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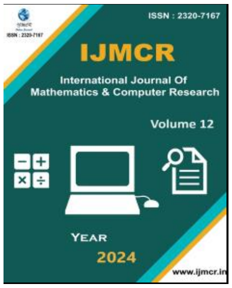

Coefficient Bounds of Bi-Univalent Function Involving Pseudo-Starlikeness Associated with Sigmoid Function Defined by Salagean Operator via Chebyshev Polynomial

GBOLAGADE, A. M.[1], ASIRU, T. M.[2], AWOLERE, I. T.[3]

[1,2]Department of Mathematics and Computing Science Education, Emmanuel Alayande University of Education, Oyo, Nigeria

[3]Department of Mathematical Sciences, Ondo State University of Science and Technology, Okitipupa, Nigeria

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Corresponding Author: **GBOLAGADE, A. M.**

ABSTRACT

Some special classes of univalent functions play an important role in geometric function theory because of their geometric properties. Many of such classes have been introduced and studied; some became well known, for example, the classes of convex, starlike, close to convex, strongly convex and strongly starlike functions. Previous studies by Awolere and Oladipupo (2018) now served as motivation and background to investigate certain classes of analytic, univalent and bi-univalent functions in terms of their coefficient bounds involving salagean and sigmoid functions via Chebyshev poynomial. The classes $H^n(\lambda,\beta,\gamma(s),\phi(z,t))$ are newly established classes for which coefficient bounds will be determined. The aim of the present work is to investigate coefficient bound for class $H^n(\lambda,\beta,\gamma(s),\phi(z,t))$ of pseudo-starlikeness associated with sigmoid functions defined by Salagean operator via Chebyshev polynomial, Fekete-szego problem will also be established and the Hankel of the function will be determined.

KEYWORDS: Bi-univalent, Sigmoid function, Salagean Operator, Chebyshev, polynomial

INTRODUCTION

The theory of a special function does not have a specific definition but it is of incredibly important to scientist and engineers who are concerned with Mathematical calculations and have a wide application in physics, Computer, engineering etc. Recently, the theory of special function has been outshining by other fields like real analysis, functional analysis, algebra, topology, differential equations. Bi-univalent function is a complex function which its inverse exist and it arises from univalent functions which is a branch of complex analysis. The concept of univalence of an analytical function g(z) in a simply connected domain refers to the fact that g(z) does not take the same value twice (Pommerenke,1983). It has found its use in solving a broad range of problems in hydrodynamic, aerodynamics, thermodynamics, electrodynamics, natural science and neural network.

Let function g(z) be regular in the unit disk $U=\{z:|z|<1\}$ and the function g(z) has a Maclaurin series expansion

$$g(z)=b_0+b_1z+b_2z^2+b_3z^3+b_4z^4+\ldots=\sum_{k=2}^{\infty}b_kz^k$$

Normalizing condition the class of analytic function on D that satisfies the $f(0)=0$ and $f'(0)=1$, then we have

$$f(z)=z+a_2z^2+a_3z^3+a_4z^4\ldots=z+\sum_{k=2}^{\alpha}a_kz^k \qquad (1)$$

where

$$a_k = \frac{b_k}{b_1}, k = 2,3,4,\ldots$$

In fact, with the univalence of g(z), the supposition that $b_1 \neq 0$ consequently hold true, otherwise for sufficiently small value of z the function f(z) takes the same value at least twice in the neighborhood of b_0 (Hayman, 1958). A function f(z) is said to be bi-univalent if its inverse exist. A function is analytic if and only if its Taylor series about x_0 converges to the function in some neighborhood for every x_0 in its domain. It's a function that is locally convergent power series. There exist both real analytic functions and complex analytic functions, categories that are similar in some ways, but different in others. Functions of each type are infinitely differentiable, but complex analytic function exhibit properties that do not hold generally for real analytic functions.

Chebyshev polynomial

Chebyshev polynomials have become increasingly important in numerical analysis, from both theoretical and practical points of view; there are four kinds of Chebyshev polynomials. The majority of books and research papers dealing with specific orthogonal polynomial of Chebyshev family, contain mainly results of Chebyshev polynomial of first and second kinds $T_n(x)$ and $U_n(x)$ and their numerous uses in different applications, see Atinkaya and Yacin (2016), Fadipe-Joseph, Kadir, Akinwumi, Adeniran (2018).

Fekete-Szego theorem

Fekete-Szego, (1933) proved that

$$\left|a_2^2 - \mu a_3\right| \leq \begin{cases} 4\mu - 3, \mu \geq 1 \\ 1 + \exp\left(\dfrac{-2\mu}{1-\mu}\right), 0 \leq \mu \leq 1 \\ 3 - 4\mu, \mu \leq 0 \end{cases} \tag{2}$$

Holds for the function $f \in S$ and the result is sharp. The problem of finding the sharp bounds for the non-linear functional $\left|a_3 - \mu a_2^2\right|$ of any compact family of function is popularly known as Fekete–Szego problem. Several authors at different time have applied the classical Fekete – Szego to various classes of functions to obtain various sharp bounds the likes of Ravichandran (2004), Selvaraj & Thirupathi (2014), Frastin & Darus (2003); Mohd & Darus (2012).

