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An Optimal Energy Control for Infinite Dimensional Systems with Applications

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Abstract

This paper investigates an optimal energy problem for infinite dimensional dynamic systems governed by partial differential equations. A new concept of an optimal energy problem is proposed, and its existence and uniqueness in Banach space are studied and obtained in terms of semigroup of linear operators and geometric properties of Banach space. Finally, an application of the optimal control to a Euler-Bernoulli robot beam is discussed, and it can be seen that the optimal energy control proposed in this paper is viable to the robot system.

Keywords: Optimal Energy Control, Infinite Dimensional System , Semigroup of Linear Operators.

AMS Subject Classification: 93C20 35B35

1 Introduction

Optimal control theory is a branch of control theory that deals with finding a control for a dynamical system over a period of time such that an objective function is optimized.

It has numerous applications in science, engineering and operations research. In the past of couple decades, the optimal control problems for infinite dimensional dynamic systems appear to be very important and significant in both theory and practice. We have worked on control theory for infinite dimensional dynamic systems[1]−[5], and obtained a number of quite meaningful results on stabilities and stabilizations. In the present paper, a new concept of optimal energy control is proposed, and its existence and uniqueness in Banach space are studied and obtained in terms of semigroup of linear operators and geometric properties of Banach space. Finally, an application of the optimal control to a Euler-Bernoulli robot beam is discussed, and it can be seen that the optimal energy control proposed in this paper is viable to the robot system.and dealt with an approach of semigroup of linear operators and in the frequency domain for infinite dimensional dynamic system governed by partial differential equations. Finally, we apply the optimal control theorem proposed to an Euler-Bernoulli robot system and show that the optimal energy control in this paper is viable to the robot system.

2 An Optimal Energy Control

In this section, we shall propose a new concept of an optimal energy control for the infinite dimensional system defined as follows:

$$\begin{aligned} &\tfrac{dy}{dt} = \mathcal{A}y(t) + \mathcal{B}u((y(t), t) + f(y(s), s) \\ &y(0) = y_0 \end{aligned} \tag{2.1}$$

where both state space $\mathcal{H}$ and control space $\mathcal{Y}$ are Hilbert spaces, the state function $y(t)$ on $[0, T]$ is valued in H, $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup $S(t)$, $t \geq 0$. $\mathcal{B}$ is a bounded linear operator from $L^2([0, T] : \mathcal{Y})$ to $L^2([0, T] : \mathcal{H})$, $u(y(t), t)$ is a control of the system, and $f(y(s), s)$ is a function in $L^2([0, T] : \mathcal{H})$.

Let us now investigate a specific optimal control, the minimum energy control of the system (2.1). We know that the minimum energy control in an abstract Banach space is just, in general, the minimum norm control. So, in an essential mathematics point of view, the topic about existence and uniqueness of the minimum energy control should be significant with a priority to be considered and studied.

Since $mathcal A$ is the infinitesimal generator of a C_0-semigroup $S(t)$, $t \geq 0$, we see from the semigroup theory of linear operators that for every control element $u(y(\cdot), \cdot) \in L^2([0,T] : \mathcal{Y})$, the system (2.1) has an unique mild solution

$$y(t) = S(t)y_0 + \int_0^t S(t-s)[\mathcal{B}(u(y(s),s)) + f(y(s),s)]ds \tag{2.2}$$

let $\varphi(\cdot)$ be an arbitrary element in $C([0,T], \mathcal{H})$, and

$$\rho = inf_{u \in L^2([0,T],\mathcal{Y})} \|\varphi(t) - S(t)y_0 - \int_0^t S(t-s)[\mathcal{B}u(y(s,s) + f(y(s),s)]ds\| \, ,$$

and define the admissible control set of the system (2.1) as follows

$$U_{ad} = \{u \in L^2([0,T], \mathcal{Y}) : \|\varphi(t) - S(t)y_0 - \int_0^t S(t-s)[\mathcal{B}u(y(s),s) + f(y(s),s)]\| \leq \rho + \epsilon\} \tag{2.3}$$

where ϵ is any positive number.

It can be seen from (2.2) that U_{ad} is not empty and contains infinitely many elements related to φ and ϵ. The minimum energy control problem is actually to find the element u, satisfying

$$\|u_0\| = min\{\|u\| : u \in U_{ad}\} \tag{2.4}$$

where u_0 is said to be a minimum energy control element.

Lemma 2.1 *The admissible control set U_{ad} defined by (2.2) is a closed convex set in Hilbert space $L^2([0,T] : \mathcal{Y})$.*

Proof. Convexity. For any u_1, $u_2 \in U_{ad}$ and a real number λ, $0 < \lambda < 1$, it is easy to see from (2.2) that

$$\|\varphi(t) - S(t)y_0 - \int_0^t S(t-s)[\mathcal{B}u_i(y(s),s) + f(y(s),s)])\| \quad \leq \quad \rho + \epsilon \quad i = 1,2 \tag{2.5}$$

and hence

$$\begin{aligned} & \|\varphi(t) - S(t)y_0 - \int_0^t S(t-s)[\mathcal{B}(\lambda u_1(y(s),s) + (1-\lambda)u_2(y(s),s)) + f(y(s),s)]ds\| \\ \leq \quad & \lambda\|\varphi(t) - S(t)y_0 - \int_0^t S(t-s)[\mathcal{B}u_1(y(s),s) + f(y(s),s)])ds\| \\ & +(1-\lambda)\|\varphi(t) - S(t)y_0 - \int_0^t S(t-s)[\mathcal{B}u_2(y(s),s) + f(y(s),s)]ds\| \\ \leq \quad & \lambda\rho + (1-\lambda)\epsilon = \rho + \epsilon. \end{aligned} \tag{2.6}$$

Consequently, we see from $\lambda u_1 + (1-\lambda)u_2 \in L^2([0,T];\mathcal{Y})$ and inequality (3.6) that $\lambda u_1 + (1-\lambda)u_2 \in U_{ad}$, Therefore, U_{ad} is a convex subset of $L^2([0,T],\mathcal{Y})$.

