Mathematical modeling of peristaltic blood flow through a vertical blood vessel using prandtl fluid model

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Abstract:

This paper deals with the non – Newtonian Prandtl fluid flow in a vertical tube. Blood flow is considered for the peristaltic blood flow. The study is motivated towards investigating the physiological flow of blood in the circulatory system. The velocity profile, flow rate, pressure gradient are investigated by using appropriate analytical and numerical method. The series solutions of axial velocity and pressure gradient are given by using regular perturbation technique when prandtl number is small. The computational results are present in graphical form. It is observe that, the pressure gradient increases with increasing values of ε, ϕ, and α. This paper also revealed that size of bolus reduces with increasing the amplitude ratio whereas it is unaltered with other parameters. Physiological implications of this mathematical and theoretical modeling to blood flow situations are also included in brief.

Key words: Peristaltic flow, Non-Newtonian fluid, Vertical tube, Velocity, Volumetric flow rate, Pressure rise.

Introduction:

A peristaltic flow has wide application in the field of medical and engineering sciences. Peristaltic flows are significant in both theoretical and industrial perspectives. Peristaltic flows occurs in many biological systems such as urine transport from kidney to bladder through ureter, flows of many other glandular ducts, and in the cardiovascular system peristaltic flow is useful to understand the circulation of blood in small blood vessels. Past many years researchers and scientist focused on the peristaltic transport of Newtonian and non – Newtonian through tubes/channels. In the small blood vessels peristaltic flow of prandtl fluid give better understanding in comparison to Newtonian fluid. In this series many researchers work on the peristaltic flow on non – Newtonian fluid in different type of small tubes, there are so many researchers who is develop a theory based on pulsatile peristaltic flow to understand the flow depend upon the time. [3]Dhermendra tripathi describe the peristaltic flow of visco elastic fluid flow of chyme in small intestine which is inclined cylindrical tube. [10]Maiti and Misra present a paper in which the peristaltic transport of a couple stress fluids in a porous channel is presented. The study is investigating the physiological flow of blood in the micro circulatory system, by taking account of the particle size effect. There are so many researchers who are investigated the magnetic effect of peristaltic transport of Newtonian and non – Newtonian fluid flows in the tube. In such a way [4]Shehawey Saleh and Husseney studied the peristaltic pumping by sinusoidal travelling wave in the porous walls of a two dimensional channel fluid under the effect of transverse magnetic field. [1]Abd-Alla, Yahya and Osaimi discussed the peristaltic flow of a micropolar fluid in a flexible tube with visco elastic under effect of magnetic field and rotation. [7]Suryanarayana Reddy also describe the non – linear peristaltic pumping of Johnson – Segalmam fluid in an asymmetric channel under effect of magnatic field. There are so many other research articles in which the peristaltic flow through a porous medium is studied. [6]Mekheimer work on the peristaltic transport with long wavelength approximation and low Reynolds number assumptions through a porous medium in an annulus filled with an incompressible viscous and Newtonion fluid. In the present article we use the peristaltic flow of a prendtl fluid in a vertical tube where prendtl no. is very small. The velocity profile, flow rate, pressure gradient are investigated by using appropriate analytical and numerical method. It is observe that, the pressure gradient increases with increasing values of ε, ϕ, and α. Also it is observed that, the pumping is more for prandtl fluid than that of Newtonian fluid.

Formulation of the problem:

In the present model we consider the peristaltic flow of viscous incompressible non – Newtonian fluid through a porous medium in a vertical tube. The flow is generated by sinusoidal wave propagating with constant speed C along with the outer wall of the tube. Blood flow is taken as Prandtl fluid model in a two dimensional channel of width 2a. The flow is induced by periodic wave of length λ and amplitude b, t is the time, λ is the wavelength (Fig.1) the equation of the wall is given by,

$$Y = \pm H(Z,t) = \pm a \pm b sin\frac{2\pi}{\lambda}(Z-ct) \qquad (1)$$

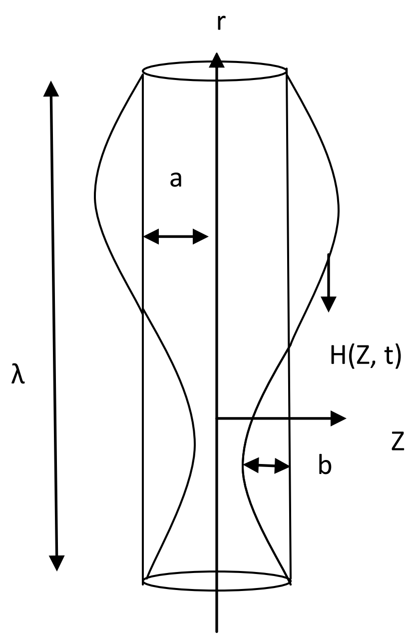

Fig. 1 The Physical Model

The flow is unsteady in the fixed frame (Z, r), However in the co-ordinate system moving with the propagation velocity c (wave frame (Z, r)), the boundary shape is stationary. The transformation from fixed frame to the wave frame is given by,

