International Journal of Mathematics and Computer Research

ISSN: 2320-7167

Volume 12 Issue 02 February 2024, Page no. – 4062-4068

Index Copernicus ICV: 57.55, Impact Factor: 8.316

DOI: 10.47191/ijmcr/v12i2.09

Jordan Dedekind Chain Condition on Pre-A* Posets with Graphical Aspects

Prof (Dr.) Jonnalagadda Venkateswara Rao[1*], Joseph Owuor Owino[2]

[1,2] Department of Mathematics, School of Science and Technology, United States International University-Africa, Kasarani, Thika Road, 14634-00800, Nairobi, Kenya.

ARTICLE INFO

Published Online:
23 February 2024

Corresponding Author:
Dr. Jonnalagadda Venkateswara Rao

ABSTRACT

Jordan Dedekind chain condition on pre-A* posets is studied. We discuss isomorphism of pre-A* posets, ascending chain condition and descending chain condition on pre-A* posets. We also prove that a pre-A* poset possessing a least element of locally finite length which satisfies Jordan Dedekind chain condition has a dimension function.

KEYWORDS: Pre-A* algebra, poset, chain, isomorphism, ascending chain condition, descending chain condition, Jordan Dedekind chain condition.

MSC Classification: 06E05 , 06EXX , 06E10 , 05 C 45

1 INTRODUCTION

An equational 3-valued generalization of a boolean algebra and C-algebra have been introduced by Fernando [1]. The aforesaid generalization based on logic functions "and ", " or " and " not ". The algebra of disjoint alternatives (Ada) $(A,\wedge,\vee,(-)^{\sim},(-)_{\pi},0,1,2)$ has been studied by Manes [2], building a foundation on C-algebras. An A*-algebra $(A,\wedge,\vee,(-)^{\sim},(-)_{\pi},0,1,2)$ and its equivalence with algebra of disjoint alternatives as well as the association of C-algebras and 3-ring has been done by Rao [3]. Moreover, A*-clones, If-Then-Else structure over A*-algebra, pre-A* algebra $(A,\wedge,\vee,(-)^{\sim})$ and ideal of A*-algebra have been investigated [3, 4]. Kalyani [5, 6] have characterized a partial order relation on pre-A* algebra, thereby studying representation of a pre-A* algebra by a partial order. In this work, we prove that if a Pre-A* poset has least element of locally finite length which satisfies the Jordan Dedekind chain condition, then the pre-A* poset has a dimension function. The theorem is based on ascending and descending chain conditions.

2 Preliminaries

In this part of the paper, we define the important terms that we have used throughout.

Definition 1 (Pre-A* algebra)**.** *[4, Definition 1] A Pre-A* algebra is an algebra* $(A \; 6{=} \; \emptyset,\wedge,\vee,(-)^{\sim})$ *where A has a 1;* $\wedge$ *and* $\vee$ *are binary operations and* $(-)^{\sim}$ *is a unary operation such that the following conditions are satisfied:*

(i). $x^{\sim\sim} = x$ *for all* $x \in A$,
(ii). $x \wedge x = x$, *for all* $x \in A$,
(iii). $x \wedge y = y \wedge x$, *for all* $x,y \in A$,
(iv). $(x \wedge y)^{\sim} = x^{\sim} \vee y^{\sim}$ *for all* $x,y \in A$,
(v). $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ *for all* $x,y,z \in A$, *(vi).* $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ *for all* $x,y,z \in A$,
(vii). $x \wedge y = x \wedge (x^{\sim} \vee y)$ *for all* $x,y \in A$.

Example 1. *(i). Let* $(\mathbf{3} = \{0,1,2\},\wedge,\vee,(-)^{\sim})$ *be a pre-A* algebra. The operations are defined using the laws:* $2^{\sim} = 2$, $1 \wedge x = x$ *for all* $x \in \mathbf{3}$, $0 \vee x = x$ *for all* $x \in \mathbf{3}$ *and* $2 \wedge x = 2 \vee x = 2$ *for all* $x \in \mathbf{3}$. *Hence we obtain Table 1.*

Table 1: An illustration of the properties in Example 1(i).

∧	0	1	2
0	0	0	2
1	0	1	2
2	2	2	2

(a) ∧.

∨	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

(b) ∨.

x	$x^{\sim}$
0	1
1	0
2	2

(c) $(-)^{\sim}$.

(ii). Let $(\mathbf{2} = \{0,1\},\wedge,\vee,(-)^{\sim})$ *be a pre-A* algebra. The operations are defined using the laws:* $1 \wedge x = x$ *for all* $x \in \mathbf{2}$, $0 \vee x = x$ *for all* $x \in \mathbf{2}$. *Hence we obtain Table 2.*

Table 2: An illustration of the properties in Example 1(ii).

∧	0	1
0	0	0
1	0	1

(a) ∧.

∨	0	1
0	0	1
1	1	1

(b) ∨.

x	$x^{\sim}$
0	1
1	0

(c) $(-)^{\sim}$.

Definition 2. *[4, Partial order] A relation R on a set A is partial order if:*

(i). For all $a \in A$, aRa (reflexivity)

(ii). For all $a,b \in A$, aRb and bRa , then $a = b$ (anti-symmetry)

(iii). For all $a,b,c \in A$, aRb and bRc, then aRc (transitivity).

The pair (A,R) is a partially ordered set (poset).