Closedness. Suppose $\{u_n\} \subset U_{ad}$, and $\lim_{n\to\infty} \|u_n - u^*\| = 0$. Let us now show that $u^* \in U_{ad}$. In fact, from the definition of U_{ad} we see that

$$\|\varphi(t) - S(t)y_0 - \int_0^t S(t-s)(\mathcal{B}u_n(y(s),s) + f(y(s),s)ds\| \leq \rho + \epsilon, \quad n = 1, 2, \cdots$$

It should be noted that $S(t),\ t \geq 0$ is a C_0-semigroup in Hilbert space $\mathcal{H}$, then there is a constant $M > 0$ such that $\sup\limits_{0\leq t\leq T} \|S(t)\| \leq M$. On the other hand, since $y(s)$ is differentiable on $[0,T]$, it is continuous on $[0,T]$, and hence $\{y(s) : s \in [0,T]\}$ is a bounded set in $L^2([0,T] : \mathcal{Y})$. Therefore, there is a constant $N > 0$ such that $\|\mathcal{B}u(y(s),s)\| \leq N \quad (0 \leq s \leq T)$, and furthermore,

$$\begin{array}{ll} & \|\varphi(t) - S(t)y_0 - \int_0^t S(t-s)(\mathcal{B}u^*(y(s),s) + f(y(s0,s))ds\| \\ \leq & \|\varphi(t) - S(t)y_0 - \int_0^t S(t-s)(\mathcal{B}u_n(y(s),s) + f(y(s),s))ds\| \\ & +\|\int_0^t S(t-s)\mathcal{B}[u_n(y(s),s) - u^*(y(s),s)]\| \\ \leq & \rho + \epsilon + M\|u_n - u^*\| \cdot NT \end{array} \tag{2.7}$$

Letting $n \to \infty$ leads to

$$\|\varphi(t) - S(t)y_0 - \int_0^t S(t-s)[\mathcal{B}u^*(y(s),s) + f(y(s),s]ds\| \leq \rho + \epsilon$$

which implies that $u^* \in U_{ad}$, and thus U_{ad} is a closed set. The proof is complete.

Theorem 2.1 *There exists an unique minimum energy control element in the admissible control set U_{ad} of the system (1.1)*

Proof. Since $L^2([0,T] : \mathcal{Y})$ is a Hilbert space, it is naturally a strict convex Banach Space. From the preceding Lemma, we have seen that U_{ad} is a closed convex set in $L^2([0,T] : \mathcal{Y})$, it follows from [2] that there is an unique element $u_0 \in U_{ad}$ such that

$$\|u_0\| = min \ \ \{\|u\| : u \in U_{ad}\}$$

According to the definition (2.3), u_0 is just the desired minimum energy control element of the system (2.1). The proof is complete.

Finally, we shall show that the minimum energy control element can be approached.

Theorem 2.2 *Suppose that u_0 is the minimum energy control element of the system (2.1), then there exists a sequence $\{u_n\} \subset U_{ad}$ such that $\{u_n\}$ converges strongly to u_0 in $L^2([0,T]:\mathcal{Y})$, namely,*

$$\lim_{n\to\infty} \|u_n - u_0\| = 0$$

Proof. Let $\{u_n\}$ be a minimized sequence in the admissible control set U_{ad}, then it follows that

$$\|u_{n+1}\| \leq \|u_n\|, \quad n = 1, 2, \cdots \tag{2.8}$$

and

$$\lim_{n\to\infty} \|u_n\| = inf\{\|u\| : u \in U_{ad}\} \tag{2.9}$$

It is obvious that $\{u_n\}$ is a bounded sequence in $L^2([0,T];\mathcal{Y})$, and so there is a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\{u_{n_k}\}$ weakly converges to an element $\widetilde{u}$ in $L^2([0,T];\mathcal{Y})$ (see [8]).

Since U_{ad} is a closed convex set in $L^2([0,T];\mathcal{Y})$ (see Lemma 2.1), we see from Mazur's Theorem that U_{ad} is a weakly closed set in $L^2([0,T]:\mathcal{Y})$, thus $\tilde{u} \in U_{ad}$. Combining (3.2) and employing the properties of limits of weakly convergent sequence on norm yield

$$\begin{aligned} inf\{\|u\| : u \in U_{ad}\} \leq \|\tilde{u}\| &\leq \underline{\lim}_{k\to\infty} \|u_{n_k}\| \\ &= \lim_{n_k\to\infty} \|u_{n_k}\| = \lim_{n\to\infty} \|u_n\| = \inf\{\|u\|; u \in U_{ad}\}. \end{aligned} \tag{2.10}$$

Thus, we have

$$\lim_{n\to\infty} \|u_n\| = \|\tilde{u}\| \tag{2.11}$$

and

$$\|\tilde{u}\| = inf\{\|u\| : u \in U_{ad}\}. \tag{2.12}$$

Since $\{u_{n_k}\}$ is weakly convergent to $\tilde{u}$, it follows from (2.3) that $\{u_{n_k}\}$ converges to $\tilde{u}$. Therefore, in view of Theorem 2.1 and (2.4) we see that $\tilde{u} = u_0$, namely, $\tilde{u}$ is the minimum energy control element. Thus, $\{u_{n_k}\}$ strongly converges to the minimum energy control element in $L^2([0,T]:\mathcal{Y})$. Without loss of generality, we can rewrite $\{u_{n_k}\}$ by $\{u_n\}$, then the conclusion of theorem is now obtained.

The Theorem 2.2 points out that the minimum energy control element can be approached by a weakly convergent sequence in the control space, which provides the theoretical basis of approximate computation for finding the minimum energy control element.

3 Application of Optimal Control to a Flexible Robot System

We shall start this section with the following flexible robot system[4],[5]:

$$\begin{cases} \ddot{y}(t,x) + 2\delta EI\rho^{-1}y''''(t,x) + EI\rho^{-1}y''''(t,x) = -x\ddot{\theta}(t), \\ \ddot{\varphi}(t,x) - 2\delta(GJ/\rho k^2)\dot{\varphi}''(t,x) - (GJ/\rho k^2)\varphi''(t,x) = 0, \\ m[(l+c)\ddot{\theta}(t) + \ddot{y}(t,l) + c\ddot{y}'(t,l) + e\ddot{\varphi}(t,l)] = EIy'''(t,l) - 2\delta EI\dot{y}'''(t,l) \\ m[(l+c)\ddot{\theta}(t) + \ddot{y}(t,l) + c\ddot{y}'(t,l) + e\ddot{\varphi}(t,l)] + J_0[\ddot{\theta}(t,l)] = -EI\ddot{y}(t,l) - 2\delta EI\dot{y}''(t,l) \\ me[(l+c)\ddot{\theta} + \ddot{y}(t,l) + c\ddot{y}'(t,l) + e\ddot{\varphi}(t,l)] + J_T\ddot{\varphi}(t,l) = -GJ\varphi'(t,l) - 2\delta GJ\dot{\varphi}(t,l) \\ y(t,0) = 0, \quad \dot{y}(t,0) = 0; \quad (t,0) = 0. \end{cases} \tag{3.1}$$

where the meaning of the symbols used above in system (3.1) are the same as in [4] and [5].

We shall choose the space $H^1 = L^2(0,l) \times L^2(0,l) \times R^3$ as the state space of system (3.1), which is a Hilbert space equipped with inner product

$$\langle w, v\rangle = \rho\int_0^l [w_1(x)\overline{v_1(x)}w_2(x)\overline{v_2(x)}]dx + \sum_{i=3}^{5} w_i\overline{v}_i$$

where $w = (w_1, \cdots, w_5)^T$, $v = (v_1, \cdots, v_5)^T$.

Define an operator $\Lambda : H \to H$ as follows

$$\Lambda w = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & M \end{pmatrix} w, \qquad M = \begin{pmatrix} m & mc & me \\ mc & J_0 + mc^2 & mce \\ me & mce & J_T + me^2 \end{pmatrix},$$

It is not difficult to verify that Λ is the positive definite linear operator on H'.

