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Methodologies of Construction of Circular Models

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ABSTRACT

This paper is focused on various methodologies of construction of circular models contributed by researchers in directional data analysis for modeling angular/periodic data. Several methods of construction are discussed in detailed here. These methods can be adopted for modeling angular/circular/periodic data which may be occurred in many practical situations in various fields.

1. INTRODUCTION

Modeling circular / periodic data is figured out using Circular Models. Due to paucity of circular models, different methods of construction are made a mention of it [Jammalamadaka and Sen Gupta (2001)]. The following is the list of various construction methods cited by them,

1. Geometrical consideration
2. Wrapping a linear distribution around the unit circle
3. Using characterization properties such as Maximum Entropy or Maximum Likelihood, etc.,
4. Offsetting a bivariate linear distribution
5. Applying inverse stereographic projection to a Linear model.

Further Girija (2010) have introduced a new construction procedure by applying Rising Sun function, yet another method based on Differential Approach is introduced by Dattatreya Rao et al (2011b). Applying the above methods, new circular models are added to the literature of Circular Statistics.

This paper is devoted to present detailed methodologies of Wrapping, Offsetting, Rising Sun function and Differential Approach in constructing circular models and also suitable illustrations are included. Further the characteristic functions of the respective illustrations along with the population characteristics are also presented. As we all know that the circular distribution is defined as a probability distribution whose total probability is concentrated on the unit circle $\{(\cos\theta, \sin\theta) / 0 \le \theta < 2\pi\}$ in the plane. Inverse Stereographic Projection is well explained in the next two chapters as new angular models are constructed by projecting on linear models.

In this chapter Sections (2) and (3) respectively deal with explaining the methodologies of Wrapping and Offsetting. Sections (4) and (5) describe the Rising Sun Circular models and differential approach for constructing new Circular models respectively. Sections (6) and (7) discuss characteristic functions of Circular models and to tabulate population characteristics respectively.

Definition 1.1: In the continuous case $g:[0, 2\pi) \to R$ is the probability density function of a circular distribution iff g has the following basic properties

- Non-negative condition, $g(\theta) \ge 0 \quad \forall \theta$
- Total probability is equal to one, $\int_0^{2\pi} g(\theta)\, d\theta = 1$
- g is periodic, $g(\theta) = g(\theta + 2k\pi)$ for any integer k (Mardia and Jupp, 2000).

Since circular model is a probability density function, it is customary to define its characteristic function in order to study population characteristics. It is well known that the characteristic function for a linear random variate X is defined as $E(e^{itx}), t \in R$ whereas for that of circular random variate Φ_C it is $E(e^{ip\theta}), p \in Z$ and $\theta \in [0,2\pi)$. The moments thus obtained are called p^{th} order trigonometric moments. Population characteristics of a circular model can be studied using trigonometric moments.

Characteristic functions of Circular Models

A brief definition of the characteristic function of a circular model is already mentioned in the earlier section. More details including illustrations are given in this section.

The characteristic function of a circular model with the probability density function $g(\theta)$ is defined as $\varphi_p(\theta) = \int_0^{2\pi} e^{ip\theta} g(\theta) d\theta \; , p \in Z$.

The value of the characteristic function φ_p at an integer *p* is also called the *p*th trigonometric moment of θ. The real part and the imaginary part of φ_p are denoted by α_p and β_p respectively. We can also view these trigonometric moments in terms of

$$\alpha_p = E(\cos p\theta), \quad \beta_p = E(\sin p\theta), \quad p \in Z$$

The first trigonometric moment namely,

$\varphi_1 = \alpha_1 + i\beta_1 = \rho_1 e^{i\mu_1}$ plays a prominent role in determining the mean direction and resultant length.

2. METHODOLOGY OF WRAPPING [Jammalamadaka and Sen Gupta (2001)]

Reducing a linear random variable to its modulo 2π is called wrapping. The methodology of wrapping by means of modulo 2π reduction is explained as follows.

