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A Single Server non-Markovian Single Vacation Queue with Two Type of Services and with an Optional Service and Application in a Web Server Model

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Abstract: A single server queue with two type of services and with single vacation has been considered. The type 1 service is a phase type service with two service phases. Both the service time distributions are generally distributed. The type 2 service has only one phase of service. In addition the server also provides an optional service. These service time distributions are also general. After completion of service, the server takes vacation if there are no customers in the queue and vacation time distribution is general. The server returns to the queue, independent of the number of customers in the queue, after taking a single vacation. For this model the probability generating function for the number of customers in the queue at different server's state are obtained using supplementary variable technique. Some performance measures and particular models are calculated. Numerical results are presented and a web server model has analyzed under the given frame work.

Keywords: Phases service – Optional service – Bernoulli process – Supplementary variable technique – Vacation – Performance measures.
AMS Subject classification number: 90B22, 60K25 and 60K30.

1. Introduction

Queueing systems constitute a key tool in modelling and performance analysis of telecommunication systems and computer systems. Poisson arrivals are in many cases a fairly realistic model for the arrival process, but exponential service times are not very common in practice. In many systems the coefficient of variation of the service times will be smaller (or greater) than 1. Therefore, it is essential to analyze the models with generally distributed service times. Due to this reason the $M/G/1$ queue has been studied in various forms by numerous authors including Gaver (1959), Keilson and Kooharian (1960), Bhat (1964), Prabhu (1965) and Cohen (1969), to mention a few.

Recently there have been several contributions considering queueing system of the $M/G/1$ type, in which the server may provide services in phases. The motivation for such type of model comes from some computer and communication networks, where messages are processed in two stages by a single server. The case where both phases of service are exponentially distributed is the so called coxian distribution. Bertsimas and Papaconstantinou (1988) considered such distribution to design a multi server queue with application in a transportation system.

The queueing system with two phase service have studied by Krishna and Lee (1990). Doshi (1991) has extended the two phase queueing system of Krishna and Lee into case of batch. Recently Artalejo and Choudhury (2004) have studied the steady state analysis of an $M/G/1$ queue with repeated attempts and two phase service.

In day to day life, one encounters numerous queueing situations in which all the arriving customers are given the essential service and only some of them may require additional optional service. Such a model was studied by Madan (1994). The other works to be noted here are Madan (2000), Medhi (2002), Al-Jararah and Madan (2003), Jinting Wang (2004), Kalyanaraman et al. (2005) and Jau-Chuan (2008).

In real life, where as soon as the server becomes free, the server shut down the service facility temporarily for a random period of time and thus the server may not be available when the customer arrives to an empty queue, the service starts only after the server returns to the queue. This random period is called vacation period and the queue is called a vacation queue. Miller (1964), first studied a model, where the server is unavailable for service during some random length of time for the M/G/1 queueing system. In the vacation queue, if the server returns to the queue after completing a random vacation period, irrespective of number of customers in the queue, called single vacation queue. Single vacation has been studied by Levy and Yechiali (1975).

In the past few years the World Wide Web has experienced unusual growth. Not only are millions browsing the Web, but also hundreds of new Web sites are added each day [Wiederspan, J. and C. Shotton (1996)]. Yet, despite the increasing number of Web (i.e., HTTP) servers in use, little is definitively known about their performance characteristics. Performance characteristics analysis is an integral part of the research area of web servers. Now a days a website receives millions and millions of hits per day and it may become overloaded as the arrival rate exceeds the capacity of the web server. To solve this problem, overload control, increasing the server's efficiency, etc. can be used. In overload control investigations for web servers, performance analysis, predict improvements when using a certain overload control strategy, see Widell (2002) or Cao and Nyberg (2002).

Several attempts have been made to create performance models for web servers. van der Mei et al. (2001) modeled web servers as tandem queueing networks. The model was used to predict web server performance metrics and was validated through measurements and simulations. Wells et al.(2001) made a performance analysis of web servers using colored Petri nets. Their model has several parameters, some of which are known. Unknown parameters are determined by simulations. Dilley et al. (1998) used layered queueing models in their performance studies. Cherkasova and Phaal (2002) used a model similar to the one presented in this paper, but with assumptions of deterministic service times and session-based workload. Beckers et al. (2001) proposed a generalized processor sharing performance model for Internet access lines which includes web servers. Their model describes the flow-level characteristics of the traffic carried. They established simple relations between the capacity, the utilization of the access line and download times of Internet objects.

In this paper, a very simple, high level view of a web server, modelled as a queueing system. A diagram of the Web server queueing model is presented in Figure 1. The file(Jobs) requests arrive at the Web server, wait in the queue, if the service is not immediate. The queue consists of a server (node), the file then proceeds to the server where the data is read from the file, processed and passed on to the internet and then to the client's browser. In addition, no request found in the queue, the server is sending for maintenance for a random period of time, called single vacation, after maintenance the server is ready for service. The server provides two types of services based on the Jobs requirements, out of which one type service has two phases the type has single phase service. But after completion of the service with single phase, the server provides another service if the customer demands.

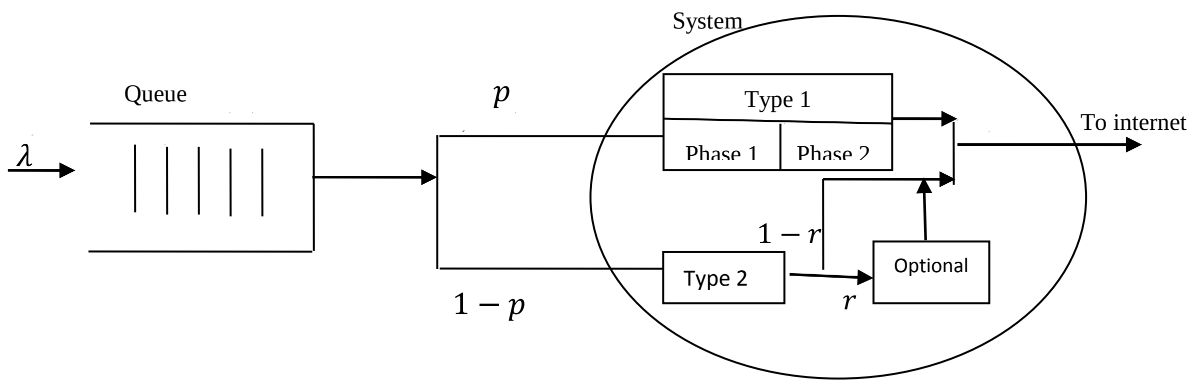

Figure 1: Queueing model of a web server

The paper is organized as follows: The corresponding mathematical model is defined in section 2 and the governing differential difference equations, the boundary conditions and the normalizing condition are given in section 3. From this model the probability generating function of the number of customers in queue irrespective of the server state is derived in section 4. Also, some performance measures related to this queueing model are derived from these probability generating functions and are given in section 5. In section 6, some particular models are derived. A numerical study is carried out in section 7 and the performance of Web server has been analysed based on the queueing model.

2. The Model

The arrival follows Poisson with rate $\lambda(>0)$ and a single server provides two type of services, respectively called type 1 service and type 2 service. Also the server provides an optional service. The entering customers selects type 1 service with probability p or type 2 service with probability $1-p$. The type 1 service is a phase type service (two phases). After completion of type 1 service, the customer leaves the system, whereas after completion of type 2 service, the customer leaves the system with probability $1-r$ or choose an optional service with probability r. After completion of optional service, the customer leaves the system. The service time distributions are general, the distribution functions are $B_{1,j}(x)$, for type 1 and j^{th} phase of service $(j=1,2)$, $B_{2,1}(x)$, for type 2 service, $B_{2,2}(x)$, for an optional service. The Laplace-Stieltjes transform (LST) for $B_{i,j}(x)$ is $B^{*}_{i,j}(\theta)$ and finite k^{th} moments are $E(B^{k}_{i,j}), k \geq 1, i, j = 1,2$.