Let $B_1 : D(B_1) \to H$:

$$B_1 w = diag\left(\frac{EI}{\rho}\frac{d^4}{dx^4}, -\frac{GJ}{\rho K^2}\frac{d^2}{dx^2}, -EI\frac{d^3}{dx^3}, EI\frac{d}{dx}, GJ\frac{d}{dx}\right)w,$$

$$D(B_1) = \{w|w = (w_1, \cdots, w_5)^T, w_1''(\cdot) \in H^2(0,l), w_2'(\cdot) \in \tilde{H}(0,l)\}$$

$$\Omega = -(x, 0, m(l+c), J_0 + mc(l+c), me(l+c))^T,$$

After defining the previous operators, we can now write again the system (3.1) as follows:

$$\begin{cases} \ddot{w}(t) + 2\delta A_1 \dot{w}(t) + A_1 w(t) = \Lambda^{-1}\Omega\ddot{\theta}(t) \\ w(0) = w_0, \dot{w}(0) = w_{10} \end{cases} \tag{3.2}$$

here $A_1 = \Lambda^{-1}B_1, D(A_1) = D(B_1); w = (w_1, \cdots, w_5)^T, w_0, w_{10}$ are the initial values of the system (3.1).

Now, we set $v = (v_1, v_2)^T$, $v_1 = w$, $v_2 = \dot{w}$

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ -A_1 & -2\delta A_1 \end{pmatrix}, \quad D(\mathcal{A}) = D(A_1) \times D(A_1), \quad \mathcal{B} = \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix},$$

$$\mathcal{F}_1(t, v(t)) = (0, \Lambda^{-1}\ddot{\theta}(t))^T. \tag{3.3}$$

then system (3.3) is equivalent to the following first order evolution equation in $\mathcal{H} = H_1 \times H_1$

$$\begin{cases} \dot{v}(t) = \mathcal{A}v(t) + \mathcal{B}u(v(t), t) + \mathcal{F}_1(t, v(t)) \\ v(0) = v_0 \end{cases} \tag{3.4}$$

where $v_0 = (w_0, w_{10})^T$.

We start a discussion with defining following operators

$$A_i f = \rho^{-1}\frac{\partial^2}{\partial x^2}\left(p_i(x)\frac{\partial^2 f(x)}{\partial x^2}\right),$$

$$D(A_i) = \Big\{f \in H^4(0,l) | f, f', f'', (p_i f'')' \text{ are absolutely continuous function on } [0,l], (p_i f'')'' \in L^2[0,l], f(0) = f'(0) = 0, (p_i f'')(l,t) = 0.\ (p_i f'')'(l,t) = 0\Big\} \quad (i = 1, 2)$$

It should be noted that f in $D(A_1)$ can be written as $f = u - \tilde{u}$, and the function $\tilde{u}$ suits following differential equation

$$\frac{\partial^2}{\partial x^2}\left(p_1(x)\ \frac{\partial^2 \tilde{u}}{\partial x^2}\right) = 0. \tag{3.5}$$

By solving equation (3.5) we find

$$\tilde{u}(x) = \int_0^x \int_l^y mg\frac{z-1}{p_1(z)}dzdy. \tag{3.6}$$

Lemma 3.1 *The operators A_1 and A_2 are positive self-adjoint operators in $L^2[0, l]$, moreover, A_1^{-1} and A_2^{-1} exist, and they are compact operators.*

Proof Apply integration by parts with the definition of A and the boundary conditions included in $D(A_1)$ to find

$$\begin{aligned} < A_1 f, f > &= \int_0^l \rho^{-1}(p_1(x)f''(x))''\overline{f(x)}dx \\ &= \rho^{-1}\int_0^l (p_1(x)f''(x))'\overline{f'(x)}dx \\ &= \rho^{-1}\int_0^l p_1(x)f''(x)\overline{f''(x)}dx \geq 0. \end{aligned}$$

Since $0 < \alpha_1 \leq p_1(x) \leq \beta_1 < \infty$, we have

$$< A_1 f, f > \geq\ \rho^{-1}\alpha_1\|f''\|^2 \geq 0, \tag{3.7}$$

and hence, A_1 is a symmetric operator.

In order to show that A_1 is self-adjoint, it suffices to show that there is a constant $c > 0$ such that $\|A_1 f\| \geq c\|f\|, \quad f \in D(A_1)$ (see [10]).

In fact, we can see from (3.7) that

$$\|A_1 f\|\|f\|\ \geq < A_1 f, f > \geq\ \rho^{-1}\alpha_1\|f''\|^2$$

Applying the boundary conditions of f in $D(A_1)$, we can get the inequality[11]

$$\int_0^l |f(x)|^2 dx \leq \frac{l^4}{12}\int_0^l |f''(x)|^2 dx,$$

and hence

$$\|A_1 f\|\|f\| \geq \rho^{-1}\alpha_1(\frac{12}{l^4})\int_0^l |f(x)|^2 dx = c\|f\|^2$$

where $c = \dfrac{12\rho^{-1}\alpha_1}{l^4} > 0$. It follows that

$$\|A_1 f\| \geq c\|f\|, \tag{3.8}$$

and so A_1 is a positively defined self-adjoint operator.

It is easy to see from (3.7) that A_1^{-1} exists. Now set $A_1 f = g$, and $f = A_1^{-1} g$, then (3.7) gives us

$$\|A_1^{-1} g\| \leq \frac{1}{c}\|g\|,$$

this means that mapping $A_1^{-1} : H^4(0,l) \to H^4(0,l)$ is bounded, and

$$\|A_1^{-1}\| \leq \frac{1}{c}.$$

Thus, A_1^{-1} is a compact operator by Sobolev embedding theorem[15].

By similar manner, it can be shown that A_2 is a positively defined self-adjoint operator, and A_2^{-1} exists as a compact operator, and the proof is complete. ■

We now choose Hilbert space $H = L^2[0,l] \times L^2[0,l]$ as a state space of equations (3.1), on which inner product and norm are defined as follows:

$$(u, v)_H = (u_1, v_1) + (u_2, v_2), \quad u, v \in H,$$

here $u = (u_1, u_2)^T, v = (v_1, v_2)^T, (\cdot, \cdot)$ is the inner product on $L^2[0,l]$. Let

$$W(t) = \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}, \ F(t) = \begin{bmatrix} 0 \\ \mathcal{F}_1(t, v(t)) \end{bmatrix}$$

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \ D(A) = D(A_1) \times D(A_2).$$

Then the equations (3.1) with the initial-boundary conditions can be written as follows:

$$\begin{gathered} \ddot{W}(t) + \eta A \dot{W}(t) + AW(t) = F(t) \\ W(0) = W_0, \dot{W}(0) = W_{10} \end{gathered} \tag{3.9}$$

For the sake of establishing an evolution equation of the system (3.1), we introduce a Hilbert space $\mathcal{H} = H \times H$, on which inner product is defined as follows:

$$\langle \vec{u}, \vec{v} \rangle = \langle Au_{(1)}, Av_{(1)} \rangle_H + \langle u_{(2)}, v_{(2)} \rangle_H, \quad \vec{u}, \vec{v} \in \mathcal{H},$$

where $\vec{u}=(u_{(1)},u_{(2)})^T,\vec{v}=(v_{(1)},v_{(2)})^T$.