If X is a r.v. defined on *R*, then the corresponding circular r.v. X_W is defined by the modulo 2π reduction

$$X_W \equiv X(\mathrm{mod}\, 2\pi) \tag{2.1}$$

It is clearly a many valued function given by

$$X_W(\theta) = \{X(\theta + 2k\pi) / k \in Z\} \tag{2.2}$$

The wrapped circular pdf $g(\theta)$ corresponding to the density function f of a linear r.v. X is defined as,

$$g(\theta) = \sum_{k=-\infty}^{\infty} f(\theta + 2k\pi) \; , \quad \theta \in [0, 2\pi) \tag{2.3}$$

The Characteristic Function of Wrapped Circular model

The pdf of a wrapped circular model can be obtained through characteristic function of the linear r.v. *X* using trigonometric moments. Using the inversion theorem of characteristic function, one can derive, circular models through trigonometric moments. These trigonometric moments can be obtained using the following Proposition [c.f. p.31, Jammalamadaka and Sen Gupta (2001)].

The density and the distribution functions of the Wrapped Exponentiated Inverted Weibull Distribution (WEIWD) [Srinivasa Subrahmanyam et al (2017)] on applying wrapping are as follows

2.1 WRAPPING OF EXPONENTIATED INVERTED WEIBULL DISTRIBUTION (WEIWD) [Srinivasa Subrahmanyam et al (2017)]

A linear random variable X is said to follow a two parameter EIWD, if the distribution function of X takes the following form

$$F(x) = \left(e^{-x^{-c}}\right)^{\lambda} \tag{2.4}$$

where c and λ both are shape parameters and $0 < x < \infty$ and $c > 0, \lambda > 0$.

Hence the probability density function of EIWD is

$$f(x) = \lambda c x^{-(c+1)} \left(e^{-x^{-c}}\right)^{\lambda} \tag{2.5}$$

where $0 < x < \infty$ and $c > 0, \lambda > 0$.

The probability density function, $g(\theta)$ of the WEIWD can be obtained as

$$g(\theta) = \sum_{k=0}^{\infty} \lambda c (\theta + 2\pi k)^{-(c+1)} \left(e^{-(\theta + 2\pi k)^{-c}}\right)^{\lambda} \tag{2.6}$$

where $\theta \in (0, 2\pi)$, $c > 0$ and $\lambda > 0$.

The graph of the pdf of WEIWD for different values of parameter c keeping the parameter λ constant at 2.0 is plotted here

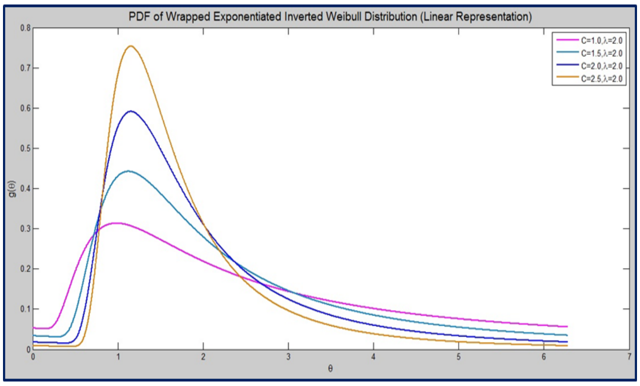

Fig 1. PDF of WEIWD (Linear Representation for different values of c)

The circular representation of pdf of WEIWD for different values of parameter c keeping the parameter λ constant at 2.0 is shown below:

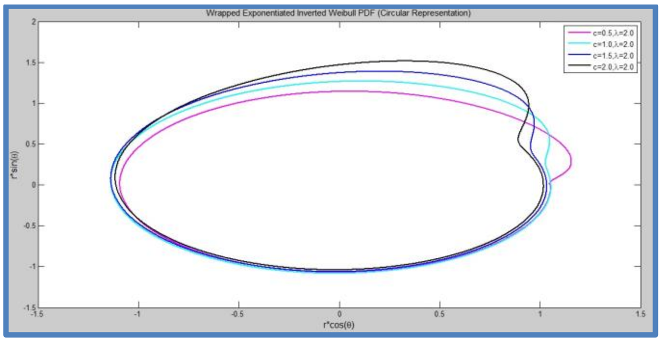

Fig 2. PDF of WEIWD (Circular Representation for different values of c)

The CDF, $G(\theta)$ of the WEIWD is

$$G(\theta) = \sum_{k=0}^{\infty} \left(e^{-\lambda(\theta+2\pi k)^{-c}} - e^{-\lambda(2\pi k)^{-c}}\right)$$

(2.7)

where $\theta \in (0, 2\pi)$ and $c > 0, \lambda > 0$.