The server takes a vacation at each time the system becomes empty. As soon as the vacation period is completed, the server immediately joins the system and starts service of the customer, if there is any customer waiting in the queue, otherwise he remains idle and waits for a new customer to arrive at the system. Vacation time distribution is also a general distribution with distribution function $V(x)$, Laplace-Stieltjes transform (LST) for $V(x)$ is $V^{*}(\theta)$ and finite moments are $E(V^{k}), k \geq 1$.

It may be noted that $B_{i,j}(x), V(x), (B_{i,j}(\infty)=1, B_{i,j}(0)=0, V(\infty)=1, V(0)=0)$ are continuous, so that $\mu_{i,j}(x)dx, (\gamma(x)dx)$ are the first order differential functions (hazard rates) of $B_{i,j}(x), (V(x))$.

For the analysis the supplementary variable (the variable is elapsed time) technique has been used.

Let $\mu_{1,j}(x)dx$ be the conditional probability of completion of the j^{th} phase of type 1 service during the interval $(x, x+dx]$, given that elapsed service time is x so that $\mu_{1,j}(x) = \dfrac{b_{1,j}(x)}{1-B_{1,j}(x)}, (j=1,2)$ and let $\mu_{2,j}(x)dx$ be the conditional probability of completion of the type 2 service and optional service during the interval $(x, x+dx]$, given that elapsed service time is x so that $\mu_{2,j}(x) = \dfrac{b_{2,j}(x)}{1-B_{2,j}(x)}, (j=1,2)$ and let $\gamma(x)dx$ be the conditional probability of completion of the vacation during the interval $(x, x+dx]$, given that elapsed vacation time is x so that $\gamma(x) = \dfrac{v(x)}{1-V(x)}$.

The following notations are introduced to define the model mathematically:

$P_n^{(1,j)}(x,t) =$ Pr{at time t, there are n customers in the queue excluding one in the type 1 service and is in the j^{th} phase of service and the elapsed service time is x}, $j = 1, 2, n \geq 0$,

$P_n^{(2,1)}(x,t) =$ Pr{at time t, there are n customers in the queue excluding one in the type 2 service and elapsed service time is x}, $n \geq 0$,

$P_n^{(2,2)}(x,t) =$ Pr{at time t, there are n customers in the queue excluding one in the optional service and elapsed service time is x}, $n \geq 0$ and

$V_n(x,t) =$ Pr{at time t, the server is on vacation with elapsed vacation time is x and the number of customers in the queue is n}, $n \geq 0$.

$Q(t) =$ Pr{at time t, there are no customers in the system and the server is idle}.

Let $P_n^{(i,j)}(x)$ $(i, j = 1,2)$, $V_n(x)$ and Q denote the corresponding steady state probabilities.

The probability generating functions for the probabilities $\{P_n^{(i,j)}(x)\}, (i, j = 1,2), \{V_n(x)\}$ are respectively defined as

$P^{(i,j)}(x,z) = \sum_{n=0}^{\infty} z^n P_n^{(i,j)}(x)$ and $V(x,z) = \sum_{n=0}^{\infty} z^n V_n(x)$.

3. The Governing Equations

The differential difference equations related to the model defined in the proceeding section are

$$\frac{d}{dx} P_0^{(1,1)}(x) + (\lambda + \mu_{1,1}(x)) P_0^{(1,1)}(x) = 0 \tag{1}$$

$$\frac{d}{dx} P_n^{(1,1)}(x) + (\lambda + \mu_{1,1}(x)) P_n^{(1,1)}(x) = \lambda P_{n-1}^{(1,1)}(x); n \geq 1 \tag{2}$$

$$\frac{d}{dx} P_0^{(1,2)}(x) + (\lambda + \mu_{1,2}(x)) P_0^{(1,2)}(x) = 0 \tag{3}$$

$$\frac{d}{dx} P_n^{(1,2)}(x) + (\lambda + \mu_{1,2}(x)) P_n^{(1,2)}(x) = \lambda P_{n-1}^{(1,2)}(x); n \geq 1 \tag{4}$$

$$\frac{d}{dx} P_0^{(2,1)}(x) + (\lambda + \mu_{2,1}(x)) P_0^{(2,1)}(x) = 0 \tag{5}$$

$$\frac{d}{dx} P_n^{(2,1)}(x) + (\lambda + \mu_{2,1}(x)) P_n^{(2,1)}(x) = \lambda P_{n-1}^{(2,1)}(x); n \geq 1 \tag{6}$$

$$\frac{d}{dx} P_0^{(2,2)}(x) + (\lambda + \mu_{2,2}(x)) P_0^{(2,2)}(x) = 0 \tag{7}$$

$$\frac{d}{dx} P_n^{(2,2)}(x) + (\lambda + \mu_{2,2}(x)) P_n^{(2,2)}(x) = \lambda P_{n-1}^{(2,2)}(x); n \geq 1 \tag{8}$$

$$\frac{d}{dx} V_0(x) + (\lambda + \gamma(x)) V_0(x) = 0 \tag{9}$$

$$\frac{d}{dx} V_n(x) + (\lambda + \gamma(x)) V_n(x) = \lambda V_{n-1}(x); n \geq 1 \tag{10}$$

$$\lambda Q = \int_0^{\infty} V_0(x)\gamma(x)dx \tag{11}$$

The boundary conditions are

$$P_0^{(1,1)}(0) = \lambda pQ + p\int_0^{\infty} V_1(x)\gamma(x)dx + p\int_0^{\infty} P_1^{(1,2)}(x)\mu_{1,2}(x)dx + (1-r)p\int_0^{\infty} P_1^{(2,1)}(x)\mu_{2,1}(x)dx$$
$$+ p\int_0^{\infty} P_1^{(2,2)}(x)\mu_{2,2}(x)dx \tag{12}$$

$$P_n^{(1,1)}(0) = p\int_0^{\infty} V_{n+1}(x)\gamma(x)dx + p\int_0^{\infty} P_{n+1}^{(1,2)}(x)\mu_{1,2}(x)dx + (1-r)p\int_0^{\infty} P_{n+1}^{(2,1)}(x)\mu_{2,1}(x)dx$$
$$+ p\int_0^{\infty} P_{n+1}^{(2,2)}(x)\mu_{2,2}(x)dx, n \geq 1 \tag{13}$$

$$P_n^{(1,2)}(0) = \int_0^{\infty} P_n^{(1,1)}(x)\mu_{1,1}(x)dx, n \geq 0 \tag{14}$$

$$P_0^{(1,2)}(0) = \lambda(1-p)Q + (1-p)\left\{\int_0^\infty V_1(x)\gamma(x)dx + \int_0^\infty P_1^{(1,2)}(x)\mu_{1,2}(x)dx + (1-r)\int_0^\infty P_1^{(2,1)}(x)\mu_{2,1}(x)dx\right\}$$

$$+ (1-p)\int_0^\infty P_1^{(2,2)}(x)\mu_{2,2}(x)dx \qquad (15)$$

$$P_n^{(1,2)}(0) = (1-p)\left\{\int_0^\infty V_{n+1}(x)\gamma(x)dx + \int_0^\infty P_{n+1}^{(1,2)}(x)\mu_{1,2}(x)dx + (1-r)\int_0^\infty P_{n+1}^{(2,1)}(x)\mu_{2,1}(x)dx\right\}$$

$$+ (1-p)\int_0^\infty P_{n+1}^{(2,2)}(x)\mu_{2,2}(x)dx, n \geq 1 \qquad (16)$$

$$P_n^{(2,2)}(0) = r\int_0^\infty P_n^{(2,1)}(x)\mu_{2,1}(x)dx, n \geq 0 \qquad (17)$$

$$V_0(0) = \int_0^\infty P_0^{(1,2)}(x)\mu_{1,2}(x)dx + (1-r)\int_0^\infty P_0^{(2,1)}(x)\mu_{2,1}(x)dx + \int_0^\infty P_0^{(2,2)}(x)\mu_{2,2}(x)dx \qquad (18)$$

$$V_n(0) = 0, n \geq 1 \qquad (19)$$

and the normalization condition is

$$Q + \sum_{n=0}^{\infty}\int_0^\infty \left[P_n^{(1,1)}(x) + P_n^{(1,2)}(x) + P_n^{(2,1)}(x) + P_n^{(2,2)}(x) + V_n(x)\right]dx = 1 \qquad (20)$$

4. The Analysis

Multiplying equations (2), (4), (6), (8) and (10) by z^n, summing from $n = 1$ to ∞ and then adding (1), (3), (5), (7) and (9), we get

$$\frac{\frac{d}{dx}P^{(i,j)}(x,z)}{P^{(i,j)}(x,z)} = -s - \mu_{i,j}(x) \qquad (21)$$

$$\frac{\frac{d}{dx}V(x,z)}{V(x,z)} = -s - \gamma(x) \qquad (22)$$

where $s = \lambda(1-z)$ and $i, j = 1,2$.