Example 2. *Posets*

(i). Let $A = \{0,1,2\}$. Then power set of A given by $2^A = \{\emptyset,\{0\},\{1\},\{2\},\{0,1\},\{0,2\},\{1,2\},A\}$ is a poset under set inclusion. For for any $A_0,A_1 \subseteq A$, we define $A_0 \leq A_1$ whenever $A_0 \subseteq A_1$. The pair $(2^A,\subseteq)$ is a poset.

(ii). Let P be the set of all real-valued functions on $A = \{0,1\}$. That is, $P = \{f \mid f : A \to \mathbb{R}\}$. For any $f_0,f_1 \in P$ let $f_0 \leq f_1$ as long as for any $t \in A$, then $f_0(t) \leq f_1(t)$. The pair $(P,\leq)$ is a poset.

Remark 1. *Let (A,R) be a poset. Then $a_0,a_1 \in A$ are comparable if either a_0Ra_1 or a_1Ra_0. Else a_0 and a_1 are incomparable. A poset in which any two members can be compared is a chain. For instance:*

(i). Let $A \neq \emptyset$ be a set with a power set 2^A. Then $(2^A,\subseteq)$ is a poset as described in Example 2 (i). In particular, if $A = \{0,1\}$, then $2^A = \{\emptyset,\{0\},\{1\},A\}$. The elements $\{0\}$ and $\{1\}$ in 2^A are not comparable by set inclusion. So, $(2^A,\subseteq)$ is merely a poset but not a chain.

(ii). The Pre-A algebra $(\mathbf{3} = \{0,1,2\},\wedge,\vee,(-)^{\sim})$ with R defined as $a_0 \leq a_1$ whenever $a_0 \wedge a_1 = a_1 \wedge a_0 = a_0$ for any $a_0,a_1 \in \mathbf{3}$ is a chain.*

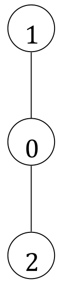
Let $(\emptyset \neq A_0,\leq)$ be any subset of a poset $(A,\leq)$. Observe that $\leq$ induces a partial ordering on A_0, and hence, $(\emptyset \neq A_0,\leq)$ is a poset (a subposet). In the same spirit, any subset of a chain is also a chain (a subchain).

Definition 3. *Let $(A_1,\leq_1)$ and $(A_2,\leq_2)$ be posets. A mapping $\varphi : A_1 \to A_2$ is an order-preserving homomorphism or isotone, if $a_1 \leq_1 a_1'$ implies that $\phi(a_1) \leq_2 \phi(a_1')$ for all $a_1,a_1' \in A_1$. If ϕ is a bijection and $a_1 \leq_1 a_1'$ if and only if $\phi(a_1) \leq_2 \phi(a_2')$ for all $a_1,a_1' \in A_1$ (that is, φ and φ^{-1} are order-preserving homomorphisms), then φ is an isomorphism (order-preserving isomorphism). An isomorphism from A_1 to itself is called an automorphism.*

Definition 4. *Let $(A,\leq)$ be a poset. An element $a \in A$ is said to be the least (greatest) element of A if $a \leq x$ $(x \leq a)$ for all $x \in A$. If A is a Pre-A* algebra, then $c \in A$ is a central element if $c \vee c^{\sim} = 1$. In the Pre-A* algebra $(\mathbf{3} = \{0,1,2\},\wedge,\vee,(-)^{\sim})$, 0 and 1 are central elements whereas 2 is a non-central element. A Pre-A* algebra A which satisfies the conditions of a poset is a pre-A* poset. We illustrate as follows:*

(i). If A is a pre-A algebra with 1, 0, and 2, then $x \leq 1$ $(x \wedge 1 = 1 \wedge x = x)$ for all $x \in A$, and $2 \leq x$ $(x \wedge 2 = 2 \wedge x = 2)$. This shows that 1 is the greatest element and 2 is the least element of the poset, since $2 \leq x \leq 1$. The Hasse diagram for the poset $(A,\leq)$ is given in Figure 1 below.*

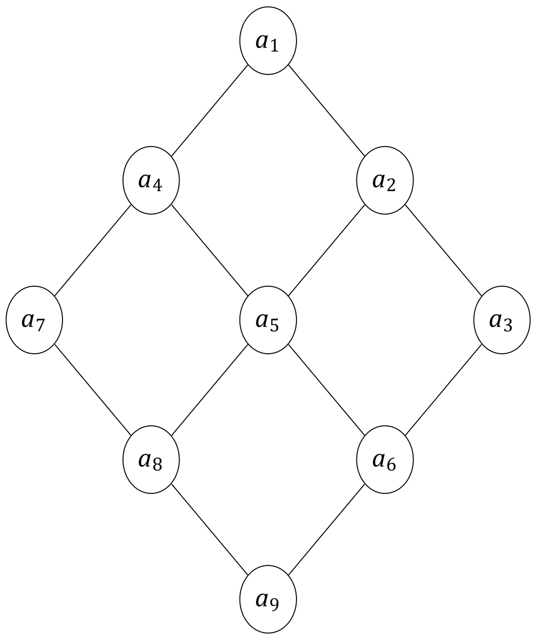
Fig. 1: Hasse diagram of the poset $(A,\leq)$.



The chromatic number of the graph in Figure 1 is 2. It is thus a bipartite graph. This graph has two nodes of odd degree, making it have an Euler path but no Euler circuit.