At different values for parameters c and λ the graph of CDF for WEIWD is obtained as below

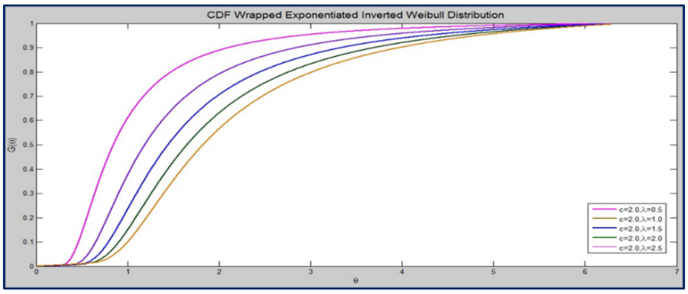

Fig 3. CDF of WEIWD

The characteristic function of the wrapped Exponentiated Inverted Weibull distribution is hence given by

$$\phi_w(p) = \int_0^{2\pi} e^{ip\theta} g(\theta)\, d\theta$$

$$\phi_w(p) = \int_0^{2\pi} e^{ip\theta} \left(\sum_{k=0}^{\infty} \lambda c(\theta+2\pi k)^{-(c+1)} \left(e^{-(\theta+2\pi k)^{-c}}\right)^{\lambda}\right) d\theta$$

(2.8)

To obtain trigonometric moments for WEIWD, the equation (2.8) has to be evaluated. That can be achieved from the characteristic function of EIWD.

$\phi_X(t) = \int_0^{\infty} e^{itx} \; f(x)dx\,, \quad t \in R$

$$\phi_X(t) = \int_0^{\infty} e^{itx} \lambda c x^{-(c+1)} \left(e^{-x^{-c}}\right)^{\lambda} dx$$

(2.9)

Taking $x^{-c} = u$ and carrying out necessary transformations, the equation (2.9) will reduce to

$$\phi_X(t) = \int_0^{\infty} e^{-v} \sum_{k=0}^{\infty} \frac{\left(it\left(\frac{v}{\lambda}\right)^{(-1/c)}\right)^k}{k!} dv = \sum_{k=0}^{\infty} \frac{(it\lambda^{1/c})^k}{k!} \int_0^{\infty} e^{-v} v^{-k/c} dv$$

$$= \sum_{k=0}^{\infty} \frac{\left(it\lambda^{1/c}\right)^k}{k!} \Gamma(1-k/c)$$

(2.10)

where $c > 0$ and $\lambda > 0$

The characteristic function of the Exponentiated Inverted Weibull distribution is feasible only when the right hand side of the equation (2.10) is convergent. But it can be noticed that the series in (2.10) fails to converge at least for some values of c. For example when $c = \frac{1}{n}, n > 0, n \in Z^+$.To overcome this for evaluating the characteristic function of Exponentiated Inverted Weibull distribution, for obtaining the trigonometric moments of Wrapped Exponentiated Inverted Weibull distribution, a numerical integration method known as n – point Gauss – Laguerre quadrature formula as given in Rao and Mitra (1975) is applied.

After evaluating the characteristic function of the WEIWD using the above said method for $p \in$, the real and imaginary parts α_p and β_p respectively are obtained. The following are the graphs for the characteristic function of the WEIWD showing the real part and imaginary part separately for different values of parameters c and λ.

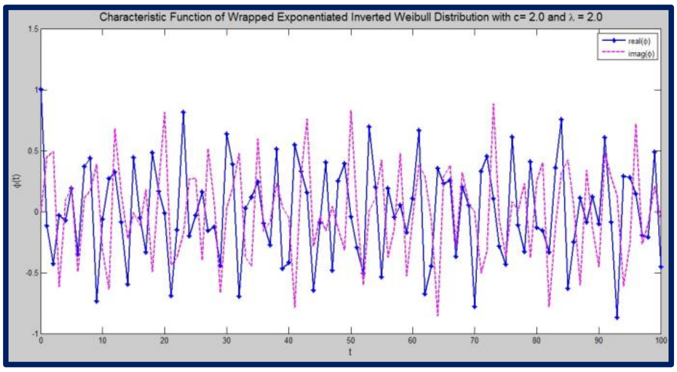

Fig 4 Characteristic Function of WEIWD

at $c = 2$ and $\lambda = 2$

Population Characteristics of WEIWD

The Population Characteristics for the WEIWD are computed. For computation of characteristics some arbitrary values are taken for the parameters c and λ.

