambda-1)(s+t) + (1+q)\right](1+q)^{m-1}a_2 = \frac{B_1c_1}{2}\cos\beta$$

$$\begin{aligned}
e^{i\beta}\frac{1}{b}\Big\{ &\left[(\lambda-1)(s^2+st+t^2)(1+q+q^2)^{m-1} + (1+q+q^2)^m\right]a_3 \\
&- \frac{\lambda(\lambda-1)}{2}(s+t)^2(1+q)^{2(m-1)}a_2^2 \\
&+ (\lambda-1)(s+t)(1+q)^{m-1}[(1+q)^m + (\lambda-1)(s+t)(1+q)^{m-1}]a_2^2\Big\} \\
&= \left(\frac{B_1}{2}(c_2 - \frac{c_1^2}{2}) + \frac{B_2c_1^2}{4}\right)\cos\beta .
\end{aligned}$$

Or, equivalently

$$a_2 = \frac{e^{-i\beta} B_1c_1 b\cos\beta}{2\left[(1+q) + (\lambda-1)(s+t)\right](1+q)^{m-1}},$$

$$\begin{aligned}
a_3 = &\frac{e^{-i\beta}\, b\left(\frac{B_1C_2}{2} - \frac{B_1C_1^2}{4} + \frac{B_2C_1^2}{4}\right)\cos\beta}{(1+q+q^2)^m + (\lambda-1)(s^2+st+t^2)(1+q+q^2)^{m-1}} - \\
&\frac{\{[(1+q)^m + (\lambda-1)(s+t)(1+q)^{m-1}](1-\lambda)(s+t)(1+q)^{m-1}\}a_2^2}{(1+q+q^2)^m + (\lambda-1)(s^2+st+t^2)(1+q+q^2)^{m-1}} - \\
&\frac{\frac{\lambda(1-\lambda)}{2}(s+t)^2(1+q)^{2(m-1)}a_2^2}{(1+q+q^2)^m + (\lambda-1)(s^2+st+t^2)(1+q+q^2)^{m-1}}.
\end{aligned}$$

On simple computation, we have

$$a_3-\mu a_2^2=\frac{e^{-i\beta}\ b(\frac{B_1C_2}{2}-\frac{B_1C_1^2}{4}+\frac{B_2C_1^2}{4})\cos\beta}{(1+q+q^2)^m+(\lambda-1)(s^2+st+t^2)(1+q+q^2)^{m-1}}-\left\{\frac{[(1+q)^m+(\lambda-1)(s+t)(1+q)^{m-1}](1-\lambda)(s+t)(1+q)^{m-1}}{(1+q+q^2)^m+(\lambda-1)(s^2+st+t^2)(1+q+q^2)^{m-1}}-\frac{\frac{\lambda(1-\lambda)}{2}(s+t)^2(1+q)^{2(m-1)}}{(1+q+q^2)^m+(\lambda-1)(s^2+st+t^2)(1+q+q^2)^{m-1}}-\mu\right\}\times\left\{\frac{e^{-i\beta}\ B_1c_1b\cos\beta}{2\left[(1+q)+(\lambda-1)(s+t)\right](1+q)^{m-1}}\right\}^2.$$

Therefore

$$a_3-\mu a_2^2=\frac{B_1e^{-i\beta}\ b\cos\beta}{2\kappa_1}\left\{c_2-\vartheta c_1^2\right\}$$

where

$$\vartheta=\frac{1}{2}\left\{1-\frac{B_2}{B_1}-\frac{\kappa_1\kappa_2B_1b\cos\beta\ \ e^{-i\beta}}{\left[(1+q)+(\lambda-1)(s+t)\right]^2(1+q)^{2(m-1)}}\right\}.$$

On rearranging the terms and taking modulus both sides, our result now follows by application of Lemma1.2. The result is sharp for the functions

$$\left\{\frac{[(s-t)z]^{1-\lambda}R_q^{m+1}f(z)}{[R_q^mf(sz)-R_q^mf(tz)]^{1-\lambda}}\right\}=\phi(z^2)$$

and

$$\left\{\frac{[(s-t)z]^{1-\lambda}R_q^{m+1}f(z)}{[R_q^mf(sz)-R_q^mf(tz)]^{1-\lambda}}\right\}=\phi(z).$$

This completes the proof of the theorem.

Corollary 2.2.

Let $\varphi(z)=1+B_1z+B_2z^2+\cdots$ *. with* $B_1=6\ 0$*. If* $f\in A$ *satisfies the differential inequality*

$$\alpha\le Re\left\{1+\frac{1}{b}\left[\frac{zf'(z)}{f(z)}-1\right]\right\}<\gamma. \tag{2.6}$$

$$|\ a_3-\mu a_2^2\ |\le\frac{(\beta-\alpha)}{\sqrt{2}\pi}\sqrt{1-\cos\left(\frac{n(1-\alpha)}{\beta-\alpha}\right)}\max\left\{1;\frac{B_2}{B_1}+(1-2\mu)bB_1\right\}$$

Then. The result is sharp.