Pseudo-starlikeness functions

More Recently Babalola (2013) defined a new subclass λ−pseudo starlike function of order β (0≤β <1) satisfying the analytic condition

$$R\left(\frac{z(f'(z))^{\lambda}}{f(z)}\right) > \beta \qquad \left(z \in U, \lambda \geq 1 \in R\right)$$

and denoted by $L_{\lambda}(\beta)$. Further note that

If λ= 1, we have the class of starlike functions of order β, satisfying the condition

$$R\left(\frac{zf'(z)}{f(z)}\right) > \beta \qquad (z \in U) \tag{3}$$

denoted by $S^*(\beta)$

If β= 0, we simply write L instead of L(0). Babalola (2013) remarked that though for λ >1, these classes of λ−pseudo-starlike functions clone the analytic representation of starlike functions, it is not yet known the possibility of any inclusion relations between them. Thereafter numerous researchers have study pseudo-starlike functions in different direction. For further information see Laxmi and Sharma (2017), Awolere and Ibrahim-Tiamiyu (2017),. Murugurusundaramoorthy and Janani (2015).

METHOD AND TOOLS

In this present work, several methods shall be employed such as Salagean derivative operator $D^n f(z) = D(d^{n-1} f(z))$, Coefficient bounds, Fekete-szego problems and differentiation, taylor series expansion will also be used to transform pseudo-starlike functions. Salagean (1983) introduces the following differential operator:

$$D^0 f(z) = f(z)$$
$$D^1 f(z) = D(D^0 f(z)) = zf^1(z))'$$
$$D^n f(z) = D(D^{n-1} f(z)) = z(D^{n-1} f(z))' \qquad (4)$$

Where $(n \in \mathrm{N}_0 = \{1,2,\ldots\})$

Lemma 1: If a function $p \in P$ is given by $P(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \ldots, z \in U$

Then $|p_k| \leq 2, k \in N$ where P is the family of all functions analytic in U for which

P(0) = 1 and Re(p(z))>0 (Miller & Mocanu, 2000, Miller 1975)

For the purpose of this work the following lemma shall be recalled.

Lemma 2: If $\omega(z) = b_1 z + b_2 z^2 + \ldots,\ b_1 \neq 0,$ is analytic and satisfies $|\omega(z)| < 1$ on unit disk E, then for each $0 < r < 1$, $|\omega'(z)| < 1$ and $|\omega(re^{1\theta})| < 1$ unless $\omega(z) = e^{1\theta} z$ for some real number θ **Definition 1:** A function $f \in \sum$ is said to be the class $HT_{\Sigma}^{\gamma}(\lambda, \phi(z,t), \gamma(s)), \lambda \geq 1,$

$\gamma(s) = \dfrac{2}{1+e^{-s}}, s \geq 0, t \in [-1,1]$ If it satisfies the following conditions:

$$\frac{z[f_{\gamma}'(z)]^{\lambda}}{f_{\gamma}(z)} \prec \phi(z,t), \qquad z \in E.$$

$$\frac{\omega[g_{\gamma}'(\omega)]}{g_{\gamma}(\omega)} \prec \phi(\omega,t) \qquad \omega \in E,$$

Where g is an extension of $f^{-1} \in E$.

Definition 2: A function $f \in \sum$ is said to be the class $\mathrm{H}^n(\lambda, \beta, \gamma(s), \phi(z,t)$ $\lambda \geq 1$, $n \geq 0, \beta = 0$, $\gamma(s) = \dfrac{2}{1+e^{-s}}, s \geq 0, t \in [-1,1]$, $|c_i| \leq 1$ and $|d_1| \leq 1$. If it satisfies the following conditions

$$\Re e\left(\frac{zD^n f'(z)^{\lambda}}{D^n f(z)}\right) > \beta \qquad z \in E \quad \text{and} \quad \Re e\left(\frac{\omega D^n g'(\omega)^{\lambda}}{D^n g(\omega)}\right) > \beta \qquad \omega \in E$$

where g is an extension of $f^{-1} \in E$.

Remark **1:** If $\phi(z,t) = \left(\dfrac{1}{1-2tz+z^2}\right)^{\beta}$, then the class $HT_{\Sigma}^{\gamma}(\lambda, \phi(z,t), \gamma(s))$ reduces to the class $HT_{\Sigma}^{\gamma}(\lambda, \beta, \gamma(s)),\ 0 < \beta \leq 1$ and satisfies the following conditions:

$$\left|\arg \frac{z[f_{\gamma}'(z)]^{\lambda}}{f_{\gamma}(z)}\right| < \frac{\beta\pi}{2}, \qquad z \in E \text{ and } \left|\arg \frac{\omega[g_{\gamma}'(\omega)]}{g_{\gamma}(\omega)}\right| < \frac{\beta\pi}{2}, \qquad \omega \in E,$$

where g is an extension of $f^{-1} \in E$

Remark 2: If $s = 0$, then the class $HT_{\Sigma}^{\gamma}(\lambda, \phi(z,t), \gamma(s))$ reduces to the class $ST_{\Sigma}^{\gamma}(\lambda, \phi(z,t))$ and satisfies the following conditions:

$$\frac{z[f_{\gamma}'(z)]^{\lambda}}{f_{\gamma}(z)} \prec \phi(z,t), \qquad z \in E \text{ and } \frac{\omega[g_{\gamma}'(\omega)]}{g_{\gamma}(\omega)} \prec \phi(\omega,t) \qquad \omega \in E$$

where g is an extension of $f^{-1} \in E$.