3. METHODOLOGY OF OFFSETTING

As mentioned earlier, the method of transforming a bivariate linear r.v. to its directional component is called OFFSETTING and the respective directional component is called offset distribution. This is done by accumulating probabilities over all different lengths for a given direction. Jammalamadaka and Sen Gupta (2001) established that the bivariate random vector (X, Y) can be transformed into polar coordinates (R,θ) and is integrated over R for a given θ. If $f(x, y)$ denotes the joint density distribution on the plane, then the resulting circular offset distribution, say $g(\theta)$ is given by

$$g(\theta)=\int_0^\infty f(r\cos\theta, r\sin\theta)\, r\, dr \qquad (3.1)$$

The density and the distribution functions of the Offset Pearson Type II model [Radhika et al (2013)] by offsetting a bivariate linear model are as follows

The Offset Pearson Type II Model

The Bivariate Pearson Type II distribution arises in many statistical problems including analysis of variance and experimental design in general and two-stage estimation procedures, but is rarely used to fit data. The Offset Pearson Type II **(OP-II)** distribution is derived from the Bivariate Pearson Type II distribution [Balakrishnan and Chin (2008), p. 371].

The pdf $g(\theta)$ and the cdf $G(\theta)$ of the Offset Pearson Type II model for the Bivariate Pearson Type II distribution with parameters $q > 1$ and ρ where $|\rho|<1$ and $\theta\in[0,2\pi)$ are respectively given by

$$g(\theta)=\frac{\sqrt{1-\rho^2}}{2\pi(1-\rho\sin 2\theta)}$$

(3.2)

and

$$G(\theta)=\begin{cases}\frac{1}{2\pi}\left\{\tan^{-1}\left(\frac{\tan\theta-\rho}{\sqrt{1-\rho^2}}\right)-\tan^{-1}\left(\frac{-\rho}{\sqrt{1-\rho^2}}\right)\right\}, & \theta\in\left[0,\frac{\pi}{2}\right)\\ \frac{1}{2\pi}\left\{\pi+\tan^{-1}\left(\frac{\tan\theta-\rho}{\sqrt{1-\rho^2}}\right)-\tan^{-1}\left(\frac{-\rho}{\sqrt{1-\rho^2}}\right)\right\}, & \theta\in\left(\frac{\pi}{2},\frac{3\pi}{2}\right)\\ \frac{1}{2\pi}\left\{2\pi+\tan^{-1}\left(\frac{\tan\theta-\rho}{\sqrt{1-\rho^2}}\right)-\tan^{-1}\left(\frac{-\rho}{\sqrt{1-\rho^2}}\right)\right\}, & \theta\in\left(\frac{3\pi}{2},2\pi\right)\end{cases}$$

(3.3)

The Characteristic Function of the Offset Pearson Type II model [Radhika et al (2013)]

The characteristic function of the Offset Pearson Type II distribution is derived here.

$$\varphi_p=\int_0^{2\pi} e^{ip\theta}\, g(\theta)\, d\theta$$

$$=\int_0^{2\pi}\frac{e^{ip\theta}\sqrt{1-\rho^2}}{2\pi\sigma\left(1-\rho\sin 2\left(\frac{\theta-\mu}{\sigma}\right)\right)}\, d\theta$$

$$=\int_{\frac{-\mu}{\sigma}}^{\frac{2\pi-\mu}{\sigma}}\frac{\sqrt{1-\rho^2}\, e^{ip(\sigma t+\mu)}\sigma}{2\pi\sigma(1-\rho\sin 2t)}\, dt$$

taking $\frac{\theta-\mu}{\sigma}=t$

$$=\frac{\sqrt{1-\rho^2}}{2\pi\sigma i}\int_C\frac{e^{ip\mu}z^{\sigma p-1}\sigma dz}{1-\frac{\rho}{2i}\left(z^2-\frac{1}{z^2}\right)}$$

, $z=e^{it}$ and C is closed unit circle.