Proof. Let

$$\phi(z)=1+\frac{\beta-\alpha}{\pi}i\log\left(\frac{1-e^{2\pi i((1-\alpha)/(\beta-\alpha))}z}{1-z}\right).$$

Clearly, it can be seen that $\varphi(z)$ maps U onto a convex domain conformally and is of the form

$$h(z)=1+\sum_{n=1}^{\infty}B_nz^n$$

Where $B_n=\frac{\beta-\alpha}{n\pi}i\left(1-e^{2n\pi i((1-\alpha)/(\beta-\alpha))}\right)$. From the equivalent subordination condition proved by Kuroki and Owa in [6], the inequality (2.6) can be rewritten in the form

$$1+\frac{1}{b}\left[\frac{zf'(z)}{f(z)}-1\right]\prec\phi(z).$$

Following the steps as in Theorem 2.1, we get the desired result.

Corollary 2.3.

[17] *Let* $\varphi(z) = 1 + B_1z + B_2z^2 + \cdots$. *With* B_1 6= 0. *If f satisfies the following subordination condition*

$$1+\frac{1}{b}\left[\frac{zD_qf(z)}{f(z)}-1\right]\prec\phi(z)\ (b\in\mathcal{C}-\{0\})$$

then

$$|\,a_3-\mu a_2^2\,|\le\frac{|\,B_1b\,|}{([3]_q-1)}\max\left\{1;\left|\frac{B_2}{B_1}+\frac{B_1b}{[2]_q-1}\left(1-\frac{[3]_q-1}{[2]_q-1}\mu\right)\right|\right\}.$$

The result is sharp.

Proof. The result follows if we let $m = \beta = 0$, $\lambda = 1$, $t \to 0$ and $s \to 1$ in Theorem3.1. The result sharp for the function

$$\frac{zD_qf(z)}{f(z)}=\phi(z^2)\qquad\text{and}\qquad\frac{zD_qf(z)}{f(z)}=\phi(z).$$

Taking $q \to 1^-$ in the corollary 2.3, we obtain the Fekete Szego¨ inequality for functions belonging to the class of starlike function of complex order b.

Corollary 2.4.

(See Ravichandran et al. [15]*) Let* $\varphi(z) = 1 + B_1z + B_2z^2 + \cdots$. *With* B_1 6= 0. *If* $f(z)$ *belongs to the class of starlike function of complex order b . Then*

$$|\,a_3-\mu a_2^2\,|\le\frac{|\,B_1\,||\,b\,|}{2}\max\left\{1;\left|\frac{B_2}{B_1}+(1-2\mu)B_1b\right|\right\}.$$

The result is sharp.

3. INITIAL COEFFICIENT BOUNDS OF BISTARLIKE FUNCTIONS

We begin this section with finding the coefficient estimates of). $\mathcal{S}_b^{\lambda,m}(\beta, s, t, h)$

Theorem 3.1.

Let $f(z)$ *be of the form (1.1) and suppose that* $f(z)$ *is in the class* $S_b^{\lambda,m}(\beta,s,t,h)$, *then*

$$|\,a_2\,|\le\sqrt{\frac{2b\,|\,B_1\,|\cos\beta}{|\,D_1\,|}}$$

and

$$|\,a_3\,|\le\frac{2b\,|\,B_1\,|\cos\beta}{|\,D_1\,|}$$

Where

$D_1 = 2(\lambda-1)(s+t)(1+q)^{m-1}[(1+q)^m+(\lambda-1)(s+t)(1+q)^{m-1}]+$
$2[(\lambda-1)(s^2+st+t^2)(1+q+q^2)^{m-1}+(1+q+q^2)^m]-\lambda(\lambda-1)(s+t)^2(1+q)^{2(m-1)}$.