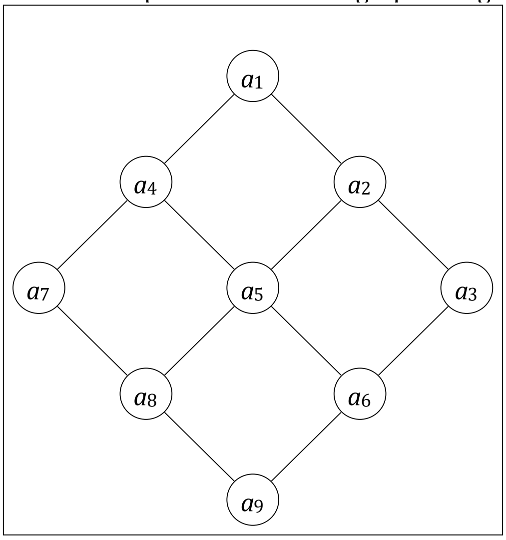
(ii). Let $A \times A = \{a_1 = (1,1), a_2 = (1,0), a_3 = (1,2), a_4 = (0,1), a_5 = (0,0), a_6 = (0,2), a_7 = (2,1), a_8 = (2,0), a_9 = (2,2)\}$ be a pre-A algebra under pointwise operation. Then $A \times A$ has four central elements and the rest noncentral. Among them, $a_9 = (2,2)$ satisfies the property that $a_9 = a_9^{\sim}$. The Hasse diagram of the poset $(A \times A,\leq)$ is given in Figure 2 below, where a_7 is the top element.*

Fig. 2: Hasse Diagram for $A \times A$.



The chromatic number of the graph in Figure 2 is 2,making it a bipartite graph. It is a planar graph since there are no edge crossings, and the planar representation is provided in Figure 3.

Fig. 3: Planar representation of the graph in Figure 2.



It's a planar graph with 5 regions. Here we can observe that $x \leq a_1$, $x \wedge a_1 = a_1 \wedge x = x$, and $a_9 \leq x$ ($x \wedge a_9 = a_9 \wedge x = a_9$) for all $x \in A \times A$. This means that a_1 is the greatest element and a_9 is the least element of $A \times A$.

(iii). We have that $\mathbf{2}\times\mathbf{3} = \{a_1 = (1,1), a_2 = (1,0), a_3 = (1,0), a_4 = (0,1), a_5 = (0,2), a_6 = (1,2)\}$ *is a Pre-A* algebra under pointwise operation, having four central elements, two non-central elements, and no element satisfying the property that* $a^{\sim} = a$. *The Hasse diagram for* $(\mathbf{2}\times\mathbf{3}, \leq)$ *is given below in Figure 4. Observe that* $x \leq a_1$, *that is,* $x \wedge a_1 = a_1 \wedge x = x$, *and* $a_5 \leq x$ ($x \wedge a_5 = a_5 \wedge x = a_5$) *for all* $x \in \mathbf{2}\times\mathbf{3}$. *This shows that* a_1 *is the greatest element and* a_5 *is the least element of* $\mathbf{2}\times\mathbf{3}$. *The graph has a chromatic number of 3, hence it is not a bipartite. It's a planar graph with 4 regions as shown in Figure 5.*

Fig. 4: The Hasse diagram for **2 × 3**.

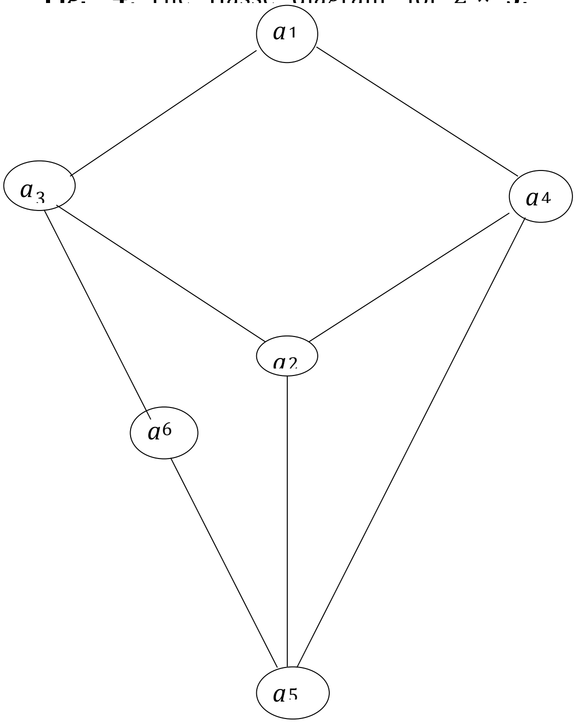
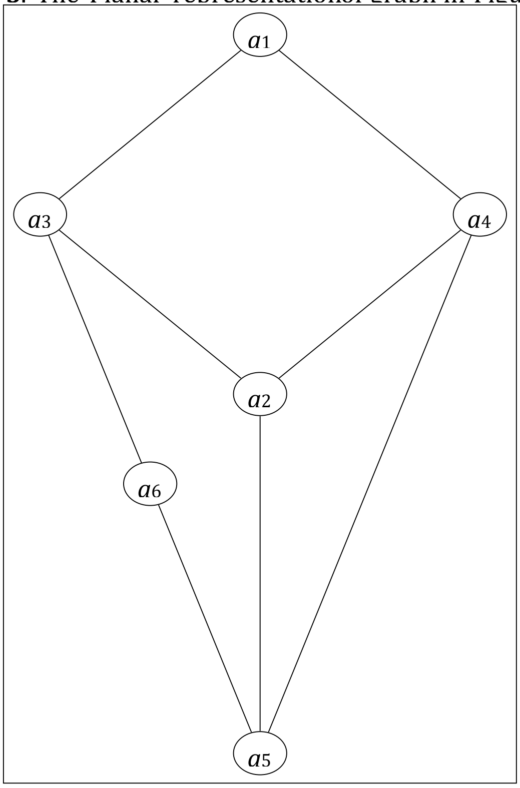


Fig. 5: The Planar representation of graph in Figure 4.