To evaluate this integration the notion of Residue at a finite point is applied.

$$f(z)=\frac{\sqrt{1-\rho^2}}{2\pi i}\left(\frac{e^{ip\mu}z^{\sigma p-1}}{1-\frac{\rho}{2i}\left(z^2-\frac{1}{z^2}\right)}\right)$$ is

analytic in C except at its singular points within C, so that

$\int_C f(z)\, dz=2\pi i$ (sum of residues at the singular points within C). If α^2 and β^2 are the roots of the denominator, then the poles of f within C are $\pm\beta$, where $\beta^2=\left(\frac{\left(1-\sqrt{1-\rho^2}\right)}{\rho}\, i\right)$ and clearly $|\alpha^2|>1$

.

$\int_C f(z)\, dz=2\pi i$ {sum of residues at the singular points within C}

$$= 2\pi i\, e^{ip\mu} \left\{ \frac{-\sqrt{1-\rho^2}}{\pi} \left(\frac{\beta^{\sigma p} + (-\beta)^{\sigma p}}{2\rho\left(\beta^2 - \alpha^2\right)} \right) \right\}$$

$$= 2\, i\, e^{ip\mu} \left\{ \frac{-\sqrt{1-\rho^2}}{2\ \rho \left(\frac{-2i\sqrt{1-\rho^2}}{\rho} \right)} \left(\beta^{\sigma p} + (-\beta)^{\sigma p} \right) \right\}$$

$$\int_C f(z)\, dz = e^{ip\mu} \frac{\beta^{\sigma p} + (-\beta)^{\sigma p}}{2}$$

$$\varphi_p = e^{ip\mu} \frac{\beta^{\sigma p} + (-\beta)^{\sigma p}}{2}$$ for $p \in Z$, $\theta, \mu \in [0,2\pi)$ and $|\rho| < 1$ (3.4)

The graph of the characteristic function of the Offset Pearson Type II model is plotted.

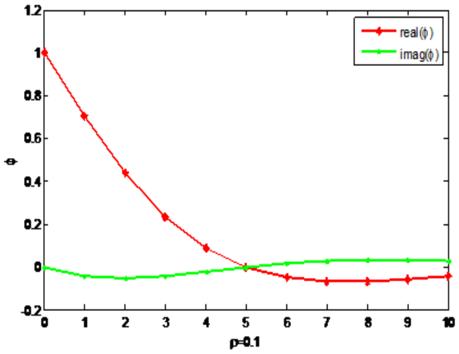

Fig 5. Graph of the Characteristic function of Offset Pearson Type II distribution with ρ = 0.1

The population characteristics of the Offset Pearson Type II model for various values of ρ are evaluated in MATLAB.

4. THE RISING SUN CIRCULAR MODELS

The Rising Sun function (**RSF**) of a bounded function $f: [a,b] \to R$ is defined by

$$f_\Theta(x) = \sup\{f(t) : x \le t \le b\}$$ [Van Rooij and Schikoff (1982)] (4.1)

It is easy to show that

- when f is nonnegative then f_Θ is nonnegative
- when f is continuous then f_Θ is continuous
- f_Θ is monotonically decreasing , hence $f_\Theta = f$ when f is decreasing and f_Θ is the smallest monotonically decreasing function such that $f_\Theta = f$.

Imagine the Rising Sun on x – axis. Then $\{(x,y) \in R^2 : y \ge f_\Theta(x)\}$ is illuminated by the sun whereas $\{(x,y) \in R^2 : y < f_\Theta(x)\}$ is covered by darkness. The set $\{(x, f(x)) : f(x) = f_\Theta(x)\}$ is the collection of those points of the graph of f that receive light from the sun.

A new construction procedure of a class of Circular Models using RSF is obtained in the following theorem. These distributions are named as 'Rising Sun Circular models'.