Integration of the equations (21) and (22), leads to

$$P^{(i,j)}(x,z) = C_{i,j}(1 - B_{i,j}(x))e^{-sx} \qquad (23)$$

$$V(x,z) = C(1 - V(x))e^{-sx} \qquad (24)$$

Taking $x = 0$ in equations (23), (24), the constants $C_{i,j}$, $(i, j = 1,2)$, C are obtained as

$$C_{i,j} = P^{(i,j)}(0,z) \qquad (25)$$

$$C = V(0,z) \qquad (26)$$

Using equations (25), (26) in (23), (24), we get

$$P^{(i,j)}(x,z) = P^{(i,j)}(0,z)(1 - B_{i,j}(x))e^{-sx} \qquad (27)$$

$$V(x,z) = V(0,z)(1 - V(x))e^{-sx} \qquad (28)$$

Multiplying equation (13) by z^n, summing from $n = 1$ to ∞ and then adding (12), using (11), (18), (27), (28), we get

$$zP^{(1,1)}(0,z) = \lambda p(z-1)Q + pV^*(s)V(0,z) + pB_{1,2}^*(s)P^{(1,2)}(0,z) + p(1-r)B_{2,1}^*(s)P^{(2,1)}(0,z)$$

$$+ pB_{2,2}^*(s)P^{(2,2)}(0,z) - pV_0(0) \qquad (29)$$

Similar performance on equations (14)-(19), we get

$$P^{(1,2)}(0,z) = B^*_{1,1}(s)P^{(1,1)}(0,z) \tag{30}$$

$$zP^{(2,1)}(0,z) = (1-p)[\lambda(z-1)Q + V^*(s)V(0,z) + B^*_{1,2}(s)P^{(1,2)}(0,z) + (1-r)B^*_{2,1}(s)P^{(2,1)}(0,z)] + (1-p)[B^*_{2,2}(s)P^{(2,2)}(0,z) - V_0(0)] \tag{31}$$

$$P^{(2,2)}(0,z) = rB^*_{2,1}(s)P^{(2,1)}(0,z) \tag{32}$$

$$V(0,z) = V_0(0) \tag{33}$$

Using equations (30), (32), (33) in (29), (31), we get

$$[z - pB^*_{1,1}(s)B^*_{1,2}(s)]P^{(1,1)}(0,z) = \lambda p(z-1)Q + p(1-r+rB^*_{2,2}(s))B^*_{2,1}(s)P^{(2,1)}(0,z) + p(V^*(s)-1)V_0(0) \tag{34}$$

$$[z-(1-p)(1-r+rB^*_{2,2}(s))B^*_{2,1}(s)]P^{(2,1)}(0,z) = (1-p)[\lambda(z-1)Q + B^*_{1,1}(s)B^*_{1,2}(s)P^{(1,1)}(0,z)] + (1-p)(V^*(s)-1)V_0(0) \tag{35}$$

From equation (34) and (35), we get

$$[z - pB^*_{1,1}(s)B^*_{1,2}(s) - (1-p)(1-r+rB^*_{2,2}(s))B^*_{2,1}(s)]P^{(1,1)}(0,z) = \lambda p(z-1)Q + p(V^*(s)-1)V_0(0) \tag{36}$$

$$[z - pB^*_{1,1}(s)B^*_{1,2}(s) - (1-p)(1-r+rB^*_{2,2}(s))B^*_{2,1}(s)]P^{(2,1)}(0,z) = \lambda(1-p)(z-1)Q + (1-p)(V^*(s)-1)V_0(0) \tag{37}$$

The unknown $P_0^{(1,2)}(x)$ is obtained using the PGF's, we get

$\sum_{n=0}^{\infty} z^n P_n^{(1,2)}(x) = P^{(1,2)}(0,z)[1-B_{1,2}(x)]e^{-sx}$ by equation (23)

$= \dfrac{pB^*_{1,1}(s)[\lambda(z-1)Q + (V^*(s)-1)V_0(0)][1-B_{1,2}(x)]e^{-sx}}{D(z)}$ by equation (30), (36)

where $D(z) = z - pB^*_{1,1}(s)B^*_{1,2}(s) - (1-p)B^*_{2,1}(s)(1-r+rB^*_{2,2}(s))$

Putting $z = 0$ in above equation, we get

$$P_0^{(1,2)}(x) = \frac{pB^*_{1,1}(\lambda)[\lambda Q + (1-V^*(\lambda))V_0(0)][1-B_{1,2}(x)]e^{-\lambda x}}{[pB^*_{1,1}(\lambda)B^*_{1,2}(\lambda) + (1-p)(1-r+rB^*_{2,2}(\lambda))B^*_{2,1}(\lambda)]} \tag{38}$$

Similarly

$$P_0^{(2,1)}(x) = \frac{(1-p)[\lambda Q + (1-V^*(\lambda))V_0(0)][1-B_{2,1}(x)]e^{-\lambda x}}{[pB^*_{1,1}(\lambda)B^*_{1,2}(\lambda) + (1-p)(1-r+rB^*_{2,2}(\lambda))B^*_{2,1}(\lambda)]} \tag{39}$$

$$P_0^{(2,2)}(x) = \frac{r(1-p)B^*_{2,1}(\lambda)[\lambda Q + (1-V^*(\lambda))V_0(0)][1-B_{2,2}(x)]e^{-\lambda x}}{[pB^*_{1,1}(\lambda)B^*_{1,2}(\lambda) + (1-p)(1-r+rB^*_{2,2}(\lambda))B^*_{2,1}(\lambda)]} \tag{40}$$

Using equations (38)-(40) in (18), we get

$$V_0(0) = \frac{\lambda Q}{V^*(\lambda)} \tag{41}$$

Using equation (41) in (33), (36), (37), we get

$$V(0,z) = \frac{\lambda Q}{V^*(\lambda)} \tag{42}$$

$$P^{(1,1)}(0,z) = \frac{\lambda p[V^*(\lambda)(z-1) + (V^*(s)-1)]Q}{V^*(\lambda)D(z)} \tag{43}$$

$$P^{(2,1)}(0,z) = \frac{\lambda(1-p)[V^*(\lambda)(z-1) + (V^*(s)-1)]Q}{V^*(\lambda)D(z)} \tag{44}$$

Using equations (43), (44) in (30), (32), we get

$$P^{(1,2)}(0,z) = \frac{\lambda p B_{1,1}^{*}(s)[V^{*}(\lambda)(z-1) + (V^{*}(s)-1)]Q}{V^{*}(\lambda)D(z)} \tag{45}$$

$$P^{(2,2)}(0,z) = \frac{\lambda r(1-p)B_{2,1}^{*}(s)[V^{*}(\lambda)(z-1) + (V^{*}(s)-1)]Q}{V^{*}(\lambda)D(z)} \tag{46}$$