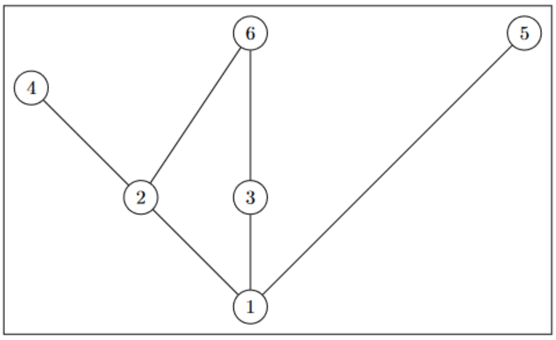


Definition 5. *Let A 6= ∅ be a set. Then* $a \in A$ *is a minimal element if A has no* a^0 *such that* $a^0 < a$ *(or* $a^0 \leq a$ *implies* $a^0 = a$*). Similarly,* $a \in A$ *is a maximal element if there exists no* $a^0 \in A$ *has such that* $a < a^0$ *(or* $a \leq a^0$ *implies* $a = a^0$*).*

(i). Consider the pre-A posets represented in the Figures 1 and 2. In Figure 1, one can observe that 2 is the minimal element and 1 is the maximal element. In Figure 2, one can observe that* a_9 *is the minimal element and* a_1 *is the maximal element.*

(ii). Consider the poset $P = \{1,2,3,4,5,6\}$ *with the definition* $x \leq y$ *if and only if x divides y for all* $x,y \in P$. *See the Hasse diagram in Figure 6 below. In P, 1 is a minimal element and 4, 5, and 6 are maximal elements because 4, 5, and 6 have no x in P to divide (that is, there is no x in P such that* $4 < x$, $5 < x$,
and $6 < x$*), whereas 1, 2, and 3 have some x to divide (that is* $1 \leq x$ *for all x in P,* $2 \leq 4$, $2 \leq 6$, *and* $3 \leq 6$*).This graph has a chromatic number of 3 and thus is not bipartite.*

Fig. 6: Hasse Diagram representing the relations in P Definition 5 (ii)



Definition 6. *Let A be a Pre-A* algebra. The length of a chain of the form* $a_0 < a_1 < ... < a_{r-1}$ *consisting of r elements is a non-negative integer* $r - 1$. *This corresponds to a path of length* $r - 1$, *connecting the vertices* a_0 *and* a_{r-1}. *The length* $l(A)$ *of a poset* $(A,\leq)$ *is the least upper bound (lub) of the lengths of all subchains of A, that is* $l(A) = lub\{l(C) \mid C$ *is a chain in A*$\}$.

(i). *Let us consider the poset represented by the Figure 1. In this poset,* $2 \leq 0 \leq 1$ *is the chain of length 2. Then, the length of the poset A is given by* $l(A) = lub\{l(C) \mid C$ *is a chain in A*$\} = lub\{2\} = 2$.

(ii). *Consider the poset* $(A \times A,\leq)$ *represented in Figure 2. The chains are:* $a_9 \leq a_8 \leq a_7 \leq a_4 \leq a_1$; $a_9 \leq a_6 \leq a_3 \leq a_2 \leq a_1$; $a_9 \leq a_8 \leq a_5 \leq a_2 \leq a_1$; $a_9 \leq a_6 \leq a_5 \leq a_4 \leq a_1$. *These are analogous to paths of length 4 connecting vertices* a_9 *and* a_1.

(iii). *In the Pre-A* poset* **2**×**3**, *from Hasse diagram in Figure 4, the chains connecting* a_5 *and* a_1 *are given by:* $a_5 \leq a_4 \leq a_1$; $a_5 \leq a_2 \leq a_4 \leq a_1$; $a_5 \leq a_6 \leq a_3 \leq a_1$; *and* $a_5 \leq a_2 \leq a_3 \leq a_1$.

If C is a subchain in A, then $l(C) \leq l(A)$. *The poset* $(A,\leq)$ *is of finite length if* $l(A)$ *is finite. Observe that all finite posets are of finite lengths. A poset A is of locally finite length if every one of its intervals is of finite length.*