Theorem 4.1: [Girija (2010)]: If g is the pdf and G is the cdf of a random variable of a circular distribution then the Rising Sun function g_Θ , gives rise to the pdf g_c of a circular model. The distribution function of g_c is given by

$$G_c = \begin{cases} \frac{1}{K}\left[\theta_1\, g(\theta_1) + G(\theta) - G(\theta_1)\right] & \text{for } \theta_1 < \theta \\ \frac{1}{K}\left[\theta\, g(\theta_1)\right] & \text{for } \theta_1 \ge \theta \end{cases}$$

(4.2)

THE RISING SUN VON MISES MODEL [Radhika (2014)]

The von Mises distribution was introduced by von Mises (1918), in order to study the deviations of measured atomic weights from integral values. A technique is proposed in which each of these microphone-pair determined azimuths are further combined into a mixture of the von Mises distributions, thus producing a practical probabilistic representation of the microphone array measurement. It is shown that this distribution is inherently multimodal and that the system at hand is non-linear, which required a discrete representation of the distribution function by means of particle filtering [Ivan Markovi´c, and Ivan Petrovi´c (2010)]. Procedures for the estimation of parameters of the proposed distribution include the method of moments, and pseudo likelihood; the efficiency of the latter is investigated in two and three dimensions. The methods are applied to real protein data of conformational angles.

The pdf of von Mises distribution [Jammalamadaka and Sen Gupta (2001)] is

$$g(\theta) = \frac{1}{2\pi I_0(k)} \exp\left(k\cos(\theta - \mu)\right),$$

(4.3)

where I_0 denotes the Modified Bessel function of the first kind and order zero, which can be defined by

$$I_0(k) = \frac{1}{2\pi}\int_0^{2\pi} e^{k\cos\theta}\, d\theta .$$

The function I_0 has power series expansion

$$I_0(k) = \sum_{r=0}^{\infty} \frac{1}{(r!)^2} \left(\frac{k}{2}\right)^{2r}$$

The parameter μ is the mean direction and the parameter k is known as the concentration parameter. The mean resultant length ρ is $A(k)$ where A is the function defined by

$$A(k) = \frac{I_1(k)}{I_0(k)}$$

The Rising Sun function of the von Mises distribution is given by

$$g_\Theta(\theta) = \operatorname{Sup}\left(g(t) : \theta \le t < 2\pi\right)$$

$$= \operatorname{Sup}\left(\frac{1}{2\pi I_0(k)} \exp\left(k\cos(t-\mu)\right) : \theta \le t < 2\pi\right)$$

(4.4)

Normalizing this function with the constant $K_1 = \int_0^{2\pi} g_\Theta(\theta)\ d\theta$ the pdf of the Rising Sun von Mises distribution (**RSVM**) is obtained.

$$g_c(\theta) = \frac{\operatorname{Sup}\left(\frac{1}{2\pi I_0(k)} \exp\left(k\cos(t-\mu)\right) : \theta \le t < 2\pi\right)}{\int_0^{2\pi} g_\Theta(\theta)\ d\theta}$$

(4.5)

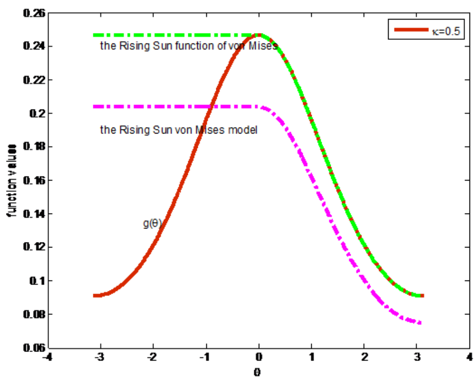

Fig. 6 Graph of von Mises pdf and the Rising Sun von Mises pdf

The Characteristic Function of the Rising Sun Circular model

Result 4.2: Let g and φ_p be the pdf and the characteristic function of a circular model and θ_1 be the mode of g. Then the characteristic function of the corresponding Rising Sun circular model with pdf g_c is

$$\varphi_\Theta(p) = \begin{cases} 1 & \text{for } p = 0 \\ \dfrac{\varphi_p(\theta)}{k} + \dfrac{g(\theta_1)}{kp} i\left(1 - e^{ip\theta_1}\right) - \dfrac{1}{k}\displaystyle\int_0^{\theta_1} e^{ip\theta} g(\theta)\, d\theta & \text{for } p \ne 0 \end{cases}$$

(4.6)

where g_Θ is the Rising Sun function of g, and

$$k = \int_0^{2\pi} g_\Theta(\theta)\ d\theta$$

The graph of the characteristic function of the Rising Sun von Mises model is presented here.