Integration of equations (27) and (28) by parts with respect to x and then using equations (42)-(46), we get

$$P^{(1,1)}(z) = \int_0^\infty P^{(1,1)}(x,z)dx$$

$$= \frac{\lambda p[V^{*}(\lambda)(z-1) + (V^{*}(s)-1)](1-B_{1,1}^{*}(s))Q}{sV^{*}(\lambda)D(z)} \tag{47}$$

$$P^{(1,2)}(z) = \int_0^\infty P^{(1,2)}(x,z)dx$$

$$= \frac{\lambda p B_{1,1}^{*}(s)[V^{*}(\lambda)(z-1) + (V^{*}(s)-1)](1-B_{1,2}^{*}(s))Q}{sV^{*}(\lambda)D(z)} \tag{48}$$

$$P^{(2,1)}(z) = \int_0^\infty P^{(2,1)}(x,z)dx$$

$$= \frac{\lambda(1-p)[V^{*}(\lambda)(z-1) + (V^{*}(s)-1)](1-B_{2,1}^{*}(s))Q}{sV^{*}(\lambda)D(z)} \tag{49}$$

$$P^{(2,2)}(z) = \int_0^\infty P^{(2,2)}(x,z)dx$$

$$= \frac{\lambda r(1-p)B_{2,1}^{*}(s)[V^{*}(\lambda)(z-1) + (V^{*}(s)-1)](1-B_{2,2}^{*}(s))Q}{sV^{*}(\lambda)D(z)} \tag{50}$$

$$V(z) = \int_0^\infty V(x,z)dx$$

$$= \frac{\lambda(1-V^{*}(s))Q}{sV^{*}(\lambda)} \tag{51}$$

The idle probability Q is obtained using the equation (20) as

$$Q = \frac{(1-\rho)V^{*}(\lambda)}{V^{*}(\lambda) + \lambda E(V)} \tag{52}$$

where

$\rho = \lambda p[E(B_{1,1}) + E(B_{1,2})] + \lambda(1-p)[E(B_{2,1}) + rE(B_{2,2})]$.

The utilization factor $(\rho) < 1$ is the stability condition under which steady state solution exists.

Equations (47)-(51) together with equation (52) are respectively, the probability generating functions of the number of customers in the queue when the server is, serving phase 1 service, serving phase 2 service, serving type 2 service, serving an optional service and the sever is on vacation.

The probability generating function for the number of customers in the queue irrespective of server state is

$$U(z) = Q + \sum_{i=1}^{2}\sum_{j=1}^{2} P^{(i,j)}(z) + V(z)$$

$$= \frac{[(z-1)V^{*}(\lambda) - 1 + V^{*}(s)](1-\rho)}{[V^{*}(\lambda) + \lambda E(V)]D(z)}$$

5. The Performance Measures

Using straightforward calculations the following performance measures have been obtained:
(i) The mean number of customers in the queue is

$$L_q = \lim_{z\to 1} \frac{d}{dz} U(z)$$

$$= \lim_{z\to 1} \frac{d}{dz}\left[\frac{N(z)(1-\rho)}{D(z)[V^*(\lambda)+\lambda E(V)]}\right] \text{ where } N(z) = (z-1)V^*(\lambda) - 1 + V^*(s)$$

$$L_q = \lim_{z\to 1}\left[\frac{[D(z)N'(z) - D'(z)N(z)](1-\rho)}{2(D(z))^2[V^*(\lambda)+\lambda E(V)]}\right]$$

Since this limit gives $\frac{0}{0}$ form, so applying L'Hospitals rule twice, we get

$$L_q = \frac{(1-\rho)[D'(1)N''(1) - D''(1)N'(1)]}{2[V^*(\lambda)+\lambda E(V)](D'(1))^2}$$

$$= \frac{\lambda^2 E(V^2)}{2[V^*(\lambda)+\lambda E(V)]} + \frac{\lambda^2}{2(1-\rho)}\{p[E(B_{1,1}^2) + 2E(B_{1,1})E(B_{1,2}) + E(B_{1,2}^2)] + (1-p)[E(B_{2,1}^2)$$
$$+ 2rE(B_{2,1})E(B_{2,2}) + rE(B_{2,2}^2)]\}$$

(ii) The mean number of customers in the queue when the server is busy is

$$L_{qb} = \lim_{z\to 1}\frac{d}{dz}\left[\sum_{i=1}^{2}\sum_{j=1}^{2} P^{(i,j)}(z)\right]$$

$$= \left[\frac{\lambda(1-\rho)}{[V^*(\lambda)+\lambda E(V)]}\right]\lim_{z\to 1}\frac{d}{dz}[A_1 A_2]$$

where $A_1 = \frac{V^*(\lambda)(z-1) + V^*(s) - 1}{s}$ and

$$A_2 = \frac{1 - pB_{1,1}^*(s)B_{1,2}^*(s) - (1-p)B_{2,1}^*(s)(1 - r + rB_{2,2}^*(s))}{D(z)}$$

$$L_{qb} = \left[\frac{\lambda(1-\rho)}{[V^*(\lambda)+\lambda E(V)]}\right]\lim_{z\to 1}[A_1 A_2' + A_1' A_2]$$

$$= \frac{\lambda^2 \rho E(V^2)}{2[V^*(\lambda)+\lambda E(V)]} + \frac{\lambda^2}{2(1-\rho)}\{p[E(B_{1,1}^2) + 2E(B_{1,1})E(B_{1,2}) + E(B_{1,2}^2)] + (1-p)[E(B_{2,1}^2)$$
$$+ 2rE(B_{2,1})E(B_{2,2}) + rE(B_{2,2}^2)]\}$$

(iii) The mean number of customers in the queue when the server is on vacation is

$$L_{qv} = \lim_{z\to 1}\frac{d}{dz}V(z)$$

$$= \left[\frac{\lambda(1-\rho)}{[V^*(\lambda)+\lambda E(V)]}\right]\lim_{z\to 1}\frac{d}{dz}\left[\frac{1-V^*(s)}{s}\right]$$

$$= \left[\frac{\lambda(1-\rho)}{[V^*(\lambda)+\lambda E(V)]}\right]\lim_{z\to 1}\left\{\frac{s[-V^*(s)s'] - s'[1-V^*(s)]}{s^2}\right\}$$

Since this limit gives $\frac{0}{0}$ form, so applying L'Hospitals rule twice, we get

$$L_{qv} = \frac{\lambda^2 E(V^2)(1-\rho)}{2[V^*(\lambda) + \lambda E(V)]}$$

(iv) The probability that the server is idle is

$$Q = \frac{(1-\rho)V^*(\lambda)}{V^*(\lambda) + \lambda E(V)}$$

(v) The probability that the server is busy is

$$P_b = \rho$$

(vi) The probability that the server is on vacation is

$$P_v = \frac{\lambda E(V)(1-\rho)}{V^*(\lambda) + \lambda E(V)}$$

(vii) The mean response time is

$$M = \frac{L}{\lambda} = \frac{(L_q + \rho)}{\lambda}$$

$$= \frac{\lambda E(V^2)}{2[V^*(\lambda) + \lambda E(V)]} + \frac{\lambda}{2(1-\rho)}\{p[E(B_{1,1}^2) + 2E(B_{1,1})E(B_{1,2}) + E(B_{1,2}^2)] + (1-p)[E(B_{2,1}^2)$$

$$+ 2rE(B_{2,1})E(B_{2,2}) + rE(B_{2,2}^2)]\} + p[E(B_{1,1}) + E(B_{1,2})] + (1-p)[E(B_{2,1}) + rE(B_{2,2})]$$

6. Some Particular Models

In this section, two particular models are derived by taking known distributions to service time and vacation time. For model 1, vacation time distribution is Erlang k whereas for model 2 the vacation time distribution is hyper exponential but both models the service time distribution is negative exponential with parameters $\mu_{1,1}$ for phase 1 service, $\mu_{1,2}$ for phase 2 service, $\mu_{2,1}$ for type 2 service, $\mu_{2,2}$ for an optional service.