Remark 2. *The interval in a poset is the set of all the elements in between the least and greatest elements, including them. For example,* $a_9 \leq a_8 \leq a_7 \leq a_4 \leq a_1$ *is a chain connecting* a_9 *and* a_1. *So, the elements of this chain are the elements of the interval* $[a_9,a_1]$ *and hence the length of the interval is computed in a similar way as defined above. In the poset* $(A{\times}A,\leq)$ *represented in Figure 2, we can observe that all intervals connecting* a_9 *and* a_1 *are of finite length. Therefore, the poset* $(A \times A,\leq)$ *is of locally finite length. Any* $x \in A$ *is an upper (lower) bound of* $R \subset (A,\leq)$ *if* $a \leq x$ $(x \leq a)$ *for all a in R. If R has at least one upper (lower) bound, then R is said to be bounded above (below). A subset R of a poset* $(A,\leq)$, *which is both bounded and below, is said to be a bounded subset of* $(A,\leq)$. *If* $(A,\leq)$ *is a poset, then for a in A,* $(a] = \{x \in A \mid x \leq a\}$ *is the set of all lower bounds of a in* $(A,\leq)$. *Similarly,* $[a) = \{x \in A \mid a \leq x\}$ *is the set of all upper bounds of a in* $(A,\leq)$. *Consider the following illustration on posets and boundedness.*

(i). *In the poset* $(A \times A = \{a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9\},\leq)$ *shown in Figure 2, consider the subset* $R = \{a_5,a_6,a_8,a_3\}$. *Here* $a_9 \leq a_8$, $a_9 \leq a_6$, $a_9 \leq a_5$, $a_9 \leq a_3$. *So* a_9 *is a lower bound of R.*

(ii). *Consider the subset* $S = \{a_4,a_2\}$ *in the poset* $(A \times A,\leq)$ *in Figure 2. Since* $a_5 \leq a_4$ *and* $a_5 \leq a_2$ *and* $a_6 \leq a_4$ *and* $a_6 \leq a_2$, *the set* $\{a_5,a_6\}$ *is a lower bound of* $S = \{a_4,a_2\}$.

(iii). *As for the set* $\{a_1,a_2,a_4\}$ *in Figure 2,* a_1 *has no upper bound. In the set* $\{a_8,a_9,a_6\}$, a_9 *has no lower bound.*

Definition 7. *Let R be any subset of a poset* $A = \{0,1,2\}$. *If there exists a lower (upper) bound of R that is also an element of R, then it is called the least (greatest) element of R. Let us denote it by 2 (1). That is, if a is the least (greatest) element of R, then* $a \leq x$ $(x \leq a)$ *for all x in R and a in R. The elements other than 2 and 1 are called inner elements of R. The least and greatest elements together will be called bound elements. Consider the poset* $A{\times}A$ *as shown in Figure 2 and* $R = \{a_1,a_2,a_3,a_4\}$. *The element* a_1 *is the maximal in R and* $x \leq a_1$ *for all x in R. So, R has an upper bound and R has a greatest element. An element x is a greatest lower bound (least upper bound) or infimum (supremum) of R if x is the greatest (least) element of all lower (upper) bounds of R. Consider the subset* $R = \{a_8,a_5,a_6\}$ *in the poset* $A{\times}A$ *as shown in Figure 2. Then* $Sup(\{a_8,a_6\}) = a_5$. *If* $R = \{a_8,a_9,a_6\}$, *then* $Inf(\{a_8,a_6\}) = a_9$.

Definition 8. *Let* $(A,\leq)$ *be a poset. Then A satisfies the minimum (maximum) condition or descending chain condition (DCC) (ascending chain condition (ACC)) if for any descending (ascending) sequence* $a_1 \geq a_2 \geq a_3 \geq ...$ $(a_1 \leq a_2 \leq a_3 \leq ...)$ *of elements of A, there exists a positive integer n such that* $a_n = a_{n+1} = ...$ *Every finite poset satisfies ACC and DCC. One can observe ACC and DCC in the finite posets* $(A{\times}A = \{a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9\},\leq)$ *in Figure 2 and* $(A = \{0,1,2\},\leq)$ *shown in Figure 1.*

Definition 9. *Let* $(A,\leq)$ *be a Pre-A* poset. For any* $a,b \in A$ *such that* $a < b$, *let C be a subchain of A having a as the least element and b as the greatest element. Then we say that C is situated between the elements a and b, or C connects a and b. Graphically, we note that the path C is situated between a and b, or C connects a and b.*

Definition 10. *Let A be a Pre-A* algebra. If for every pair of elements a,b such that* $a \leq b$ *in a Pre-A* poset* $(A,\leq)$, *it is true that all maximal chains connecting the elements a and b are of the same length, then* $(A,\leq)$ *is said to satisfy the Jordan Dedekind Chain condition (JDCC), that is, all the maximal paths connecting the elements a and b are of the same length.*

Definition 11. *Let* $(A,\leq)$ *be a pre-A* poset which is bounded below. Let* $a \in A$. *We define the height* $h(a)$ *or dimension of the element a as the length of the maximal chain that connects the least element* a^0 *and a. That is,* $h(a) = l([a^0,a])$.

Definition 12. *Let* $(A,\leq)$ *be a pre-A* poset which is bounded below and of locally finite length satisfying JDCC. Then* $x \prec y$ *if and only if* $x \leq y$ *and* $h(x) + 1 = h(y)$. *(Here* $x \prec y$ *will be called "x covered by y" and it means that there is no element z between x and y such that* $x \leq z \leq y$*). For instance, from Figure 2:*

(i). *It can be observed that* $a_2 \prec a_1$ *if and only if* $a_2 \leq a_1$ *and* $h(a_2) + 1 = h(a_1)$. *This implies,* $h(a_1) = 3 + 1 = 4$.

(ii). *Further,* $a_9 \prec a_6$ *if and only if* $a_9 \leq a_6$ *and* $h(a_9) + 1 = h(a_6)$. *This implies,* $h(a_6) = 0 + 1 = 1$.