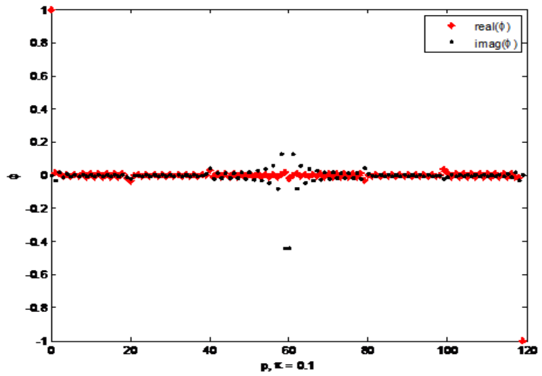

Fig 7 Graph of the Characteristic function of the Rising Sun von Mises distribution

As the pdf of the Rising Sun von Mises model is not in closed form, the values of the characteristic function can be evaluated using numerical methods in MATLAB. The population characteristics for the Rising Sun von Mises model are computed based on their respective trigonometric moments.

5. DIFFERENTIAL APPROACH FOR CONSTRUCTING CIRCULAR MODELS [Dattatreya Rao et al (2011b)]

CARDIOID DISTRIBUTION THROUGH A DIFFERENTIAL EQUATION

By making use of certain assumptions on arbitrary constants in the general solution of a differential equation, we construct the pdf of the Cardioid model.

Theorem 5.1: The solution of the initial value problem

$$\frac{d^2y}{d\theta^2} + y = \frac{1}{2\pi}, \qquad y(0) = \frac{1 + 2\rho\cos\mu}{2\pi}, \qquad y'(0) = \frac{\rho\sin\mu}{\pi}$$

admits

i) the particular integral which is a Uniform distribution on the unit circle

and

ii) probability density function of the Cardioid distribution

$$y(\theta) = \frac{1}{2\pi}\left(1 + 2\rho\cos(\theta - \mu)\right)$$

where $-\pi \le \theta, \mu < \pi$ and $|\rho| < 0.5$.

On these lines other circular models could be tried by changing initial conditions.

REFERENCES

1. Balakrishnan, N. & Chin – Diew Lai (2008). Continuous Bivariate Distributions, Springer.
2. Dattatreya Rao, A.V., Girija, S.V.S. and Phani, Y. (2011b). Differential Approach to Cardioid Distribution, Computer Engineering and Intelligen Systems, ISSN 2222-1719, Vol 2, No.8, pp.1-6.
3. Girija, S.V.S. (2010).Construction of New Circular Models, Project number 36254, VDM VERLAG, Germany. ISBN 978-3-639-27939-9.
4. Ivan Markovi´c & Ivan Petrovi´c, (2010). Applying von Mises Distribution to Microphone Array Probabilistic Sensor Modelling, Robotics (ISR), 41st International Symposium on and 2010 6th German Conference on Robotics (ROBOTIK).
5. Jammalamadaka, S.R. and Sen Gupta, A. (2001). "Topics in Circular Statistics", World Scientific Press, Singapore.
6. Mardia, K. V. and Jupp, P. E. (2000). "Directional Statistics", John Wiley, Chichester.
7. Radhika, A.J.V., Girija, S.V.S. & Dattatreya Rao, A.V. (2013). *On Univariate Offset Pearson Type II Model – Application To Live Data*, International Journal of Mathematics and Statistics Studies, Vol.1, No. 1, 2013, pp. 1-9.
8. Rao, C.R. and Mitra, S.K. (1975), "Formulae and Tables for Statistical Work", Statistical Publishing Society.
9. Srinivasa Subrahmanyam P., A.V.Dattatreya Rao and S.V.S.Girija (2017). On Wrapping of Exponentiated Inverted Weibull Distribution, IJIRST –International Journal for Innovative Research in Science & Technology| Volume 3 | Issue 11 | April 2017, 18 – 25.
10. Van Rooij, A.C.M. & Schikof, W.H. (1982). A Second Course on Real Functions, *Cambridge University Press, Cambridge*.
11. von Mises, R. (1918). Uber die "Ganzzahligkeit" der Atomgewichte and verwandte Frogmen. Phys.Z., 19, 490 - 500.