Chebyshev polynomial

The Chebyshev polynomial of the first and second kinds is well known. In the case of a real variable x in (-1, 1), they are defined by

$T_n(x) = \cos n\theta$,

$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}$,

Where the subscript n denotes the polynomial degree and where $x = \cos\theta$,

$$H(z,t) = \frac{1}{1-2tz+z^2}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{\sin(n+1)\alpha}{\sin\alpha} z^n \ (z \in D).$$

Thus

$$H(z,t) = 1 + 2\cos\alpha z + (3\cos^2\alpha - \sin^2\alpha)z^2 + (4\cos^3\alpha - 4\cos\alpha\sin\alpha)z^3 + \dots\dots\dots \ (z \in D).$$

Following, we write

$$H(z,t) = 1 + U_1(t)z + U_2(t)z^2 + U_3(t)z^3 + \dots\dots\dots\dots \ \left(z \in D, t \in (-1,1)\right),$$

Where $U_{n-1} = \frac{\sin(n \arccos t)}{\sqrt{1-t^2}}$ $(n \in N)$ are the Chebyshev polynomials of the second kind. Also it is known that

$U_n(t) = 2tU_{n-1}(t) - U_{n-2}(t),$

and

$$U_1(t) = 2t$$
$$U_2(t) = 4t^2 - 1$$
$$U_3(t) = 8t^3 - 4t$$
$$U_4(t) = 16t^4 - 12t^2 + 1$$
$$U_5(t) = 32t^5 - 32t^4 + 6t \qquad (5)$$
$$U_6(t) = 64t^6 - 80t^4 + 24t^2 - 1$$

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The Chebyshev polynomials $T_n(t)$, $t \in [-1,1]$, of the first kind have the generating function of the form

$$\sum_{n=0}^{\infty} T_n(t)z^n = \frac{1 - tz}{1 - 2tz + z^2} (z \in D)$$

However, the Chebyshev polynomials of the first kind T_n (t) and the second kind U_u (t) are well connected by the following relationships

$$\frac{dT_n(t)}{dt} = nU_{n-1}(t),$$
$$T_n(t) = U_n(t) - tU_{n-1}(t),$$
$$2T_n(t) = U_n(t) - U_{n-2}(t).$$

Main Results

Theorem 1: Let $f \in \mathrm{H}^n(\lambda, \beta, \gamma(s), \phi(z,t))$ then

$$|a_2| \le \frac{2t\sqrt{2t}(1-\beta)}{\sqrt{\left|4t^2(2\lambda^2-\lambda)(1-\beta)2^{2n}-(2\lambda-1)^2(4t^2-1)2^{2n}\right|\gamma^2(s)}},$$

$$|a_3| \le \frac{4t^2(1-\beta)^2}{(2\lambda-1)^2 3^n\gamma(s)} + \frac{2t(1-\beta)}{(3\lambda-1)3^n\gamma(s)},$$

$$|a_4| \le \frac{4t^3(9-34\lambda-8\lambda^3)(1-\beta)^3}{3(2\lambda-1)^3(4\lambda-1)4^n\gamma(s)} + \frac{20t^3(1-\beta)}{(2\lambda-1)^3 4^n\gamma(s)} + \frac{10t^2(1-\beta)^2}{(2\lambda-1)(3\lambda-1)4^n\gamma(s)} + \frac{2t(1-\beta)}{(4\lambda-1)4^n\gamma(s)} + \frac{8t^2-2(1-\beta)}{(4\lambda-1)4^n\gamma(s)}$$
$$+\frac{8t^3-4t(1-\beta)}{(4\lambda-1)4^n\gamma(s)}$$

Proof: Since $\mathrm{H}^n(\lambda,\beta,\gamma(s),\phi(z,t))$. There exist two Chebyshev polynomials

$$\frac{zD^n[f_r'(z)]^\lambda}{D^n f_r(z)} = \beta + (1-\beta)H(z,t) \tag{6}$$

$$\frac{zD^n[g_r'(\omega)]^\lambda}{D^n g_r(\omega)} = \beta + (1-\beta)H(\omega,t) \tag{7}$$

Define the function $u(z)$ and $v(\omega)$ by

$$u(z) = c_1 z + c_2 z^2 + \ldots \tag{8}$$

$$v(\omega) = d_1\omega + d_2\omega^2 + \ldots \tag{9}$$

which is analytic in D with $u(0)=0$ and $|u(z)|<1$, $|v(w)|<1$ for all $z \in E$. It is well known that

$$|u(z)| = |c_1 z + c_2 z^2 + \ldots| < 1$$

$$|v(\omega)| = |d_1\omega + d_2\omega^2 + \ldots| < 1$$

and

$$|c_i| \le 1$$

$$|d_i| \le 1$$

Using (6) and (7) in (4) and (5) respectively we have

$$\frac{zD^n[f_r'(z)]^\lambda}{D^n f_r(z)} = \beta - (1-\beta)U_1(t)U(z) + (1-\beta)U_2(t)U^2(z) + \ldots \tag{10}$$

$$\frac{\omega D^n[g_r'(\omega)]^\lambda}{D^n g_r(\omega)} = \beta + (1-\beta)U_1(t)v(\omega) + (1-\beta)U_2(t)v^2(\omega) + \ldots \tag{11}$$

In the light of (5), (6), and (7) and from (10) and (11) we have,

$$1 + (2\lambda-1)2^n\gamma(s)a_2 z + \left[(3\lambda-1)3^n\gamma(s)a_3 + (2\lambda^2-4\lambda+1)2^{2n}\gamma^2(s)a_2^2\right]z^2$$
$$+\left\{(4\lambda-1)4^n\gamma(s)a_4 + (6\lambda^2-11\lambda+2)2^n3^n\gamma^2(s)a_2a_3 + \left[\frac{4\lambda(\lambda-1)(\lambda-2)}{3} + (4\lambda-2\lambda^2-1)\right]2^{3n}\gamma^3(s)a_2^3\right\}z^3 \text{ and}$$
$$+\ldots = 1 + (1-\beta)U_1(t)c_1 z + \left[(1-\beta)c_2U_1(t) + (1-\beta)c_1^3U_2(t)\right]z^2 +$$
$$\left[(1-\beta)U_1(t)c_3 + 2(1-\beta)c_1c_2U_2(t) + (1-\beta)c_1^2U_3(t)\right]z^3 + \ldots$$