Definition 13. *Let* $(A,\leq)$ *be a pre-A* poset (not necessarily bounded below) and let* $d : A \longrightarrow \mathbb{Z} \cup \{\pm\infty\}$ *be an integer valued function whose co-domain includes integers and one or both of symbols* $\pm\infty$. *Then d is called the dimension function of A if* $x \prec y$ *if and only if* $x \leq y$ *and* $d(x) + 1 = d(y)$ *for all x,y in A.*

3 MAIN RESULTS

We use this section to present main results in the manuscript.

Lemma 1. *Let A be a pre-A* algebra. Define a relation $\leq$ on A by $x \leq y$ if and only if $y \wedge x = x \wedge y = x$. Then $(A,\leq)$ is a poset.*

Proof. Since $x \wedge x = x$, $x \leq x$ for all $x \in A$. Therefore, $\leq$ is reflexive. For all $x,y \in A$, whenever $x \leq y$, and $y \leq x$, then $y \wedge x = x \wedge y = x$ and $y \wedge x = x \wedge y = y$. So $x = y$ and $\leq$ is anti-symmetric. Now let $x,y,z \in A$ such that $x \leq y$ and $y \leq z$. Then $y \wedge x = x \wedge y = x$ and $z \wedge y = y \wedge z = y$. Now, $x = x \wedge y = x \wedge y \wedge z = x \wedge z$, which means that $x \wedge z = z \wedge x = x$. Therefore, $x \leq z$. Consequently, $\leq$ is transitive and $(A,\leq)$ is a poset.

Proposition 1. *Let $A = \{0,1,2\}$ and 2^A be the power set of A. Then 2^A is a bounded set.*

Proof. For any A_0,A_1 in 2^A, define $A_0 \leq A_1$ whenever $A_0 \subseteq A_1$. Clearly, $\emptyset \subseteq A_0$ for all A_0 in 2^A. Then, $\emptyset \leq A_0$ for all $A_0 \in 2^A$. Thus, $\emptyset$ is the lower bound of 2^A. Moreover, for any $A_0 \in 2^A$, $A_0 \subseteq A$ for all $A_0 \in 2^A$. Then $A_0 \leq A$ for all $A_0 \in 2^A$. Thus, A is the upper bound of 2^A and $\emptyset \leq A_0 \leq A$ for all $A_0 \in 2^A$. Hence, 2^A is a bounded set.

Lemma 2. *Let A be any poset. If A has a lower (upper) bound, then it has at most one lower (upper) bound.*

Proof. We prove uniqueness of the lower bound and the same idea can be adopted to suit the upper bound case. Suppose that A has two lower bounds, say a and b. Since a is a lower bound of A, then

$a \leq b$. (1)

Since b is a lower bound of A, then

$b \leq a$. (2)

$b \leq a$. Since $\leq$ is antisymmetric, then by Equations (1)-(2), it follows that $a = b$. Similarly, it can be shown that A has at most one upper bound.

Corollary 1. *In a poset $(A,\leq)$, if A has a least (greatest) element, then it is the only minimal (maximal) element of A.*

Proof. Let A have a least element a. Then

$a \leq x$, (3)

for all $x \in A$ and $a \in A$. We claim that a is the minimal element of A. Assume to the contrary that a is not a minimal element of A. Then, there exists x in A such that

$x \leq a$. (4)

Since $(A,\leq)$ is a poset, Equations (3)-(4) and anti-symmetry imply that $a = x$ is the minimal element of A. Since a is the least element of A, a is unique by Lemma 2. Therefore, a is the only minimal element of A. Similarly, we prove that if A has a greatest element, then it is the only maximal element of A.

Theorem 2. *Let A_1 and A_2 be posets and let ϕ be an order isomorphism of A_1 onto A_2. If a subset R_1 of A_1 has an infimum in A_1, then the set $R_2 = \{\phi(x) \mid x \in R_1\}$ has an infimum in A_2. That is, $\inf_{A2}(R_2) = \phi(\inf_{A1}(R_1))$ or $\inf_{A2}(\phi(R_1)) = \phi(\inf_{A1}(R_1))$.*

Proof. Let $\inf_{A1}(R_1) = a$. This implies that for every x in R_1, $a \leq x$ in A_1. Since ϕ is an order isomorphism, we have that

$\phi(a) \leq \phi(x)$, (5)

in A_2 for all x in R_1. Hence, $\phi(a)$ is a lower bound of $\phi(R_1)$. We prove that

$\phi(a) = \text{glb}(\phi(R_1))$. (6)

Let $t \in A_2$ be any other lower bound of $\phi(R_1)$. Since ϕ is onto, there exists b in A_1 such that $\phi(b) = t$. Therefore, $t = \phi(b)$ is a lower bound of $\phi(R_1)$. Since x is any element of R_1, $\phi(b) \leq \phi(x)$ for all x in R_1, implies that $b \leq x$ (since ϕ is order preserving) for all x in R_1. Then b is a lower bound of R_1. Hence, $b \leq a$, since $a = \text{glb}(R_1)$. So, $\phi(b) \leq \phi(a)$. This implies,

$t \leq \phi(a)$. (7)

By (5), (6), and (7), we conclude that $\phi(a)$ is a lower bound of $\phi(R_1)$, and for any lower bound t of $\phi(R_1)$, we have that $t \leq \phi(a)$. Therefore, $\phi(a) = \text{glb}_{A2}(\phi(R_1)) = \inf_{A2}(\phi(R_1))$. That is, $\phi(\inf_{A1}(R_1)) = \inf_{A2}(\phi(R_1))$.