Let $\vec{u}=(u_{(1)},u_{(2)})^T, u_{(1)}=W, u_{(2)}=dW/dt$.

$$\mathcal{A}=\begin{bmatrix} 0 & I \\ -A & -\eta A \end{bmatrix}, \quad D(\mathcal{A})=D(A)\times D(A), \quad \mathcal{F}=\begin{bmatrix} 0 \\ F(t) \end{bmatrix}.$$

then (3.9) can be written as the following first-order evolution equation:

$$\begin{gathered} \frac{d\vec{u}(t)}{dt}=\mathcal{A}\vec{u}(t)+\mathcal{F}(t) \\ \vec{u}(0)=\vec{u}_0=(W(0),\dot{W}(0))^T. \end{gathered} \tag{3.10}$$

We shall discuss semigroup generation of operator $\mathcal{A}$ below. Let us start with investigation of spectrum of $\mathcal{A}$.

Lemma 3.2 *The operator A is a positive self-adjoint operator in Hilbert space H, and A^{-1} is a compact operator. Therefore, the spectrum $\sigma(A)$ of A consists entirely of isolated eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ with finite multiplicities so that*

$$0<\lambda_1<\lambda_2<\cdots<\lambda_n<\cdots \quad \textit{and} \quad \lambda_n\to\infty \quad (n\to\infty),$$

and the set of all normalized eigenvectors $\{\phi_{k_1},\phi_{k_2},\cdots,\phi_{k_{j_k}}\}_{k=1}^{\infty}$ constitutes a orthonormal basis of H.

Proof. Since $A=\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$ and A_1, A_2 are positive self-adjoint operators with compact inverses, A is also positive self-adjoint operator with compact inverse.

In the light of [15], we can arrive at the result that the spectrum of operator A consists entirely of isolated eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ with finite multiplicities, and

$$0<\lambda_1<\lambda_2<\cdots<\lambda_n; \quad \lambda_n\to+\infty \quad (n\to\infty).$$

If $\phi_{k_j}(j=1,2,\cdots,j_k)$ is the eigenvector for the eigenvalues λ_k of A:

$$A\phi_{k_j}=\lambda_k\phi_{k_j}, \qquad \|\varphi_{k_j}\|=1,$$

then $\{\phi_{k_1},\phi_{k_2},\cdots,\phi_{k_{j_k}}\}_{k=1}^{\infty}$ constitutes a orthonormal basis of H. ■

Theorem 3.1 Let $\sigma(\mathcal{A})$ and $\sigma_p(\mathcal{A})$ be the spectrum and the point spectrum of $\mathcal{A}$ respectively, then

(i) $$\sigma(\mathcal{A})=\sigma_p(\mathcal{A})\cup\{-1/\eta\}, \quad \sigma_p(\mathcal{A})=\{\xi_k,\mu_k\}_{k=1}^{\infty}.$$

where $\xi_k = (-\eta\lambda_k + \sqrt{(\eta\lambda_k)^2 - 4\lambda_k})/2$ and $\mu_k = (-\eta\lambda_k - \sqrt{(\eta\lambda_k)^2 - 4\lambda_k})/2$ are eigenvalues of $\mathcal{A}$, and the eigenvectors of $\mathcal{A}$ for ξ_k and μ_k are

$$\vec{\phi}_{k_j} = \frac{1}{\sqrt{\lambda_k^2 + |\xi_k|^2}} \begin{bmatrix} \phi_{k_j} \\ \xi_k \phi_{k_j} \end{bmatrix}, \quad \vec{\psi}_{k_j} = \frac{1}{\sqrt{\lambda_k^2 + |\mu_k|^2}} \begin{bmatrix} \phi_{k_j} \\ \mu_k \phi_{k_j} \end{bmatrix}, \|\vec{\phi}_{k_j}\| = \|\vec{\psi}_{k_j}\| = 1,$$
$$j = 1, 2, \cdots, j_k; \; k = 1, 2, \cdots$$

respectively.

(ii) If $\alpha \in \rho(\mathcal{A})$, then

$$(\alpha I - \mathcal{A})^{-1} = \begin{bmatrix} (\alpha^2+\eta\alpha A+A)^{-1}(\alpha+\eta A) & (\alpha^2+\eta\alpha A+A)^{-1} \\ -I+(\alpha^2+\eta\alpha A+A)^{-1}(\alpha^2+\eta\alpha A) & \alpha(\alpha^2+\eta\alpha A+A)^{-1} \end{bmatrix} \tag{3.11}$$

Proof By verifying directly, we can see that $\{\xi_k, \mu_k\}_{k=1}^{\infty} \subset \sigma_p(\mathcal{A})$, and $\vec{\phi}_{k_j}$, $\vec{\psi}_{k_j}$ are eigenvectors for ξ_k and μ_k respectively.

Since

$$\begin{aligned}
\lim_{k\to\infty} \xi_k &= \lim_{k\to\infty} \frac{-\eta\lambda_k + \sqrt{(\eta\lambda_k)^2 - 4\lambda_k}}{2} \\
&= \lim_{k\to\infty} \frac{(-\eta\lambda_k + \sqrt{(\eta\lambda_k)^2 - 4\lambda_k})(-\eta\lambda_k - \sqrt{(\eta\lambda_k)^2} - 4\lambda_k)}{2(-\eta\lambda_k - \sqrt{(\eta\lambda_k)^2 - 4\lambda_k})} \\
&= \lim_{k\to\infty} \frac{4\lambda_k}{-2(\eta\lambda_k + \sqrt{(\eta\lambda_k)^2 - 4\lambda_k})} \\
&= \lim_{k\to\infty} (-2) \frac{1}{\eta + \sqrt{\eta^2 - \frac{4}{\lambda_k}}} \\
&= (-2)\frac{1}{\eta+\eta} = -\frac{1}{\eta} < 0,
\end{aligned}$$

and similarly,

$$\begin{aligned}
\lim_{k\to\infty} \mu_k &= \lim_{k\to\infty} \frac{(-\eta\lambda_k - \sqrt{(\eta\lambda_k)^2 - 4\lambda_k})(-\eta\lambda_k + \sqrt{(\eta\lambda_k)^2} - 4\lambda_k)}{2(-\eta\lambda_k + \sqrt{(\eta\lambda_k)^2 - 4\lambda_k})} \\
&= \lim_{k\to\infty} \frac{4\lambda_k}{2(-\eta\lambda_k + \sqrt{(\eta\lambda_k)^2 - 4\lambda_k})} \\
&= \lim_{k\to\infty} (2) \frac{1}{-\eta + \sqrt{\eta^2 - \frac{4}{\lambda_k}}} \\
&= -\infty,
\end{aligned}$$

we have

$$-\frac{1}{\eta} \in \sigma(\mathcal{A}) \quad \text{and} \quad \{\xi_k, \mu_k\}_{k=1}^{\infty} \cup \{-\frac{1}{\eta}\} \subseteq \sigma(\mathcal{A}).$$