$$1-(2\lambda-1)2^n\gamma(s)a_2w+\left[(2\lambda^2+2\lambda-1)2^{2n}\gamma^2(s)a_2^2-(3\lambda-1)3^n\gamma(s)a_3\right]w^2$$
$$+\left\{-(4\lambda-1)4^n\gamma(s)a_4+(6\lambda^2+9\lambda-3)2^n3^n\gamma^2(s)a_2a_3-\left[\frac{4\lambda(\lambda-1)(\lambda-2)}{3}+10\lambda^2+2\lambda-2\right]2^{3n}\gamma^3a_2^3\right\}w^3+\dots$$
$$=1+(1-\beta)U_1(t)d_1w+\left((1-\beta)d_2U_1(t)+(1-\beta)d_1^2U_1(t)\right)w^2+$$
$$\left((1-\beta)d_3U_1(t)+2(1-\beta)d_1d_2U_2(t)+(1-\beta)d_1^3U_3(t)\right)w^3+\dots,$$

This yields the following relations

$$(2\lambda-1)2^n\gamma(s)a_2=(1-\beta)U_1(t)c_1 \tag{12}$$
$$(3\lambda-1)3^n\gamma(s)a_3+(2\lambda^2-4\lambda+1)2^{2n}\gamma^2(s)a_2^2=(1-\beta)c_2U_1(t)+(1-\beta)c_1^2U_2(t) \tag{13}$$
$$(4\lambda-1)4^n\gamma(s)a_4+(6\lambda^2-11\lambda+2)2^n3^n\gamma^2(s)a_2a_3 \tag{14}$$
$$+\left[\frac{4\lambda(\lambda-1)(\lambda-2)}{3}+(4\lambda-2\lambda^2-1)\right]2^{3n}\gamma^3(s)a_2^3=(1-\beta)c_3U_1(t)+2(1-\beta)c_1c_2U(t)+(1-\beta)c_1^3U_3(t)$$

$$-(2\lambda-1)2^n\gamma(s)a_2=(1-\beta)U_1(t)d_1. \tag{15}$$
$$(2\lambda^2+2\lambda-1)2^{2n}\gamma^2(s)a_2^2-(3\lambda-1)3^n\gamma(s)a_3=(1-\beta)d_2U_1(t)+(1-\beta)d_1^2U_1(t) \tag{16}$$

and

$$-(4\lambda-1)4^n\gamma(s)a_4+(6\lambda^2+9\lambda-3)2^n3^n\gamma^2(s)a_2a_3 \tag{17}$$
$$-\left[\frac{4\lambda(\lambda-1)(\lambda-2)}{3}+10\lambda^2+2\lambda-2\right]2^{3n}\gamma^3(s)a_2^3=(1-\beta)d_3U_1(t)+2(1-\beta)d_1d_2U_2(t)+(1-\beta)d_1^3U_3(t)$$

from (12) and (13)

$$c_1=-d_1 \tag{18}$$

$$(2\lambda-1)2^n\gamma(s)a_2=(1-\beta)U_1(t)c_1$$
$$-(2\lambda-1)2^n\gamma(s)a_2=(1-\beta)U_1(t)d_1$$
$$a_2=\frac{(1-\beta)U_1(t)c_1}{(2\lambda-1)2^n\gamma(s)}=\frac{-(1-\beta)U_1(t)d_1}{(2\lambda-1)2^n\gamma(s)} \tag{19}$$

and

$$(2\lambda-1)^2 2^{2n}\gamma^2(s)a_2^2=(1-\beta)^2U_1^2(t)c_1^2$$
$$(2\lambda-1)^2 2^{2n}\gamma^2(s)a_2^2=(1-\beta)^2U_1^2(t)d_1^2$$
$$2(2\lambda-1)^2 2^{2n}\gamma^2(s)a_2^2=(1-\beta)^2U_1^2(t)\left[c_1^2+d_1^2\right] \tag{20}$$
$$\left[c_1^2+d_1^2\right]=\frac{2(2\lambda-1)^2 2^{2n}\gamma^2(s)a_2^2}{(1-\beta)^2U_1^2(t)}$$

Adding (13) and (16) and making use of equation (20)

$$(3\lambda-1)3^n\gamma(s)a_3+(2\lambda^2-4\lambda+1)2^{2n}\gamma^2(s)a_2^2=(1-\beta)c_2U_1(t)+(1-\beta)c_1^2U_2(t)$$
$$(2\lambda^2+2\lambda-1)2^{2n}\gamma^2(s)a_2^2-(3\lambda-1)3^n\gamma(s)a_3=(1-\beta)d_2U_1(t)+(1-\beta)d_1^2U_2(t)$$

$$[(3\lambda-1)3^n\gamma(s)a_3+(2\lambda^2-4\lambda+1)2^{2n}\gamma^2(s)a_2^2]+[(2\lambda^2+2\lambda-1)2^{2n}\gamma^2(s)a_2^2-(3\lambda-1)3^n\gamma(s)a_3]$$
$$=[(2\lambda^2-4\lambda+1)+(2\lambda^2+2\lambda-1)]2^{2n}\gamma^2(s)a_2^2$$
$$=[4\lambda^2-2\lambda]2^{2n}\gamma^2(s)a_2^2$$