Proof. Let $f \in S_b^{\lambda,m}(\beta,s,t,h)$ and g denote that inverse of f to U . It follows from the Definition 1.1 that there exist functions $p(z), q(z) \in$ P(the class of function with positive real part), such that

$$\text{(3.1)}\qquad e^{i\beta}\left[1+\frac{1}{b}\left\{\frac{[(s-t)z]^{1-\lambda}R_q^{m+1}f(z)}{[R_q^mf(sz)-R_q^mf(tz)]^{1-\lambda}}-1\right\}\right]=p(z)\cos\beta+i\sin\beta$$

and

$$\text{(3.2)}\qquad e^{i\beta}\left[1+\frac{1}{b}\left\{\frac{[(s-t)w]^{1-\lambda}R_q^{m+1}g(w)}{[R_q^mg(sw)-R_q^mg(tw)]^{1-\lambda}}-1\right\}\right]=q(w)\cos\beta+i\sin\beta$$

$$\left[s,t\in\mathcal{C}\text{with } s\ne t,\ |\,t\,|\le1;\ b\in\mathbb{C}-\{0\};\ \lambda\ge0\ \beta\in\left(\frac{-\pi}{2},\frac{\pi}{2}\right)\right]$$

Where $p(z) \prec h(z)$ and $q(w) \prec g(w)$ have the forms $p(z) = 1 + p_1z + p_2z^2 + \cdots$

And $q(w) = 1 + q_1w + q_2w^2 + \cdots$ respectively. It follows from (3.1) and (3.2), we deduce

$$e^{i\beta}\ \frac{1}{b}[(\lambda-1)(s+t)+(1+q)](1+q)^{m-1}a_2=p_1\cos\beta\ , \tag{3.3}$$

$$e^{i\beta}\ \frac{1}{b}\Big\{[(\lambda-1)(s^2+st+t^2)(1+q+q^2)^{m-1}+(1+q+q^2)^m]a_3- \frac{\lambda(\lambda-1)}{2}(s+t)^2(1+q)^{2(m-1)}a_2^2 + (\lambda-1)(s+t)(1+q)^{m-1}[(1+q)^m+(\lambda-1)(s+t)(1+q)^{m-1}]a_2^2\Big\}=p_2\cos\beta\ , \tag{3.4}$$

and

$$-e^{i\beta}\ \frac{1}{b}[(\lambda-1)(s+t)+(1+q)](1+q)^{m-1}a_2=q_1\cos\beta\ , \tag{3.5}$$

$$e^{i\beta}\ \frac{1}{b}\Big\{2[(\lambda-1)(s^2+st+t^2)(1+q+q^2)^{m-1}+(1+q+q^2)^m]a_2^2- \frac{\lambda(\lambda-1)}{2}(s+t)^2(1+q)^{m-1}a_2^2+(\lambda-1)(s+t)(1+q)^{m-1}[(1+q)^m+(\lambda-1)(s+t)(1+q)^{m-1}]a_2^2- [(\lambda-1)(s^2+st+t^2)(1+q+q^2)^{m-1}+(1+q+q^2)^m]a_3\Big\}=q_2\cos\beta\ . \tag{3.6}$$

From (3.3) and (3.5) we obtain

$p_1=-q_1$.

By adding (3.4) and (3.6), we get

$$e^{i\beta}\ \frac{1}{b}\Big\{2(\lambda-1)(s+t)(1+q)^{m-1}[(1+q)^m+(\lambda-1)(s+t)(1+q)^{m-1}]+ 2[(\lambda-1)(s^2+st+t^2)(1+q+q^2)^{m-1}+(1+q+q^2)^m]-\lambda(\lambda-1)(s+t)^2(1+q)^{2(m-1)}\Big\}a_2^2 =(p_2+q_2)\cos\beta\ . \tag{3.7}$$

Since p, $q\in h(\mathrm{U})$, applying Lemma1.1, we have

$$|\ p_m\ |=|\ \frac{p^m(0)}{m!}\ |\le|\ \mathcal{B}_1\ |,\ m\in\mathcal{N} \tag{3.8}$$

and

$$|\ q_m\ |=|\ \frac{q^m(0)}{m!}\ |\le|\ \mathcal{B}_1\ |,\ m\in\mathcal{N}. \tag{3.9}$$

Applying (3.8) , (3.9) and Lemma1.1 for the coefficients p_1, p_2, q_1 and q_2, we readily get

$$|\ a_2\ |\le\sqrt{\frac{2b\ |\ B_1\ |\ \cos\beta}{|\ D_1\ |}}$$

Subtracting (3.6) from (3.4) we have

$$a_3=\frac{e^{-i\beta}\ b(p_2+q_2)\cos\beta}{D_1}+\frac{e^{-i\beta}\ b(p_2-q_2)\cos\beta}{2[(\lambda-1)(s^2+st+t^2)(1+q+q^2)^{m-1}+(1+q+q^2)^m]}.$$

Applying (3.8), (3.9)and Lemma1.1 once again for the coefficients p_1, p_2, q_1 and q_2, we readily get

$$|\ a_3\ |\le\frac{2b\ |\ B_1\ |\ \cos\beta}{|\ D_1\ |}$$

This completes the proof of Theroem3.1.

Remark 3.1. We note that the results analogous to Altınkaya and Yal¸cın[1] can be obtained if we let $m=0$, $q\to1^-$ and $b=1$ in Theorem3.1.

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