On the other hand, let $\alpha \neq \xi_k, \mu_k, -\frac{1}{\eta}$, and $f(\lambda) = \alpha^2\lambda^{-1} + \eta\alpha + 1$, then $f(A) = \alpha^2 A^{-1} + \eta\alpha + I$ by functional calculus. Since the extended spectrum, $\sigma_e(A)$ of A is equal to $\sigma(A) \cup \{\infty\} = \{\lambda_k\}_{k=1}^{\infty} \cup \{\infty\}$, and $\alpha \neq \xi_k$, μ_u and $-\frac{1}{\eta}$, we have

$$\begin{aligned} f(\lambda_k) &= \alpha^2\lambda_k^{-1} + \eta\alpha + 1 \neq 0, \quad \text{otherwise } \ \alpha = \xi_k \text{ or } \mu_k; \\ f(\infty) &= \lim_{k\to\infty} f(\lambda_k) = \eta\alpha + 1 \neq 0, \quad \text{otherwise } \alpha = -\frac{1}{\eta}. \end{aligned}$$

It follows from the spectral mapping theorem[15] that $0 \notin f(\sigma_e(A)) = \sigma(f(A))$, namely $0 \in \rho(f(A))$. This implies that the inverse of $\alpha^2 + \eta\alpha A + A$ exists, and

$$(\alpha^2 + \eta\alpha A + A)^{-1} = A^{-1}(\alpha^2 A^{-1} + \eta\alpha + I)^{-1} = A^{-1}[f(A)]^{-1}$$

is a bounded linear operator on H. Therefore, the operator defined by

$$T = \begin{bmatrix} (\alpha^2 + \eta\alpha A + A)^{-1}(\alpha + \eta A) & (\alpha^2 + \eta\alpha A + A)^{-1} \\ -I + (\alpha^2 + \eta\alpha A + A)^{-1}(\alpha^2 + \eta\alpha A) & \alpha(\alpha^2 + \eta\alpha A + A)^{-1} \end{bmatrix}$$

is a bounded linear operator on $\mathcal{H}$. A simple computation shows that

$$(\alpha I - \mathcal{A})T = I_{\mathcal{H}}, \quad T(\alpha I - A) = I_{D(A)},$$

and $\alpha \in \rho(\mathcal{A})$. This implies that $\sigma(\mathcal{A}) \subseteq \{\xi_k, \mu_k\}_{k=1}^{\infty} \cup \{-\frac{1}{\eta}\}$. Thus, $\sigma(\mathcal{A}) = \{\xi_k, \mu_k\}_{k=1}^{\infty} \cup \{-\frac{1}{\eta}\}$, and $(\alpha I - \mathcal{A})^{-1} = T$. The proof of the theorem is complete. ■

Corollary 3.1. $\mathcal{A}$ is a closed linear operator.

Proof From Theorem 1, we know that $\rho(\mathcal{A}) \neq \emptyset$. Since for any $\alpha \in \rho(\mathcal{A}), (\alpha I - \mathcal{A})^{-1}$ is a bounded linear operator, it is closed. It follows that $\lambda I - \mathcal{A} = [(\lambda I - \mathcal{A})^{-1}]^{-1}$ is closed, and hence $\mathcal{A}$ is a closed operator.

Corollary 3.2 $\sup\{Re\ \lambda | \lambda \in \sigma(\mathcal{A})\} = -\omega < 0$.

Proof It can be seen from the expressions of ξ_k and μ_k that $Re\ \xi_k < 0$ and $Re\ \mu_k < 0 \quad (k = 1, 2, \cdots)$, and

$$\lim_{k\to\infty} = \xi_k = -\frac{1}{\eta} \quad \text{and} \quad \lim_{k\to\infty} \mu_k = -\infty$$

and therefore,

$$\lim_{k\to\infty} Re\ \xi_k = -\frac{1}{\eta} \quad \text{and} \quad \lim_{k\to\infty} Re\ \mu_k = -\infty.$$

It follows that $\sup\{Re\, \sigma_n | \sigma_n \in \sigma_p(\mathcal{A})\} \overset{\text{def}}{=} -\omega_1 < 0$. Thus, we conclude from (i) of Theorem 3.1 that $\sup\{Re\ \mu | \mu \in \sigma(\mathcal{A})\} = \max\{-\omega_1, -\frac{1}{\eta}\} \overset{\text{def}}{=} -\omega < 0$. ■

Lemma 3.3 The family of the following vectors

$$\left\{\binom{\phi_{k_1}}{0},\binom{0}{\phi_{k_1}},\cdots,\binom{\phi_{k_{j_k}}}{0},\binom{0}{\phi_{k_{j_k}}}\right\}_{k=1}^{\infty}. \tag{3.12}$$

constitute an orthonormal basis of $\mathcal{H}$.

Proof Since $\{\phi_{k_1},\cdots,\phi_{k_{j_k}}\}_{k=1}^{\infty}$ is an orthonormal basis of H, we see from Lemma 3.2 that form any $\binom{v}{w}\in\mathcal{H},\ v,w\in H$,

$$v=\sum_{k=1}^{\infty}\sum_{j=1}^{j_k}v_{k_j}\phi_{k_j},\quad \text{where}\quad v_{k_j}=<v,\,\phi_{k_j}>_H$$

$$w=\sum_{k=1}^{\infty}\sum_{j=1}^{j_k}w_{k_j}\phi_{k_j},\quad \text{where}\quad w_{k_j}=<w,\,\phi_{k_j}>_H$$

Hence,

$$\begin{pmatrix}v\\w\end{pmatrix}=\sum_{k=1}^{\infty}\sum_{j=1}^{j_k}\begin{pmatrix}v_{k_j}\phi_{k_j}\\w_{k_j}\phi_{k_j}\end{pmatrix}=\sum_{k=1}^{\infty}\sum_{j=1}^{j_k}\left[v_{k_j}\begin{pmatrix}\phi_{k_j}\\0\end{pmatrix}+w_{k_j}\begin{pmatrix}0\\\phi_{k_j}\end{pmatrix}\right],$$

and

$$\left\|\begin{pmatrix}v\\w\end{pmatrix}\right\|^2=\sum_{k=1}^{\infty}\sum_{j=1}^{j_k}\left(\lambda_k^2|v_{k_j}|^2+|w_{k_j}|^2\right)$$

Thus, the family (3.8) of vectors is complete, and therefore it constitutes an orthonormal basis of $\mathcal{H}$. ■

Lemma 3.4 (i) Let $\eta\neq 2\lambda_k^{-\frac{1}{2}}$, $(k=1,2,\cdots)$ then the set of the pairs eigenvectors of $\mathcal{A}$, $\{\vec{\phi}_{k_1},\vec{\psi}_{k_1},\cdots,\vec{\phi}_{k_{j_k}},\vec{\psi}_{k_{j_k}}\}_{k=1}^{\infty}$, constitutes a Riesz basis of $\mathcal{H}$.