Solving the R.H.S

$$(1-\beta)c_2U_1(t)+(1-\beta)c_1^2U_2(t)+(1-\beta)d_2U_1(t)+(1-\beta)d_1^2U_2(t)$$
$$[c_2+d_2](1-\beta)U_1(t)+[c_1^2+d_1^2](1-\beta)U_2(t)$$
$$[4\lambda^2-2\lambda]2^{2n}\gamma^2(s)a_2^2=[c_2+d_2](1-\beta)U_1(t)+[c_1^2+d_1^2](1-\beta)U_2(t)$$
$$2\lambda(2\lambda-1)2^{2n}\gamma^2(s)a_2^2=[[c_2+d_2]U_1(t)+[c_1^2+d_1^2]U_2(t)](1-\beta) \tag{21}$$

Upon simplification of (21), we have

$$2\lambda(2\lambda-`)2^{2n}\gamma^2(s)a_2^2=(1-\beta)U_1(t)[c_2+d_2]+(1-\beta)U_2(t)\left[\frac{2(2\lambda-1)^2 2^{2n}\gamma^2(s)}{(1-\beta)^2U_1(t)}\right]a_2^2$$
$$2\lambda(2\lambda-`)2^{2n}\gamma^2(s)a_2^2=(1-\beta)U_1(t)[c_2+d_2]+\frac{U_2(t)}{U_1^2(t)(1-\beta)}\left[2(2\lambda-1)^2 2^{2n}\gamma^2(s)\right]a_2^2$$
$$\frac{2\lambda(2\lambda-`)2^{2n}\gamma^2(s)a_2^2}{1}-\frac{U_2(t)}{U_1^2(t)(1-\beta)}\left[2(2\lambda-1)^2 2^{2n}\gamma^2(s)\right]a_2^2=(1-\beta)U_1(t)[c_2+d_2]$$
$$\left[\frac{2\lambda(1-\beta)(2\lambda-1)2^{2n}\gamma^2(s)U_1^2(t)-2U_2(t)(2\lambda-1)^2 2^{2n}\gamma^s(s)}{(1-\beta)U_1^2(t)}\right]a_2^2=(1-\beta)U_1(t)[c_2+d_2]$$
$$\left[2\lambda(1-\beta)(2\lambda-1)2^{2n}\gamma^2(s)-2U_2(t)(2\lambda-1)^2 2^{2n}\gamma^s(s)\right]a_2^2=(1-\beta)^2U_1^3(t)[c_2+d_2]$$
$$a_2^2=\frac{(1-\beta)^2U_1^3(t)[c_2+d_2]}{2\lambda(1-\beta)(2\lambda-1)2^{2n}\gamma^2(s)-2U_2(t)(2\lambda-1)^2 2^{2n}\gamma^s(s)}$$

By making use of lemma 2, we have

$$a_2^2=\frac{16(1-\beta)^2t^3}{2\lambda(1-\beta)(2\lambda-1)2^{2n}U_1^2(t)\gamma^2(s)-2U_2(t)(2\lambda-1)^2 2^{2n}\gamma^s(s)}$$
$$a_2^2=\frac{8(1-\beta)^2t^3}{\left|4\lambda t^2(2\lambda-1)(1-\beta)2^{2n}-(4t^2-1)(2\lambda-1)^2 2^{2n}\right|\gamma^2(s)}$$
$$a_2=\sqrt{\frac{8(1-\beta)^2t^3}{\left|4\lambda(2\lambda-1)(1-\beta)2^{2n}t^2-(4t^2-1)(2\lambda-1)^2 2^{2n}\right|\gamma^2(s)}}$$
$$|a_2|\le\frac{2t\sqrt{2t}(1-\beta)}{\sqrt{\left|4\lambda(2\lambda-1)(1-\beta)2^{2n}t^2-(4t^2-1)(2\lambda-1)^2 2^{2n}\right|\gamma^2(s)}}$$

Subtracting (13) and (16), making use of (18) and (19) we observe that

$$[(3\lambda-1)3^n\gamma(s)a_3+(2\lambda^2-4\lambda+1)2^{2n}\gamma^2(s)a_2^2=(1-\beta)c_2U_1(t)+(1-\beta)c_1^2U_2(t)]$$
$$-[(2\lambda^2+2\lambda-1)2^{2n}\gamma^2(s)a_2^2-(3\lambda-1)3^n\gamma(s)a_3=(1-\beta)d_2U_1(t)+(1-\beta)d_1^2U_1(t)]$$
$$2(3\lambda-1)3^n\gamma(s)a_3+[(2\lambda^2-4\lambda+1)-(2\lambda^2+2\lambda-1)]a_2^2$$

$$2(3\lambda-1)3^n\gamma(s)a_3=(6\lambda-2)2^{2n}\gamma^2a_2^2+(1-\beta)U_1(t)[c_2-d_2]+U_2^2(t)(1-\beta)[c_1^2-d_1^2]$$

$$2(3\lambda-1)3^n\gamma(s)a_3=2(3\lambda-1)2^{2n}\gamma^2(s)a_2^2+(1-\beta)U_1(t)[c_2-d_2]+U_2^2(1-\beta)^2[c_1^2-d_1^2]$$

$$a_3=\frac{a_2^2}{3^n}+\frac{(1-\beta)U_1(t)[c_2-d_2]}{2(3\lambda-1)3^n\gamma(s)}$$

$$a_3=\frac{U_1^2(t)(1-\beta)^2[c_1^2+d_1^2]}{2(2\lambda-1)^2 3^n\gamma^2(s)}+\frac{(1-\beta)U_1(t)[c_2-d_2]}{2(3\lambda-1)3^n\gamma(s)}$$