Model 1: In this model, the vacation time distribution is Erlang k with parameter θ.

$$Q = \frac{\theta(k\theta)^k C_1}{\mu_{1,1}\mu_{1,2}\mu_{2,1}\mu_{2,2}[\lambda(k\theta+\lambda)^k + \theta(k\theta)^k]}$$

$$P_b = \frac{\lambda}{\mu_{1,1}\mu_{1,2}\mu_{2,1}\mu_{2,2}}\{p\mu_{2,1}\mu_{2,2}[\mu_{1,1} + \mu_{1,2}] + (1-p)\mu_{1,1}\mu_{1,2}[r\mu_{2,1} + \mu_{2,2}]\}$$

$$P_v = \frac{\lambda(k\theta+\lambda)^k C_1}{\mu_{1,1}\mu_{1,2}\mu_{2,1}\mu_{2,2}[\lambda(k\theta+\lambda)^k + \theta(k\theta)^k]}$$

$$L_q = \frac{\lambda^2}{\mu_{1,1}\mu_{1,2}\mu_{2,1}\mu_{2,2}C_1}\{p\mu_{2,1}^2\mu_{2,2}^2[\mu_{1,1}^2 + 2\mu_{1,1}\mu_{1,2} + \mu_{1,2}^2] + (1-p)\mu_{1,1}^2\mu_{1,2}^2[\mu_{2,2}^2 + r\mu_{2,1}(\mu_{2,1} + 2\mu_{2,2})]\}$$

$$+ \frac{\lambda^2(k+1)(k\theta+\lambda)^k}{2k\theta[\lambda(k\theta+\lambda)^k + \theta(k\theta)^k]}$$

$$L_{qb} = \frac{\lambda^2}{\mu_{1,1}\mu_{1,2}\mu_{2,1}\mu_{2,2}C_1}\{p\mu_{2,1}^2\mu_{2,2}^2[\mu_{1,1}^2 + 2\mu_{1,1}\mu_{1,2} + \mu_{1,2}^2] + (1-p)\mu_{1,1}^2\mu_{1,2}^2[\mu_{2,2}^2 + r\mu_{2,1}(\mu_{2,1} + 2\mu_{2,2})]\}$$

$$+ \frac{\lambda^3(k+1)(k\theta+\lambda)^k}{2k\theta\mu_{1,1}\mu_{1,2}\mu_{2,1}\mu_{2,2}[\lambda(k\theta+\lambda)^k + \theta(k\theta)^k]}\{p\mu_{2,1}\mu_{2,2}[\mu_{1,1} + \mu_{1,2}] + (1-p)\mu_{1,1}\mu_{1,2}[r\mu_{2,1} + \mu_{2,2}]\}$$

$$L_{qv} = \frac{\lambda^2(k+1)(k\theta+\lambda)^k C_1}{2k\theta\mu_{1,1}\mu_{1,2}\mu_{2,1}\mu_{2,2}[\lambda(k\theta+\lambda)^k + \theta(k\theta)^k]}$$

$$M = \frac{\lambda}{\mu_{1,1}\mu_{1,2}\mu_{2,1}\mu_{2,2}C_1}\{p\mu_{2,1}^2\mu_{2,2}^2[\mu_{1,1}^2 + 2\mu_{1,1}\mu_{1,2} + \mu_{1,2}^2] + (1-p)\mu_{1,1}^2\mu_{1,2}^2[\mu_{2,2}^2 + r\mu_{2,1}(\mu_{2,1} + 2\mu_{2,2})]$$
$$+ C_1\{p\mu_{2,1}\mu_{2,2}[\mu_{1,1} + \mu_{1,2}] + (1-p)\mu_{1,1}\mu_{1,2}[r\mu_{2,1} + \mu_{2,2}]\}\} + \frac{\lambda(k+1)(k\theta+\lambda)^k}{2k\theta[\lambda(k\theta+\lambda)^k + \theta(k\theta)^k]}$$

where

$$C_1 = \mu_{1,1}\mu_{1,2}\mu_{2,1}\mu_{2,2} - \lambda[p\mu_{2,1}\mu_{2,2}(\mu_{1,1} + \mu_{1,2}) + (1-p)(r\mu_{2,1} + \mu_{2,2})]$$

Model 2: In this model, the vacation time distribution is hyper exponential with parameters q_1, q_2, $(q_1 + q_2 = 1)$, θ_1 and θ_2.

$$Q = \frac{\theta_1\theta_2 C_1[\lambda(q_1\theta_1 + q_2\theta_2) + \theta_1\theta_2]}{\mu_{1,1}\mu_{1,2}\mu_{2,1}\mu_{2,2}C_2}$$

$$P_b = \frac{\lambda}{\mu_{1,1}\mu_{1,2}\mu_{2,1}\mu_{2,2}}\{p\mu_{2,1}\mu_{2,2}[\mu_{1,1} + \mu_{1,2}] + (1-p)\mu_{1,1}\mu_{1,2}[r\mu_{2,1} + \mu_{2,2}]\}$$

$$P_v = \frac{\lambda(q_1\theta_2 + q_2\theta_1)(\lambda+\theta_1)(\lambda+\theta_2)C_1}{\mu_{1,1}\mu_{1,2}\mu_{2,1}\mu_{2,2}C_2}$$

$$L_q = \frac{\lambda^2}{\mu_{1,1}\mu_{1,2}\mu_{2,1}\mu_{2,2}C_1}\{p\mu_{2,1}^2\mu_{2,2}^2[\mu_{1,1}^2 + 2\mu_{1,1}\mu_{1,2} + \mu_{1,2}^2] + (1-p)\mu_{1,1}^2\mu_{1,2}^2[\mu_{2,2}^2 + r\mu_{2,1}(\mu_{2,1} + 2\mu_{2,2})]\}$$
$$+ \frac{\lambda^2(\lambda+\theta_1)(\lambda+\theta_2)(q_1\theta_2^2 + q_2\theta_1^2)}{\theta_1\theta_2 C_2}$$

$$L_{qb} = \frac{\lambda^2}{\mu_{1,1}\mu_{1,2}\mu_{2,1}\mu_{2,2}C_1}\{p\mu_{2,1}^2\mu_{2,2}^2[\mu_{1,1}^2 + 2\mu_{1,1}\mu_{1,2} + \mu_{1,2}^2] + (1-p)\mu_{1,1}^2\mu_{1,2}^2[\mu_{2,2}^2 + r\mu_{2,1}(\mu_{2,1} + 2\mu_{2,2})]\}$$
$$+ \frac{\lambda^3(\lambda+\theta_1)(\lambda+\theta_2)(q_1\theta_2^2 + q_2\theta_1^2)}{\mu_{1,1}\mu_{1,2}\mu_{2,1}\mu_{2,2}\theta_1\theta_2 C_2}\{p\mu_{2,1}\mu_{2,2}[\mu_{1,1} + \mu_{1,2}] + (1-p)\mu_{1,1}\mu_{1,2}[r\mu_{2,1} + \mu_{2,2}]\}$$

$$L_{qv} = \frac{\lambda^2(\lambda+\theta_1)(\lambda+\theta_2)(q_1\theta_2^2 + q_2\theta_1^2)C_1}{\mu_{1,1}\mu_{1,2}\mu_{2,1}\mu_{2,2}\theta_1\theta_2 C_2}$$

$$M = \frac{\lambda}{\mu_{1,1}\mu_{1,2}\mu_{2,1}\mu_{2,2}C_1}\{p\mu_{2,1}^2\mu_{2,2}^2[\mu_{1,1}^2 + 2\mu_{1,1}\mu_{1,2} + \mu_{1,2}^2] + (1-p)\mu_{1,1}^2\mu_{1,2}^2[\mu_{2,2}^2 + r\mu_{2,1}(\mu_{2,1} + 2\mu_{2,2})]$$
$$+ C_1\{p\mu_{2,1}\mu_{2,2}[\mu_{1,1} + \mu_{1,2}] + (1-p)\mu_{1,1}\mu_{1,2}[r\mu_{2,1} + \mu_{2,2}]\}\} + \frac{\lambda(\lambda+\theta_1)(\lambda+\theta_2)(q_1\theta_2^2 + q_2\theta_1^2)}{\theta_1\theta_2 C_2}$$

where

$$C_2 = \theta_1\theta_2[\lambda(q_1\theta_1 + q_2\theta_2) + \theta_1\theta_2] + \lambda(\lambda+\theta_1)(\lambda+\theta_2)(q_1\theta_2 + q_2\theta_1)$$