(ii) If $\eta=2\lambda_{k^*}^{-\frac{1}{2}}$ for some k^* (there exist at most one k^*), then the set of eigenvectors of $\mathcal{A}$,

$\{\vec{\phi}_{k_1},\vec{\psi}_{k_1},\cdots,\vec{\phi}_{k_{j_k}},\vec{\psi}_{k_{j_k}}\}_{k_i\neq k^*}\cup\{\vec{\phi}_{k_1^*},\cdots,\vec{\phi}_{k^*_{j_{k^*}}}\}$ together with $\{\binom{0}{\phi_{k_1^*}},\cdots,\binom{0}{\phi_{k^*_{j_{k^*}}}}\}$, constitute a basis of $\mathcal{H}$.

Proof (i) Since for arbitrary $\begin{pmatrix}v\\w\end{pmatrix}\in\mathcal{H}$, we have from Lemma 3 and the definitions of $\vec{\phi}_k$ and $\vec{\psi}_k$ that

$$\begin{pmatrix}v\\w\end{pmatrix}=\sum_{k=1}^{\infty}\sum_{j=1}^{j_k}[v_{k_j}\begin{pmatrix}\phi_{k_j}\\0\end{pmatrix}+w_{k_j}\begin{pmatrix}0\\\phi_{k_j}\end{pmatrix}]=\sum_{k=1}^{\infty}\sum_{j=1}^{j_k}(a_{k_j}^{(1)}\vec{\phi}_{k_j}+a_{k_j}^{(2)}\vec{\psi}_{k_j})$$

where

$$a_{k_j}^{(1)}=(-\mu_k v_{k_j}-w_{k_j})\frac{\sqrt{\lambda_k^2+|\xi_k|^2}}{\sqrt{(\eta\lambda_k)^2-4\lambda_k}},\quad a_{k_j}^{(2)}=(\xi_k v_{k_j}-w_{k_j})\frac{\sqrt{\lambda_k^2+|\mu_k|^2}}{\sqrt{(\eta\lambda_k)^2-4\lambda_k}}$$

and thus it is an Riesz basis of $\mathcal{H}$[5] .

(ii) If $\eta = 2\lambda_{k^*}^{-\frac{1}{2}}$, we can obtain from Theorem 1 that $\xi_{k^*} = \mu_{k^*} = \lambda_{k^*}^{\frac{1}{2}}$, and the eigenvectors corresponding to ξ_{k^*} and μ_{k^*} are as follows

$$\vec{\phi}_{k_j^*} = \vec{\psi}_{k_j^*} = \frac{1}{\sqrt{\lambda_{k^*}^2 + \lambda_{k^*}}} \begin{pmatrix} \phi_{k_j^*} \\ -\lambda_{k^*}^{\frac{1}{2}} \phi_{k_j^*} \end{pmatrix} \quad (j = 1, 2, \cdots, j_{k^*})$$

Then

$$\begin{aligned} v_{k_j^*} \begin{pmatrix} \phi_{k_j^*} \\ 0 \end{pmatrix} + w_{k_j^*} \begin{pmatrix} 0 \\ \phi_{k_j^*} \end{pmatrix} &= (\lambda_{k^*}^{\frac{1}{2}} v_{k_j^*} + w_{k_j^*}) \begin{pmatrix} 0 \\ \phi_{k_j^*} \end{pmatrix} + v_{k*_j} \begin{pmatrix} \phi_{k^*} \\ -\lambda_{k^*}^{\frac{1}{2}} \phi_{k_j^*} \end{pmatrix} \\ &= b_{k_j^*}^{(1)} \begin{pmatrix} 0 \\ \phi_{k_j^*} \end{pmatrix} + b_{k_j^*}^{(2)} \vec{\phi}_{k_j^*} \quad (j = 1, 2, \cdots, j_{k^*}) \end{aligned}$$

where

$$b_{k_j^*}^{(1)} = \lambda_{k^*}^{\frac{1}{2}} v_{k_j^*} + w_{k_j^*}, \quad b_{k_j^*}^{(2)} = v_{k_j^*} \sqrt{\lambda_{k^*}^2 + \lambda_{k^*}}$$

Thus, for every $\begin{pmatrix} v \\ w \end{pmatrix} \in \mathcal{H}$, we have

$$\begin{aligned} \begin{pmatrix} v \\ w \end{pmatrix} &= \sum_{k=1}^{\infty} \sum_{j=1}^{j_k} [v_{k_j} \begin{pmatrix} \phi_{k_j} \\ 0 \end{pmatrix} + w_{k_j} \begin{pmatrix} 0 \\ \phi_{k_j} \end{pmatrix}] \\ &= \sum_{j=1}^{j_{k^*}} [b_{k_j^*}^{(1)} \begin{pmatrix} 0 \\ \phi_{k_j^*} \end{pmatrix} + b_{k_j^*}^{(2)} \vec{\phi}_{k_j^*}] + \sum_{\substack{k=1 \\ k \neq k^*}}^{\infty} \sum_{j=1}^{j_k} (a_{k_j}^{(1)} \vec{\phi}_{k_j} + a_{k_j}^{(2)} \vec{\psi}_{k_j}), \end{aligned}$$

and so the conclusion of (ii) of Lemma 3.4 is obtained. ■

The semigroup generation of $\mathcal{A}$ is stated and proved in next theorem.

Theorem 3.2 The operator $\mathcal{A}$ is an infinitesimal generator of a C_0-semigroup $T(t)$ on $\mathcal{H}$, and there are constants $M > 0$ such that

$$\|T(t)\| \leq M e^{-\omega t} \tag{3.13}$$

where $-\omega = \sup\{Re\, \mu | \mu \in \sigma(\mathcal{A})\} < 0$.