Applying Lemma 2 once again, we obtain

$$|a_3|\le\frac{4t^2(1-\beta)^2}{(2\lambda-1)^2}+\frac{2t(1-\beta)}{(3\lambda-1)3^n\gamma(s)}$$

Now from (14) and (17), it is evident that

$$(4\lambda-1)4^n\gamma(s)a_4+(6\lambda^2-11\lambda+2)2^n3^n\gamma^2(s)a_2a_3 \qquad -(4\lambda-1)4^n\gamma(s)a_4+(6\lambda^2+9\lambda-3)2^n3^n\gamma^2(s)a_2a_3$$

$$-\left[\frac{4\lambda(\lambda-1)(\lambda-2)}{3}+10\lambda^2+2\lambda-2\right]2^{3n}\gamma^3(s)a_2^3=(1-\beta)d_3U_1(t)+2(1-\beta)d_1d_2U_2(t)+(1-\beta)d_1^3U_3(t)$$

again by (13) and (15) we observe that

$$2(4\lambda-1)4^n\gamma(s)a_4+[6\lambda^2-11\lambda+2-6\lambda^2-9\lambda+3]2^n3^n\gamma^2(s)a_2a_3$$

$$+\left[\frac{8\lambda(\lambda-1)(\lambda-2)}{3}+4\lambda-2\lambda^2-1+10\lambda^2+2\lambda-2\right]2^{3n}\gamma^3(s)a_2^3$$

$$2(4\lambda-1)4^n\gamma^2(s)a_4-(20\lambda-5)2^n3^n\gamma^2(s)a_2a_3+\left[\frac{8\lambda(\lambda-1)(\lambda-2)}{3}+6\lambda+8\lambda^2-3\right]2^{3n}\gamma^3(s)a_2^3$$

$$2(4\lambda-1)4^n\gamma^2(s)a_4=5(4\lambda-1)2^n3^n\gamma^2(s)a_2a_3-\left[\frac{8\lambda(\lambda-1)(\lambda-2)}{3}+8\lambda^2+6\lambda-3\right]2^{3n}\gamma(s)a_2^3$$

$$2(4\lambda-1)4^n\gamma^2(s)a_4=5(4\lambda-1)2^n3^n\gamma^2(s)a_2a_3-\left[\frac{8\lambda(\lambda-1)(\lambda-2)}{3}+8\lambda^2+6\lambda-3\right]2^{3n}\gamma(s)a_2^3$$

$$+2(1-\beta)U_1(t)[c_3-d_3]+2(1-\beta)U_2(t)[c_1c_1-d_1d_2]+(1-\beta)U_3(t)[c_1^3-d_1^3]$$

$$2(4\lambda-1)4^n\gamma^2(s)a_4=5(4\lambda-1)2^n3^n\gamma^2(s)\frac{(1-\beta)U_1(t)c_1}{(2\lambda-1)2^n\gamma(s)}\left[\frac{U_1^2(t)(1-\beta)^2[c_1^2+d_1^2]}{2(2\lambda-1)^2\gamma^2(s)3^n}+\frac{(1-\beta)U_1(t)[c_2-d_2]}{2(3\lambda-1)3^n\gamma(s)}\right]$$

$$-\left[\frac{8\lambda(\lambda-1)(\lambda-2)}{3}+8\lambda^2+6\lambda-3\right]2^{3n}\gamma(s)\frac{(1-\beta)^3U_1^3(t)c_1^3}{(2\lambda-1)^3 2^{3n}\gamma^3(s)}+(1-\beta)U_1(t)[c_3-d_3]+2(1-\beta)U_2(t)[c_1c_1-c_1d_2]+$$

$$(1-\beta)U_3(t)[c_1^3-d_1^3]$$

$$a_4=\frac{(-8\lambda-34\lambda+9)(1-\beta)^3U_1^3(t)c_1^3}{6(2\lambda-1)^3(4\lambda-1)4^n\gamma(s)}+\frac{5(1-\beta)U_1^3(t)[c_1^2+d_1^2]}{4(2\lambda-1)^3 4^n\gamma(s)}+\frac{5(1-\beta)^2U_1^2(t)[c_1c_2-c_1d_2]}{4(2\lambda-1)(3\lambda-1)4^n\gamma(s)}$$

$$+\frac{(1-\beta)U_1(t)[c_3-d_3]}{2(4\lambda-1)4^n\gamma(s)}+\frac{2(1-\beta)U_2(t)[c_1c_2-c_1d_2]}{2(4\lambda-1)4^n\gamma(s)}+\frac{(1-\beta)U_3(t)[c_1^3-d_1^3]}{2(4\lambda-1)4^n\gamma(s)}$$

on the application of Lemma 2, above yields

$$|a_4| \le \frac{4t^3(9-34\lambda-8\lambda^3)(1-\beta)^3}{3(2\lambda-1)^3(4\lambda-1)4^n\gamma(s)} + \frac{20t^3(1-\beta)}{(2\lambda-1)^3 4^n\gamma(s)} + \frac{10t^2(1-\beta)^2}{(2\lambda-1)(3\lambda-1)4^n\gamma(s)} + \frac{2t(1-\beta)}{(4\lambda-1)4^n\gamma(s)} + \frac{8t^2-2(1-\beta)}{(4\lambda-1)4^n\gamma(s)} + \frac{8t^3-4t(1-\beta)}{(4\lambda-1)4^n\gamma(s)}$$

This completes the proof.