7. The Numerical Study and the performance of web server

In this section, some numerical results related to the web server queueing model have been calculated related to the model discussed in the above section. Fix some parameter values and vary other parameters. The service rates, the vacation rate are fixed as $\mu_{1,1} = 4, \mu_{1,2} = 5, \mu_{2,1} = 6, \mu_{2,2} = 3, \theta = 7, \theta_1 = 5, \theta_2 = 4$, $k = 6$ and the probability $r = 0.4, q_1 = 0.8, q_2 = 0.2$. The arrival rate (λ) has been varied from 0.1 to 1.0. By fixing the probability $r = 0.4$, the probability p has been varied from 0.1 (0.2) 0.9. The system performance measures the mean number of jobs in the queue (L_q), the mean number of jobs in the queue when the server is busy (L_{qb}), the mean number of jobs in the queue when the server is on vacation (L_{qv}), the probability that the server is idle (Q), the probability that the server is busy (P_b), the probability that the server is on vacation (P_v) and mean response time (M) have been calculated and are presented in figures and tables. From the figure 2 to 9, it is clear that the mean values of number of jobs in the queue at various server state and the mean response time increases for the increasing values of arrival rate. Table 1 and 2 presents the idle probability of the server, the probability value decreases as the arrival rate increases. Table 3 and 4 presents the busy probability of the server, the probability value increases as the arrival rate increases, which coincide with our expectation. Finally table 5 and 6 presents the probability that the server is on vacation. From the table it is clear that, the probability value increases as the arrival rate increases.

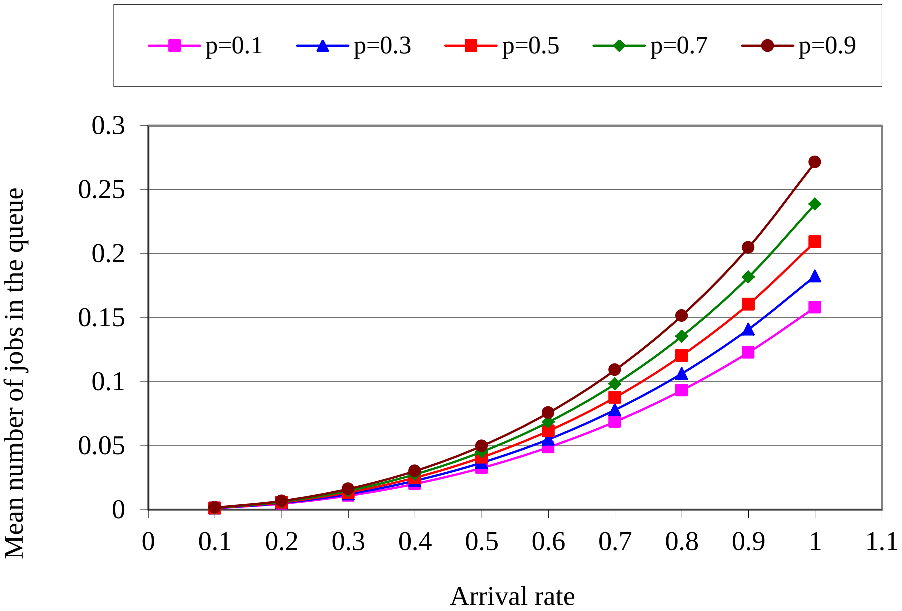

Figure 2: Arrival rate versus mean number of jobs in the queue for model 1

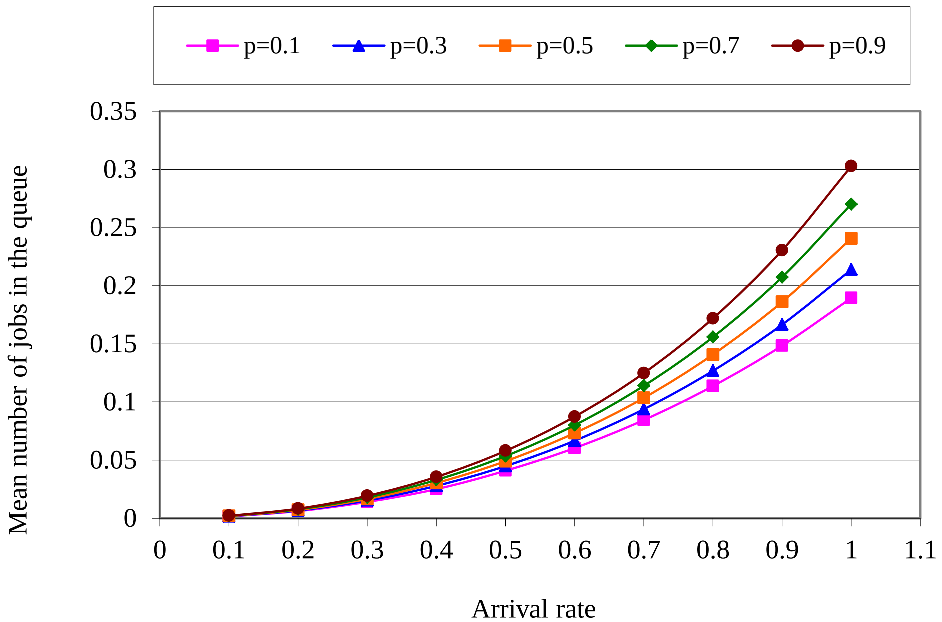

Figure 3: Arrival rate versus mean number of jobs in the queue for model 2

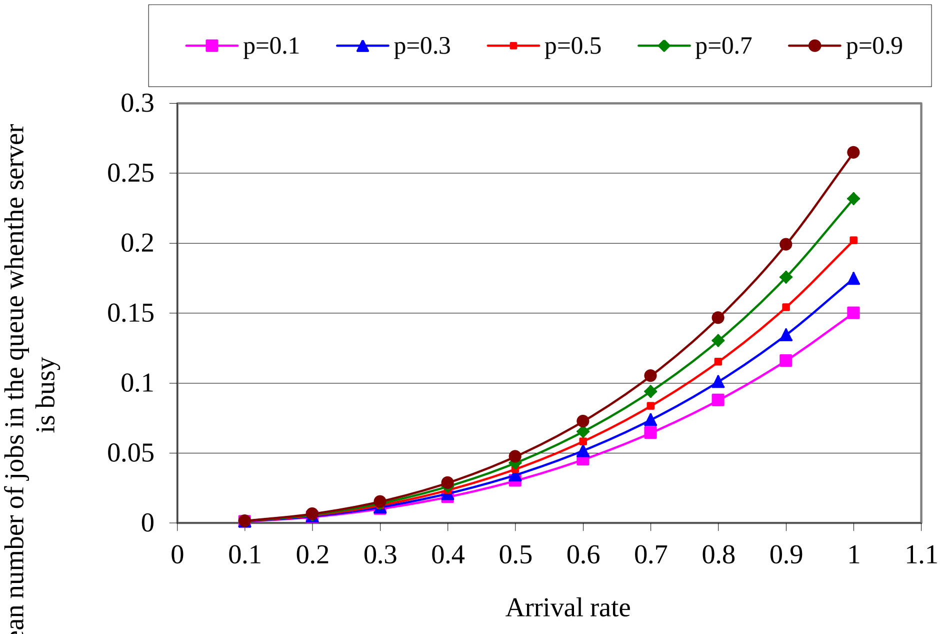

Figure 4: Arrival rate versus mean number of jobs in the queue when the server is busy for model 1