Proof We shall prove Theorem 3.2 in two different cases,

Case 1. $\eta \neq 2\lambda_k^{-\frac{1}{2}} (k = 1, 2, \cdots)$. For the sake of simplicity, we denote the eigenpairs of $\mathcal{A}$ by $\{\sigma_n, \vec{e}_n\}, n = 1, 2, \cdots$. For every real λ, $\lambda > -\omega = \sup\{Re\ \mu | \mu \in \sigma(\mathcal{A})\}$, we see that

$\lambda \in \rho(\mathcal{A})$. For any $\vec{u} \in \mathcal{H}$, since $\{\vec{e}_n\}$ constitutes a Riesz basis of $\mathcal{H}$ (see (i) of Lemma 4), $\vec{u} = \sum_{n=1}^{\infty} a_n \vec{e}_n$. A simple computation shows that

$$(\lambda I - \mathcal{A})^{-1}\vec{u} = \sum_{n=1}^{\infty} a_n \frac{1}{\lambda - \sigma_n}\vec{e}_n$$

and

$$\begin{aligned} \|[\lambda I - \mathcal{A})^{-1}]^m \vec{u}\| &= \|\sum_{n=1}^{\infty} a_n \frac{1}{(\lambda - \sigma_n)^m}\vec{e}_n\| \\ &= \|\sum_{n=1}^{\infty} a_n \frac{(\lambda+\omega)^m}{(\lambda - \sigma_n)^m}\frac{1}{(\lambda+\omega)^m}\vec{e}_n\| \end{aligned}$$

It should be noted that $|(\lambda+\omega)^m/(\lambda-\sigma_n)^m| \leq 1$ because $-\omega = \sup\{Re\ \mu | \mu \in \sigma(\mathcal{A})\}$ and $\lambda > -\omega$. Therefore,

$$\|[\lambda I - \mathcal{A})^{-1}]^m \vec{u}\| \leq \frac{1}{(\lambda+\omega)^m}\|\sum_{n=1}^{\infty} a_n \vec{e}_n\| = \frac{1}{(\lambda+\omega)^m}\|\vec{u}\|$$

We thus arrive at the following result:

$$\|[(\lambda I - \mathcal{A})^{-1}]^m\| \leq \frac{1}{(\lambda+\omega)^m}, \quad \lambda > -\omega, \quad m = 1, 2, \cdots$$

It follows from [14] that $\mathcal{A}$ is the infinitesimal generator of a C_0-Semigroup $T(t)$ on $\mathcal{H}$, and $\|T(t)\| \leq Me^{-\omega t}$, where $M \geq 1$.

Case 2. $\eta = 2\lambda_{k^*}^{-\frac{1}{2}}$ for some positive integer k^*. We see from (ii) of Lemma 3.4 that for any $\vec{u} \in \mathcal{H}$,

$$\vec{u} = \sum_{\substack{k=1 \\ k \neq k^*}}^{\infty} \sum_{j=1}^{n_k} (a_{k_j}^{(1)}\vec{\phi}_{k_j} + a_{k_j}^{(2)}\vec{\psi}_{k_j}) + \sum_{j=1}^{n_{k^*}} [b_{k_j^*}^{(1)} \begin{pmatrix} 0 \\ \phi_{k_j^*} \end{pmatrix} + b_{k_j^*}^{(2)}\vec{\psi}_{k_j^*}]$$

Since

$$\mathcal{A}\vec{\psi}_{k_j^*} = \mu_{k^*}\vec{\psi}_{k_j^*}, \quad j = 1, 2, \cdots, j_{k^*}$$

and

$$\begin{aligned} \mathcal{A}\begin{pmatrix} 0 \\ \phi_{k_j^*} \end{pmatrix} &= \begin{pmatrix} 0 & I \\ -A & -\eta A \end{pmatrix}\begin{pmatrix} 0 \\ \phi_{k_j^*} \end{pmatrix} = \begin{pmatrix} \phi_{k_j^*} \\ -\eta A\phi_{k_j^*} \end{pmatrix} = \begin{pmatrix} \phi_{k_j^*} \\ -2\lambda_{k^*}^{\frac{1}{2}} A\phi_{k_j^*} \end{pmatrix} \\ &= \begin{pmatrix} \phi_{k_j^*} \\ -\lambda_{k^*}^{\frac{1}{2}}\phi_{k_j^*} \end{pmatrix} + \begin{pmatrix} 0 \\ -\lambda_{k^*}^{\frac{1}{2}}\phi_{k_j^*} \end{pmatrix}. \end{aligned}$$

Since $\eta = 2\lambda_{k^*}^{-\frac{1}{2}}$, we refer to (i) of Theorem 3.1 to find

$$\begin{aligned}\mu_{k^*} &= (-\eta\lambda_{k^*} - \sqrt{(\eta\lambda_{k^*})^2 - 4\lambda_{k^*}})/2 = (-2\lambda_{k^*}^{\frac{1}{2}} - \sqrt{(2\lambda_{k^*}^{\frac{1}{2}}\lambda_{k^*})^2 - 4\lambda_{k^*}})/2 = (-2\lambda_{k^*}^{\frac{1}{2}} - \sqrt{4\lambda_{k^*} - 4\lambda_{k^*}})/2 \\ &= -2\lambda_{k^*}^{\frac{1}{2}}/2 = -\lambda_{k^*}^{\frac{1}{2}},\end{aligned}$$

and so

$$\begin{aligned}\mathcal{A}\begin{pmatrix} 0 \\ \phi_{k_j^*}\end{pmatrix} &= \begin{pmatrix} \phi_{k_j^*} \\ \mu_{k_j^*}\phi_{k_j^*}\end{pmatrix} + (-\lambda_{k^*}^{\frac{1}{2}})\begin{pmatrix} 0 \\ \phi_{k_j^*}\end{pmatrix} \\ &= \sqrt{\lambda_{k^*}^2 + \lambda_{k^*}}\, \vec{\psi}_{k_j^*} + (-\lambda_{k^*}^{\frac{1}{2}})\begin{pmatrix} 0 \\ \phi_{k_j^*}\end{pmatrix}.\end{aligned}$$

Hence, the space spanned by $\left\{\vec{\psi}_{k_1^*}, \cdots, \vec{\psi}_{k_{j_{k^*}}^*}\right\} \cup \left\{\begin{pmatrix} 0 \\ \phi_{k_1^*}\end{pmatrix}, \cdots, \begin{pmatrix} 0 \\ \phi_{k_{j_{k^*}}^*}\end{pmatrix}\right\}$ is an invariant subspace of $2j_{k^*}$ dimensions of $\mathcal{A}$, denoted by $\mathfrak{M}_{k^*}$. From theory of finite dimensional space, we assert that

$$\sigma(\mathcal{A}|\mathfrak{M}_{k^*}) = \sigma_p(\mathcal{A}|\mathfrak{M}_{k^*}) \subseteq \sigma_p(\mathcal{A}) \subseteq \sigma(\mathcal{A}),$$

and therefore $-\omega^* = \sup\{Re\ \mu|\mu \in \sigma(\mathcal{A}|\mathfrak{M}_{k^*})\} \leq \sup\{Re\ \mu|\mu \in \sigma(\mathcal{A})\} = -\omega$. Actually, we can arrange the vectors spanning $\mathfrak{M}_{k^*}$ as follows

$$\begin{pmatrix} 0 \\ \phi_{k_1^*}\end{pmatrix}, \vec{\psi}_{k_1^*}, \begin{pmatrix} 0 \\ \phi_{k_2^*}\end{pmatrix}, \vec{\psi}_{k_2^*}, \cdots, \begin{pmatrix} 0 \\ \phi_{k_{j_{k^*}}^*}\end{pmatrix}, \vec{\psi}_{k_{j_{k^*}}^*}.$$