Theorem 2: Let $f \in H_{\Sigma}^{n}(\lambda,\beta,\gamma(s),\phi(z,t))$ and $\ell \in R$. Then

$$\left|a_3-\ell a_2^2\right| \le \frac{4t^2(1-\beta)^2}{(2\lambda-1)^2 3^n\gamma^2(s)} + \frac{2t(1-\beta)}{(2\lambda-1)3^n\gamma(s)} - \frac{4t^2\ell(1-\beta)^2}{(2\lambda-1)^2 2^{2n}\gamma^2(s)}$$

Proof: we have

$$a_2 = \frac{(1-\beta)U_1(t)c_1}{(2\lambda-1)2^n\gamma(s)}$$

$$a_2^2 = \frac{(1-\beta)^2U_1^2(t)c_1^2}{(2\lambda-1)^2 2^{2n}\gamma^2(s)}$$

$$a_3 = \frac{(1-\beta)^2U_1^2(t)\left[c_1^2+d_1^2\right]}{2(2\lambda-1)^2 3^n\gamma^2(s)} + \frac{(1-\beta)U_1(t)\left[c_2-d_2\right]}{2(3\lambda-1)3^n\gamma(s)}$$

Upon substitution for values of a_2 and a_3, we have

$$a_3-\ell a_2^2 = \frac{(1-\beta)^2U_1^2(t)\left[c_1^2+d_1^2\right]}{2(2\lambda-1)^2 3^n\gamma^2(s)} + \frac{(1-\beta)U_1(t)\left[c_2-d_2\right]}{2(3\lambda-1)3^n\gamma(s)} - \frac{\ell(1-\beta)^2U_1^2(t)c_1^2}{(2\lambda-1)^2 2^{2n}\gamma^2(s)} \qquad (22)$$

Applying lemma 2 for the coefficients of c_1, d_1, c_2, and d_2 in (22), we have

$$\left|a_3-\ell a_2^2\right| = \frac{4t^2(1-\beta)^2}{(2\lambda-1)^2 3^n\gamma^2(s)} + \frac{2t(1-\beta)}{(2\lambda-1)3^n\gamma(s)} - \frac{4t^2\ell(1-\beta)^2}{(2\lambda-1)^2 2^{2n}\gamma^2(s)}$$

which completes the proof

Theorem 3: Let $f \in H^n(\lambda,\beta,\gamma(s),\phi(z,t)$ Then

$$\left|a_2a_4-a_2^3\right| \le \frac{(-56\lambda^3-12\lambda^2+40\lambda-33)(1-\beta)^4t^4}{3(2\lambda-1)^4(4\lambda-1)2^n4^n\gamma^2(s)} + \frac{4t^3(8\lambda-3)(1-\beta)^3}{(2\lambda-1)^2(3\lambda-1)2^n4^n\gamma^2(s)} + \frac{t^2(-68\lambda^2+44\lambda-8)(1-\beta)^2}{3(2\lambda-1)(3\lambda-1)^2(4\lambda-1)2^n4^n\gamma^2(s)} - \frac{4t(1-\beta)^2}{(2\lambda-1)(4\lambda-1)\gamma^2 2^n4^n(s)}$$

Proof: we have

$$a_2 = \frac{(1-\beta)U_1(t)c_1}{(2\lambda-1)2^n\gamma(s)}$$

$$a_3 = \frac{(1-\beta)^2U_1^2(t)\left[c_1^2+d_1^2\right]}{2(2\lambda-1)^2 3^n\gamma^2(s)} + \frac{(1-\beta)U_1(t)\left[c_2-d_2\right]}{2(3\lambda-1)3^n\gamma(s)}$$

$$a_3^2 = \frac{(1-\beta)^4U_1^4(t)\left[c_1^2+d_1^2\right]^2}{4(2\lambda-1)^4 3^{2n}\gamma^4(s)} + \frac{(1-\beta)^3U_1^3(t)\left[c_1^2+d_1^2\right]\left[c_2-d_2\right]}{2(2\lambda-1)^2(3\lambda-1)3^{2n}\gamma^3(s)} + \frac{(1-\beta)^2U_1^2(t)\left[c_2-d_2\right]^2}{4(3\lambda-1)3^{2n}\gamma^2(s)}$$

$$a_4=\frac{(-8\lambda-34\lambda+9)(1-\beta)^3U_1^3(t)c_1^3}{6(2\lambda-1)^3(4\lambda-1)4^n\gamma(s)}+\frac{5(1-\beta)U_1^3(t)\left[c_1^2+d_1^2\right]}{4(2\lambda-1)^3 4^n\gamma(s)}+\frac{5(1-\beta)^2U_1^2(t)\left[c_1c_2-c_1d_2\right]}{4(2\lambda-1)(3\lambda-1)4^n\gamma(s)}$$
$$+\frac{(1-\beta)U_1(t)\left[c_3-d_3\right]}{2(4\lambda-1)4^n\gamma(s)}+\frac{2(1-\beta)U_2(t)\left[c_1c_2-c_1d_2\right]}{2(4\lambda-1)4^n\gamma(s)}+\frac{(1-\beta)U_3(t)\left[c_1^3-d_1^3\right]}{2(4\lambda-1)4^n\gamma(s)}$$