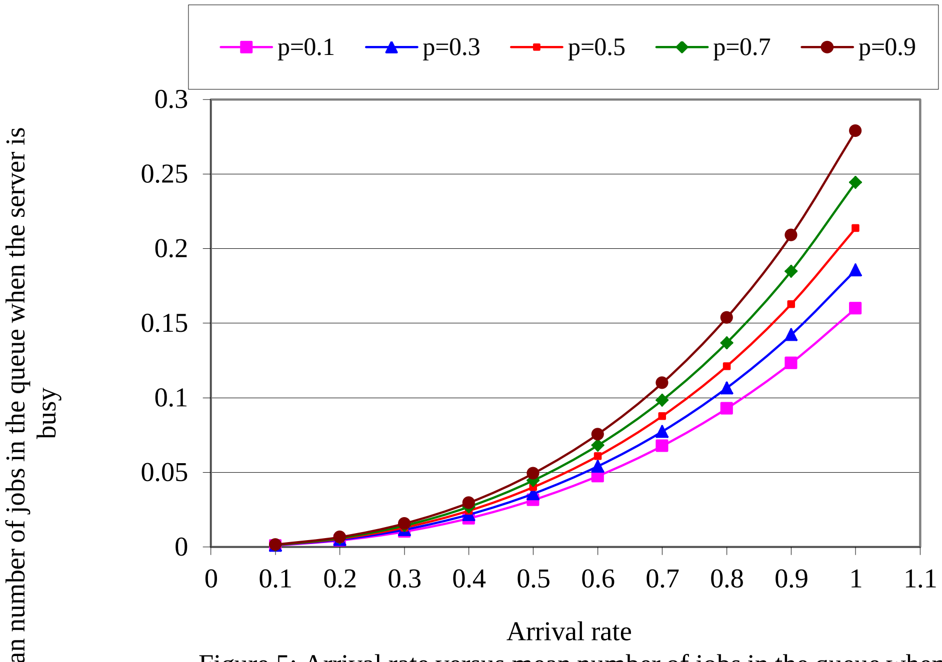

Figure 5: Arrival rate versus mean number of jobs in the queue when the sever is busy for model 2

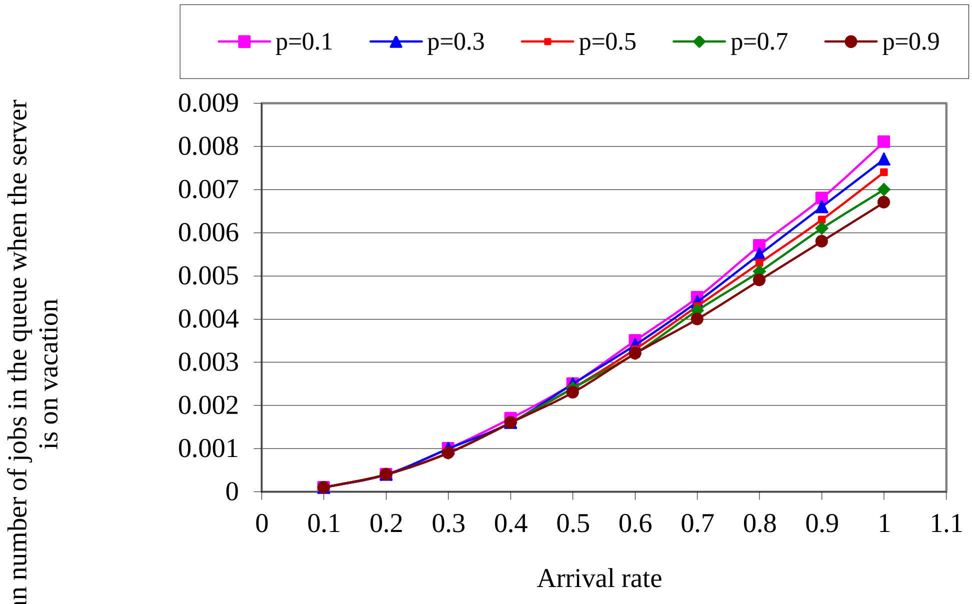

Figure 6: Arrival rate versus mean number of jobs in the queue when the server is on vacation for model 1

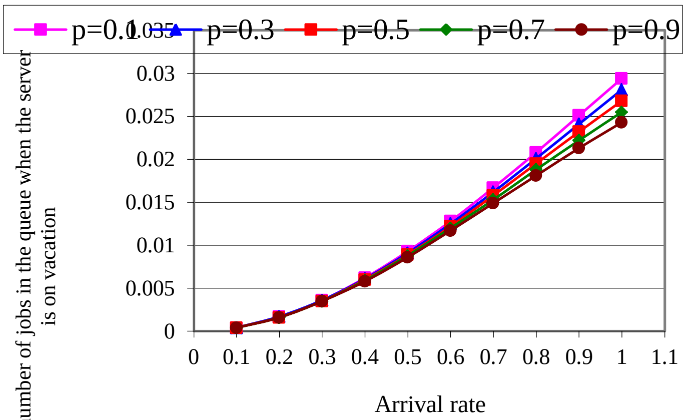

Figure 7: Arrival rate versus mean number of jobs in the queue when the server is on vacation for model 2