Set

$$\mathbb{A} = \begin{pmatrix} -\lambda_{k^*}^{\frac{1}{2}} & \sqrt{\lambda_{k^*}^2 + \lambda_{k^*}} \\ 0 & \lambda_{k^*}\end{pmatrix},$$

then $\mathcal{A}|\mathfrak{M}_{k^*}$ has the form

$$\mathcal{A}|\mathfrak{M}_{k^*} = \begin{bmatrix} \mathbb{A} & \ddots & 0 \\ 0 & & \mathbb{A}\end{bmatrix} \quad \text{(there are} j_{k^*}\, \mathbb{A}'s \text{ in the diagonal)}$$

Applying the Theorem 1.5.3 of [13], we can conclude that $\mathcal{A}$ generates a C_0-semigroup $T_1(t)$ satisfying $\|T_1(t)\| \leq M_1 e^{-\omega^* t}$, and so

$$\|T_1(t)\| \leq M_1 e^{-\omega t} \tag{3.14}$$

On the other hand, since the family $\{\vec{\phi}_{k_1}, \vec{\psi}_{k_1}, \cdots, \vec{\phi}_{k_{j_k}}, \vec{\psi}_{k_{j_k}}\}_{k \neq k^*}$ consists of the eigenvectors of $\mathcal{A}$, the subspace $\mathfrak{M}$ spanned by them is an invariant subspace of $\mathcal{A}$, and this family is just a Riesz basis of $\mathfrak{M}$ (see Lemma 4). Thus, form case 1, it is aware of the fast that $\mathcal{A}$ generates a C_0-Semigroup $T_2(t),(t \geq 0)$ in $\mathfrak{M}$. For $\mathfrak{M} \in \rho(\mathcal{A})$, we have

$$(\mu I-\mathcal{A})^{-1}\vec{\phi}_{k_j}=\frac{1}{\mu-\xi_k}\vec{\phi}_{k_j},\quad (\mu I-\mathcal{A})^{-1}\vec{\varphi}_{k_j}=\frac{1}{\mu-\mu_k}\vec{\psi}_{k_j}\quad (j=1,2,\cdots,n_k;k\neq k^*)$$

and $(\mu I-\mathcal{A})^{-1}\mathfrak{M}_k \subset \mathfrak{M}_k$, it follows from [16] that $\sigma(\mathcal{A}|\mathfrak{M}_k) \subseteq \sigma(\mathcal{A})$ and $-\omega_2=\sup\{Re\ \mu|\mu \in \sigma(\mathcal{A}|\mathfrak{M}_k)\} \leq \sup\{Re\ \mu|\mu\in\sigma(\mathcal{A})\}=-\omega$. Thus, there is M_2 such that

$$\|T_2(t)\| \leq M_2 e^{-\omega t} \tag{3.15}$$

Since $\mathfrak{M}_{k^*}$ is finite dimensional, it is a closed subspace of $\mathcal{H}$, and so $\mathcal{H}=\mathfrak{M}_{k^*}\oplus\mathfrak{M}_k$, where $\oplus$ expresses orthogonal sum in Hilbert space $\mathcal{H}$. Now, we define $T(t)\overset{\text{def}}{=}T_1(t)\oplus T_2(t)$ (obviously, $T_1(t)T_2(t)=T_2(t)T_1(t)=0$). We shall next prove an interesting result that $T(t)$ is exactly a C_0-semigroup on $\mathcal{H}$ generated by $\mathcal{A}$. The semigroup properties of $T(t)$ can be easily presented as follows:

(i) $T(0)=T_1(0)\oplus T_2(0)=I_{\mathfrak{M}_{k^*}}\oplus I_{\mathfrak{M}_k}=I_{\mathcal{H}}$

(ii)
$$\begin{aligned} T(t+s) &= T_1(t+s)\oplus T_2(t+s)=[T_1(t)T_1(s)\oplus[T_2(t)T_2(s)] \\ &= [T_1(t)\oplus T_2(t)][T_1(s)\oplus T_2(s)]=T(t)T(s) \qquad (t,s\geq 0)\end{aligned}$$

(iii) For every $x\in\mathcal{H}$, $x=x_{k^*}\oplus x_k$, where $x_{k^*}\in\mathfrak{M}_{k^*}$, $x_k\in\mathfrak{M}_k$,

$$\begin{aligned} \lim_{t\to 0^+} T(t)x &= \lim_{t\to 0^+}[T_1(t)\oplus T_2(t)](x_{k^*}\oplus x_k) \\ &= \lim_{t\to 0^+} T(t)x \lim_{t\to 0^+}[T_1(t)\oplus T_2(t)]x_{k^*}\oplus[T_1(t)\oplus T_2(t)]x_k \\ &= \lim_{t\to 0^+}[T_1(t)x_{k^*}\oplus T_2(t)x_k] \end{aligned}$$

$$\begin{aligned} &= (\lim_{t\to 0^+} T_1(t)x_{k^*})\oplus(\lim_{t\to 0^+} T_2(t)x_k) \\ &= x_{k^*}\oplus x_k \\ &= x \end{aligned}$$

(iv) For any $x\in D(\mathcal{A})$, we have $x=x_{k^*}\oplus x_k$, $x_{k^*}\in\mathfrak{M}_{k^*}$ and $x_k\in\mathfrak{M}_k$, and

$$\begin{aligned} \mathcal{A}x &= \mathcal{A}(x_{k^*}\oplus x_k)=\mathcal{A}x_{k^*}\oplus\mathcal{A}x_k \\ &= \left(\lim_{t\to 0+}\frac{T_1(t)x_{k^*}-x_{k^*}}{t}\right)\oplus\left(\lim_{t\to 0^+}\frac{(T_1(t)x_{k^*}\oplus T_2(t)x_{k^*})-(x_{k^*}\oplus x_k)}{t}\right) \\ &= \lim_{t\to 0+}\frac{Tx-x}{t} \end{aligned}$$

Thus, $T(t)$ defined by the orthogonal sum of $T_1(t)$ and $T_2(t)$ is exactly C_0-Semigroup on $\mathcal{H}$ generated by $\mathcal{A}$. Taking $M = max\{M_1, M_2\}$ from (3.10) and (3.11), leads to the following result

$$\|T(t)\| \leq Me^{-\omega t} \quad (t \geq 0)$$

The Theorem 3.2 is established now. ■//

We have now seen from the discussion above that the robot system (3.1) satisfies all conditions for the system (2.1), and hence we can apply the optimal control theorem 2.1 to the the robot system (3.1) and obtain a result as be stated as follows:

Theorem 3.3 An optimal energy control to the robot system (3.1) defined by means of the ( ) exists uniquely.

4 Conclusion

In this paper, we have investigated a kind of optimal energy control for infinite dimensional dynamic system. We proposed and proved the significant and important results that the minimum energy control of infinite dimensional space exists uniquely in terms of semigroup approach of linear operators and geometric method of Banach space. As a byproduct, we find that the minimum energy control element can be obtained by finding a weak limit of the admissible control set. In the last part of the paper, we studied a Euler-Bernoulli robot beam system by means of spectral analysis and semigroup theory, and showed that the optimal control theorem and procedure proposed in this paper are viable to the robot system.

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