Upon substitution for values of $a_{2,}a_3$ and a_4, we have

$$a_2a_4-a_3^2=\frac{(-8\lambda-34\lambda+9)(1-\beta)^4U_1^4(t)c_1^4}{6(2\lambda-1)^4(4\lambda-1)2^n4^n\gamma^2(s)}+\frac{5(1-\beta)^2U_1^4(t)\left[c_1^2+d_1^2\right]c_1}{4(2\lambda-1)^4 2^n4^n\gamma^2(s)}+\frac{5(1-\beta)^3U_1^3(t)\left[c_1c_2-c_1d_2\right]c_1}{4(2\lambda-1)^2(3\lambda-1)2^n4^n\gamma^2(s)}$$

$$+\frac{(1-\beta)^2U_1^2(t)\left[c_3-d_3\right]c_1}{2(4\lambda-1)(2\lambda-1)2^n4^n\gamma^2(s)}+\frac{2(1-\beta)^2U_1U_2(t)\left[c_1c_2-c_1d_2\right]c_1}{2(4\lambda-1)(2\lambda-1)2^n4^n\gamma^2(s)}+\frac{(1-\beta)^2U_1U_3(t)\left[c_1^3-d_1^3\right]c_1}{2(4\lambda-1)(2\lambda-1)2^n4^n\gamma(s)}$$
$$-\frac{(1-\beta)^4U_1^4(t)\left[c_1^2+d_1^2\right]^2}{4(2\lambda-1)^4 3^{2n}\gamma^4(s)}+\frac{(1-\beta)^3U_1^3(t)\left[c_1^2+d_1^2\right]\left[c_2-d_2\right]}{2(2\lambda-1)^2(3\lambda-1)3^{2n}\gamma^3(s)}+\frac{(1-\beta)^2U_1^2(t)\left[c_2-d_2\right]^2}{4(3\lambda-1)3^{2n}\gamma^2(s)} \tag{23}$$

Applying lemma 2 for the coefficients of $c_{1,}d_1$, c_2, and d_2 yields

$$\left|a_2a_4-a_2^3\right|\le\frac{\left(-56\lambda^3-12\lambda^2+40\lambda-33\right)(1-\beta)^4t^4}{3(2\lambda-1)^4(4\lambda-1)2^n4^n\gamma^2(s)}+\frac{4t^3(8\lambda-3)(1-\beta)^3}{(2\lambda-1)^2(3\lambda-1)2^n4^n\gamma^2(s)}$$
$$+\frac{t^2\left(-68\lambda^2+44\lambda-8\right)(1-\beta)^2}{3(2\lambda-1)(3\lambda-1)^2(4\lambda-1)2^n4^n\gamma^2(s)}-\frac{4t(1-\beta)^2}{(2\lambda-1)(4\lambda-1)\gamma^2 2^n4^n(s)}$$

which completes the proof.

Corollary 1: Let $f\in H^0(\lambda,0,\gamma(s),\phi(z,t)$ from theorem1, we have

$$|a_2|\le\frac{2t\sqrt{2t}}{\sqrt{\left|4t^2\left(2\lambda^2-\lambda\right)-(2\lambda-1)^2\left(4t^2-1\right)\right|\gamma^2(s)}}$$

$$|a_3|\le\frac{4t^2}{(2\lambda-1)^2\gamma(s)}+\frac{2t}{(3\lambda-1)\gamma(s)}$$

$$|a_4|\le\frac{4t^3\left(9-34\lambda-8\lambda^3\right)}{3(2\lambda-1)^3(4\lambda-1)\gamma(s)}+\frac{20t^3}{(2\lambda-1)^3\gamma(s)}+\frac{10t^2}{(2\lambda-1)(3\lambda-1)\gamma(s)}+\frac{2t}{(4\lambda-1)4\gamma(s)}+\frac{8t^2-2}{(4\lambda-1)\gamma(s)}+\frac{8t^3-4t}{(4\lambda-1)\gamma(s)}$$

Corollary 2: Let $f\in H_\Sigma^0(\lambda,\beta,\gamma(s),\phi(z,t))$ and $\ell\in\mathbb{R}$ in theorem 2, we have

$$\left|a_3-\ell a_2^2\right|\le\frac{4t^2(1-\beta)^2}{(2\lambda-1)^2\gamma^2(s)}+\frac{2t(1-\beta)}{(2\lambda-1)\gamma(s)}-\frac{4t^2\ell(1-\beta)^2}{(2\lambda-1)^2\gamma^2(s)}$$

Corollary 3: Let $f\in H^0(\lambda,\beta,\gamma(s),\phi(z,t)$ in theorem 3, we have

$$\left|a_2a_4-a_2^3\right| \le \frac{\left(-56\lambda^3-12\lambda^2+40\lambda-33\right)(1-\beta)^4t^4}{3(2\lambda-1)^4(4\lambda-1)\gamma^2(s)}+\frac{4t^3(8\lambda-3)(1-\beta)^3}{(2\lambda-1)^2(3\lambda-1)\gamma^2(s)}$$
$$+\frac{t^2\left(-68\lambda^2+44\lambda-8\right)(1-\beta)^2}{3(2\lambda-1)(3\lambda-1)^2(4\lambda-1)\gamma^2(s)}-\frac{4t(1-\beta)^2}{(2\lambda-1)(4\lambda-1)\gamma^2(s)}$$

CONCLUSION

This work is focused on defining a bi-univalent function of order β and establishing coefficient bounds of bi-univalent function involving pseudo-starlikeness associated with sigmoid functions defined by Salagean operator via Chebyshev polynomials. The results gave birth to new subclasses of bi-univalent function for class $\mathrm{H}^n(\lambda,\beta,\gamma(s),\phi(z,t))$, Coefficient bounds for class $\mathrm{H}^n(\lambda,\beta,\gamma(s),\phi(z,t))$, and relevant connection to Fekete-szego problem for the class $\mathrm{H}^n_\Sigma(\lambda,\beta,\gamma(s),\phi(z,t))$ with respect to the coefficient bounds of bi-univalent function involving pseudo-starlikeness associated with sigmoid functions defined by Salagean operator via Chebyshev polynomials were established. The consequences of the results with respects to the choices of the parameters involved made establishment of corollaries possible.

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