Theorem 3. *If a poset $(A,\leq)$ satisfies the minimum (maximum) condition, then for any x in A, there exists one element m of A such that $m \leq x$ ($x \leq m$).*

Proof. Suppose that A satisfies the minimum condition (DCC). Let $x \in A$. If x is minimal, then $x = m$. If x is not minimal, then there exists $x_1 \in A$ such that $x_1 \leq x$. If x_1 is not minimal, then there exists $x_2 \in A$ such that $x_2 \leq x_1 \leq x$. If we continue this process, then we have a descending sequence of elements of A. But by hypothesis, A satisfies the minimum condition (DCC). Therefore, the above process must be terminated at a certain stage, say x_r, and no element of A will be less than x_r. Hence, x_r is a minimal element of A. That is, $x_r \leq x$ for all x in A. Similarly, we can prove that if a poset A satisfies the maximum condition, then it has a maximal element.

Corollary 2. *Let $(A,\leq)$ be a chain. Then every subchain of $(A,\leq)$ satisfying the maximum (minimum) condition has a greatest (least) element.*

Proof. Suppose that a poset A satisfies the maximum condition. Then, every subset of A also satisfies the maximum condition. Since every subchain of a poset A is also a subset of A, every subchain of A satisfies the maximum condition. We know that a chain does not have more than one maximal element, and that maximal element is the greatest element of the chain. Therefore, every subchain of A satisfies the maximum condition. Every chain of the form $a_1 < a_2 < a_3 \ldots$ does not contain an infinite number of elements. Therefore, after a certain stage, that is, after a finite number of steps, we obtain a maximal element, which is also the greatest element. Similarly, we can prove that every subchain of a poset satisfying the minimum condition has a least element.

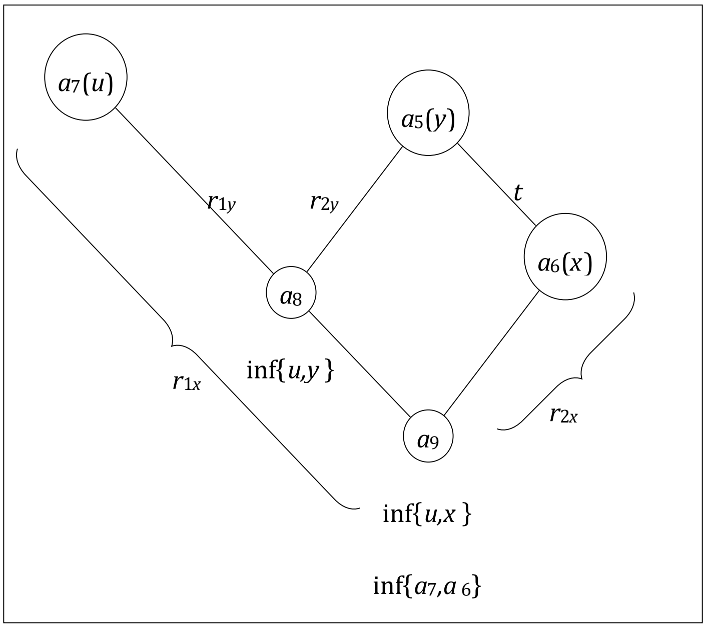
Theorem 4. *Let $(A,\leq)$ be a poset. Then $(A,\leq)$ can satisfy both the maximum and minimum conditions if and only if every one of its subchains is of finite length.*

Proof. Suppose A is a poset that satisfies both the maximum and minimum conditions. Let C be a subchain of A. We prove that C is of finite length. Since A satisfies the minimum condition, then by Corollary 2, C has a minimal element, say x_1, and hence x_1 is a least element of C. Define $C_1 = C \setminus \{x_1\}$. Then C_1 is again a subchain of A. Let x_2 be the least element of C_1. Therefore, we have $x_1 < x_2$. If we continue this process, we obtain an ascending sequence $x_1 < x_2 < x_3 \ldots$ of elements of C and hence elements of A. But A satisfies the maximum condition. Therefore, the above sequence must be finite. Therefore, C is finite. That is, every subchain of A is of finite length. Conversely, suppose that every subchain of a poset A is of finite length. We prove that A satisfies both the maximum and minimum conditions. Asuume to the contrary that A does not satisfy the minimum condition. Then there exists x_0 in A such that starting from x_0, we can obtain an infinite number of elements $x_1 > x_2 > x_3 \ldots$. Put $C = \{x_r\}_{r=0}^{\infty}$. Then clearly C is a subchain of A that is infinite, which contradicts the fact that every subchain of a poset is finite. Therefore, A satisfies the minimum condition. Similarly, we can prove that A satisfies the maximum condition.

In view of the pre-A* poset $A \times A$, with 9 elements defined as in Figure 2, we have the following theorem.

Theorem 5. *If a Pre-A* poset* $(A{\times}A = a_1, a_2, \ldots, a_9, \leq)$ *with least element of locally finite length satisfies JDCC, then it has a dimension function.*

Fig. 7: An illustration of Theorem 5.