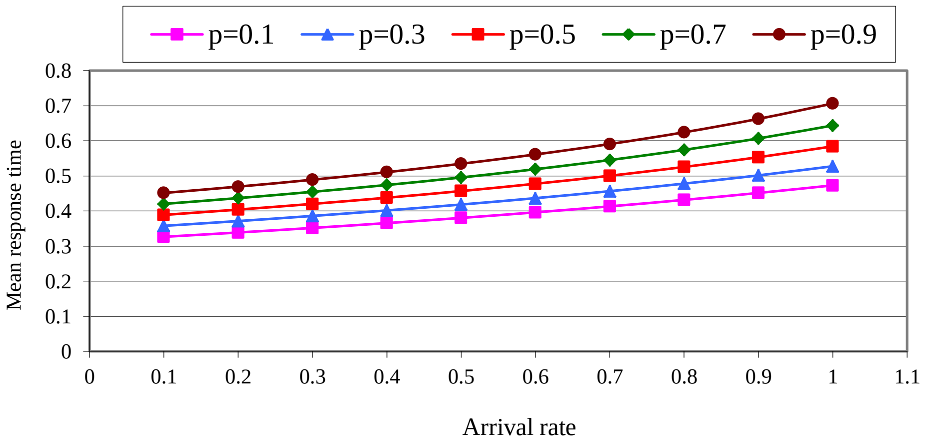

Figure 8: Arrival rate versus mean response time for model 1

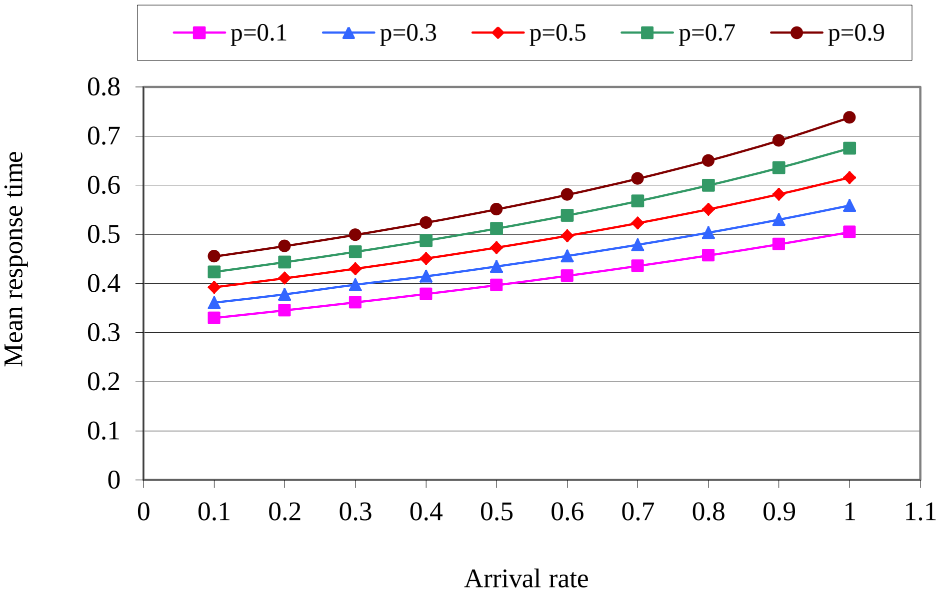

Figure 9: Arrival rate versus mean response time for model 2

Table 1: Probability that the server is idle for model 1

λ	$\mu_{1,1}=4, \mu_{1,2}=5, \mu_{2,1}=6, \mu_{2,2}=3, \theta=7, k=6, r=0.4$				
	$p=0.1$	$p=0.3$	$p=0.5$	$p=0.7$	$p=0.9$
0.1	0.9547	0.9517	0.9488	0.9458	0.9428
0.2	0.9102	0.9044	0.8986	0.8928	0.8869
0.3	0.8667	0.8581	0.8495	0.8409	0.8323
0.4	0.8241	0.8128	0.8015	0.7902	0.7789
0.5	0.7825	0.7686	0.7546	0.7407	0.7268
0.6	0.7418	0.7253	0.7088	0.6924	0.6759
0.7	0.7020	0.6831	0.6642	0.6452	0.6263
0.8	0.6631	0.6419	0.6206	0.5993	0.5780
0.9	0.6252	0.6017	0.5781	0.5545	0.5310
1.0	0.5882	0.5625	0.5367	0.5109	0.4852

Table 2: Probability that the server is idle for model 2

λ	$\mu_{1,1}=4, \mu_{1,2}=5, \mu_{2,1}=6, \mu_{2,2}=3, \theta=7, r=0.4, \theta_1=5, \theta_2=4, q_1=0.8, q_2=0.2$				
	$p=0.1$	$p=0.3$	$p=0.5$	$p=0.7$	$p=0.9$
0.1	0.9482	0.9452	0.9453	0.9394	0.9364
0.2	0.8977	0.8920	0.8862	0.8805	0.8747
0.3	0.8487	0.8402	0.8318	0.8234	0.8149
0.4	0.8011	0.7901	0.7791	0.7681	0.7571
0.5	0.7549	0.7415	0.7280	0.7146	0.7012
0.6	0.7102	0.6945	0.6787	0.6630	0.6472
0.7	0.6670	0.6491	0.6311	0.6131	0.5952
0.8	0.6253	0.6053	0.5852	0.5651	0.5451
0.9	0.5851	0.5630	0.5410	0.5189	0.4969
1.0	0.5462	0.5223	0.4984	0.4745	0.4505

Table 3: Probability that the server is busy for model 1

λ	$\mu_{1,1} = 4, \mu_{1,2} = 5, \mu_{2,1} = 6, \mu_{2,2} = 3, \theta = 7, k = 6, r = 0.4$				
	$p = 0.1$	$p = 0.3$	$p = 0.5$	$p = 0.7$	$p = 0.9$
0.1	0.0315	0.0345	0.0375	0.0405	0.0435
0.2	0.0630	0.0690	0.0750	0.0810	0.0870
0.3	0.0945	0.1035	0.1125	0.1215	0.1305
0.4	0.1260	0.1380	0.1500	0.1620	0.1740
0.5	0.1575	0.1725	0.1875	0.2025	0.2175
0.6	0.1890	0.2070	0.2250	0.2430	0.2610
0.7	0.2205	0.2415	0.2625	0.2835	0.3045
0.8	0.2520	0.2760	0.3000	0.3240	0.3480
0.9	0.2835	0.3105	0.3375	0.3645	0.3915
1.0	0.3150	0.3450	0.3750	0.4050	0.4350

Table 4: Probability that the server is busy for model 2

λ	$\mu_{1,1} = 4, \mu_{1,2} = 5, \mu_{2,1} = 6, \mu_{2,2} = 3, \theta = 7, r = 0.4, \theta_1 = 5, \theta_2 = 4, q_1 = 0.8, q_2 = 0.2$				
	$p = 0.1$	$p = 0.3$	$p = 0.5$	$p = 0.7$	$p = 0.9$
0.1	0.0315	0.0345	0.0375	0.0405	0.0435
0.2	0.0630	0.0690	0.0750	0.0810	0.0870
0.3	0.0945	0.1035	0.1125	0.1215	0.1350
0.4	0.1260	0.1380	0.1500	0.1620	0.1740
0.5	0.1575	0.1725	0.1875	0.2025	0.2175
0.6	0.1890	0.2070	0.2250	0.2430	0.2610
0.7	0.2205	0.2415	0.2625	0.2835	0.3045
0.8	0.2520	0.2760	0.3000	0.3240	0.3480
0.9	0.2835	0.3105	0.3375	0.3645	0.3915
1.0	0.3150	0.3450	0.3750	0.4050	0.4350

Table 5: Probability that the server is on vacation for model 1

λ	$\mu_{1,1} = 4, \mu_{1,2} = 5, \mu_{2,1} = 6, \mu_{2,2} = 3, \theta = 7, k = 6, r = 0.4$				
	$p = 0.1$	$p = 0.3$	$p = 0.5$	$p = 0.7$	$p = 0.9$
0.1	0.0138	0.0138	0.0137	0.0137	0.0137
0.2	0.0268	0.0266	0.0264	0.0262	0.0261
0.3	0.0388	0.0384	0.0380	0.0376	0.0372
0.4	0.0499	0.0492	0.0485	0.0478	0.0471
0.5	0.0600	0.0589	0.0579	0.0568	0.0557
0.6	0.0692	0.0677	0.0662	0.0646	0.0631
0.7	0.0775	0.0754	0.0733	0.0713	0.0692
0.8	0.0849	0.0821	0.0794	0.0767	0.0740
0.9	0.0913	0.0878	0.0844	0.0810	0.0775
1.0	0.0968	0.0925	0.0883	0.0841	0.0798

Table 6: Probability that the server is on vacation for model 2

λ	$\mu_{1,1} = 4, \mu_{1,2} = 5, \mu_{2,1} = 6, \mu_{2,2} = 3, \theta = 7, r = 0.4, \theta_1 = 5, \theta_2 = 4, q_1 = 0.8, q_2 = 0.2$				
	$p = 0.1$	$p = 0.3$	$p = 0.5$	$p = 0.7$	$p = 0.9$
0.1	0.0203	0.0203	0.0202	0.0201	0.0201
0.2	0.0393	0.0390	0.0388	0.0385	0.0383
0.3	0.0568	0.0563	0.0557	0.0551	0.0546
0.4	0.0729	0.0719	0.0709	0.0699	0.0689
0.5	0.0876	0.0860	0.0845	0.0829	0.0813
0.6	0.1008	0.0985	0.0963	0.0940	0.0918
0.7	0.1125	0.1094	0.1064	0.1034	0.1003
0.8	0.1227	0.1187	0.1148	0.1109	0.1069
0.9	0.1314	0.1265	0.1215	0.1166	0.1116
1.0	0.1388	0.1327	0.1266	0.1205	0.1145

8. Conclusion

In this paper, a non-Markovian queueing model with single vacation has been analyzed. This model is correlated with a web server model. Two particular models have been derived, the corresponding nymerical results is also given.

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