Proof. Let $(A \times A = \{a_1, a_2, \ldots, a_9\}, \leq)$ be a poset of locally finite length satisfying

JDCC. If $A{\times}A$ contains an element u such that $\inf\{u,x\}$ exists for all $x \in A$, then a dimension function can be defined on A in the following way: For any element u, define

$$d(u) = d_0, \quad (8)$$
$$d(x) = d_0 - r_{1x} + r_{2x}, \quad (9)$$
$$d(y) = d_0 - r_{1y} + r_{2y}. \quad (10)$$

The lengths of maximal chains connecting $\inf\{u,x\}$ with u and x are r_{1x} and r_{2x} respectively. We have to show that d satisfies the following: $x \prec y$ if and only if $x \leq y$ and $d(x) + 1 = d(y)$ for all x,y in $A \times A$. Let $x,y \in A \times A$ such that $x \prec y$.

Therefore, $\inf\{u,x\} \leq \inf\{u,y\}$. There exists a maximal chain between $\inf\{u,x\}$ and u which includes $\inf\{u,y\}$. Then the length of the maximal chain between $\inf\{u,x\}$ and $\inf\{u,y\}$ is $r_{1x} - r_{1y}$ as illustrated in Figure 7. Let us denote the length of the maximal chain between x and y by t. By definition, we have:

$$d(y) = d_0 - r_{1y} + r_{2y}, \quad (11)$$
$$d(x) = d_0 - r_{1x} + r_{2x}. \quad (12)$$

Therefore,

$d(y) - d(x) = (r_{1x} - r_{1y}) + (r_{2y} - r_{2x})$. (13)

Since our poset satisfies JDCC, the lengths of all maximal chains between $\inf\{u,x\}$ and y are equal, that is,

$r1x - r1y + r2y = r2x + t.$ (14)

From Equations (13) and (14), we have

$d(y) - d(x) = t.$ (15)

Therefore, $x \prec y$ if and only if $x \leq y$ and there is no z such that $x \leq z \leq y$. This is equivalent to saying that $x \leq y$ and the lengths of maximal chains connecting x and y is 1. That is, $x \leq y$ and $t = 1$ and hence, $x \leq y$ and by Equation (15), we have that $d(y) = d(x) + 1$. Thus, $x \prec y$ if and only if $x \leq y$ and $d(x) + 1 = d(y)$ for all $x,y \in A$. This completes the proof.

Remark 3 (Graphical aspect of Figure 7)**.** *The graph has a chromatic number of* 2 *and is bipartite. Furthermore, the graph in Figure 7 is a subgraph of the bipartite graph* $G = A \times A$ *in Figure 3. Additionally, the graph in Figure 7 is a planar graph with* 2 *regions.*

4 TECHNICAL ASPECT

The significance of the crucial element 2 in a Pre-A* algebra is to extend the Boolean two valued logic to three valued logic. That is, if the Boolean elements 0 and 1 stand for false and true statements respectively, then 2 stands for divergence as neither true nor false. The best example of this is a traffic signal system the two basic signals (green light and red light) that can be further extended to another signal (different from green light and red light) depending on a particular situation. The similar logic can be observed in washing machines (Fuzzy logic control systems). The present work is an algebraic study of this logic.

5 APPLICATIONS

Our study on pre-A* posets directly links to graph theory. Given the applications of bipartite graphs in areas such as cancer detection, advertising, e-commerce rankings, prediction of preferences (such as food and movies), and matching problems (such as the stable marriage problem), this work has a lot of potential real-life applications. In fact, we have developed a new way to study graphs using pre-A* poset structures. The use of bipartite graphs to represent binary relations between disjoint sets invites applications of pre-A* posets to medicine and biology, such as bipartite life cycles and bipartite patella for a split kneecap.

6 CONCLUSIONS

We have proven that if a pre-A* poset *A* has least element of locally finite length and satisfies the Jordan Dedekind chain conditions, then *A* has a dimension function. We have also established the relationship between pre-A* algebras and graph theory, by giving an analysis of Hasse diagrams. The graph in Figure 7 is subgraph of the one in Figure 2 and both have chromatic number of 2, making them bipartite graphs. The graph in Figure 3 is a planar graph with 5 regions whereas its subgraph in Figure 7 is planar graph with 2 regions.

DECLARATIONS

- Funding: There was no funding.
- Conflict of interest : There is no conflict of interest.
- Consent for publication : All authors have consented.
- Availability of data and materials : No data was used.
- Code availability : No code was used.
- Authors' contributions: All the authors contributed to the work.

REFERENCES

1. Fernando, G., Craig, C.S.: The algebra of conditional logic. Algebra Universalis 27
2. Manes, E.G.: Ada and the equational theory of if-then-else. Algebra Universalis 30, 373–394 (1993)
3. Koteswara, R.P.: A*-Algebra, an If-Then-Else Structures (Doctoral Thesis). Nagarjuna University, A.P., India, (1994)
4. Venkateswara, R.J.: On A*-algebras (Doctoral Thesis). Nagarjuna University, A.P., India, (2000)
5. Kalyani, D., Rami, R.B., Venkateswara, R.J., Satyanarayana, A.: Characterization of a partial order relation on pre-a*- algebra. International Journal of Mathematical Archive 4(11), 87–296 (2013)
6. Kalyani, D., Rami, R.B., Venkateswara, R.J., Satyanarayana, A.: Representation of pre a*- algebra by a partially order. Journal of Scientific and Innovative Mathematical Research 1(3), 200–210 (2013)