Z=Z – ct, r = R, w = W – c. (2)

The constitute equation for the Prandtl fluid is

$$\tau = \frac{Asin^{-1}\left(\frac{1}{C}\left[\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial r}\right)^2\right]^{1/2}\right)}{\left[\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial r}\right)^2\right]^{1/2}}\frac{\partial u}{\partial z} \qquad (3)$$

The equations governing the flow in the wave frame of reference are

$$\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{\partial w}{\partial z} = 0 \qquad (4)$$

$$\rho\left(u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial r}+\frac{1}{r}\frac{\partial}{\partial r}\left(\alpha\frac{\partial w}{\partial r}+\frac{\epsilon}{6}\left(\frac{\partial w}{\partial r}\right)^3\right)+\frac{\partial^2 u}{\partial z^2} \quad (5)$$

$$\rho\left(u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z}+\frac{1}{r}\frac{\partial}{\partial r}\left(\alpha\frac{\partial w}{\partial r}+\frac{\epsilon}{6}\left(\frac{\partial w}{\partial r}\right)^3\right)+\frac{\partial^2 w}{\partial z^2}+\rho g\alpha(T-T_0) \quad (6)$$

Introducing the non – dimensional variables defined by

$$\bar{z}=\frac{z}{\lambda},\bar{w}=\frac{w}{c},\bar{u}=\frac{u}{c\delta},\bar{r}=\frac{r}{a},\bar{P}=\frac{Pa^2}{\mu c\lambda},=\frac{T-T_0}{T_0}, \quad (7)$$

$$h=\frac{H(z)}{a}=1+\emptyset\sin(2\pi z),\emptyset=\frac{b}{a},Gr=\frac{g\alpha a^2 T_0}{v^2},\beta=\frac{a^2 Q_0}{kT_0}$$

Using (7), Equation (4 – 7) become

$$\frac{\partial P}{\partial r}=0 \quad (8)$$

$$\frac{\partial P}{\partial z}=\frac{1}{r}\frac{\partial \tau_{rz}}{\partial r}+Gr\theta \quad (9)$$

Where $\tau_{rz}=\alpha\frac{\partial w}{\partial r}+\frac{\epsilon}{6}\left(\frac{\partial w}{\partial r}\right)^3$ (10)

Using Eq. (9) and (10) we get

$$\frac{\partial P}{\partial z}=\frac{1}{r}\frac{\partial}{\partial r}\left(\alpha\frac{\partial w}{\partial r}+\frac{\epsilon}{6}\left(\frac{\partial w}{\partial r}\right)^3\right)+Gr\theta \quad (11)$$

$$\frac{\partial^2\theta}{\partial r^2}+\frac{1}{r}\frac{\partial\theta}{\partial r}+\beta=0 \quad (12)$$

The corresponding non – dimensional boundary conditions are

$\frac{\partial w}{\partial r}=0$ at r = 0 (13)

w = -1 at r = h (14)

$\frac{\partial\theta}{\partial r}=0$ at r = 0 (15)

θ = 0 at r = h (16)

The volume flow rate in the wave frame is

$$q=2\int_0^h wr\,dr \quad (17)$$

Instantaneous volume flow rate in the frame of reference is given by

$$Q(r,t)=2\int_0^h WR\,dr=2\int_0^h(w+1)rdr=q+h^2 \quad (18)$$

Time mean flow over a period T(=λ/c) of the peristaltic wave is defined as

$$\bar{Q}=\frac{1}{T}\int_0^1 Q(r,t)dt=q+\int_0^1 h^2\,dr=q+1+\frac{\emptyset^2}{2} \quad (19)$$

Solution of the problem:

Solving Eq. 12 using boundary condition 15 – 16, we get

$$\theta=\frac{\beta}{4}[h^2-r^2] \quad (20)$$

Considering the Womersley parameter to be very small, the velocity w, and pressure P, and flow rate q as

$$u=u_0+\epsilon u_1+o(\epsilon^2) \quad (21)$$

$$P=P_0+\epsilon P_1+o(\epsilon^2) \quad (22)$$

$$q = q_0 + \epsilon q_1 + o(\epsilon^2) \tag{23}$$

Substituting Eq. (20) – (23) into the Eq. (11) and solving Eq. (11) with the boundary conditions (13) – (14), and equating the coefficient of like powers of ϵ to 0 and neglecting the terms of $\epsilon^2$and higher order, we obtain,

$$w_0 = \frac{1}{6\alpha}(r^3 - h^3)\frac{\partial p_0}{\partial z} + \frac{G_r \beta_1}{4\alpha}\left[\frac{7h^5}{60} - \frac{h^2 r^3}{6} + \frac{r^5}{20}\right] - 1 \tag{24}$$

$$w_1 = \frac{r^3}{6\alpha}\left(\frac{\partial p_1}{\partial z}\right) - \frac{1}{2\alpha}\left[\frac{r^7}{168\alpha^3}\left(\frac{\partial p_0}{\partial z}\right)^3 + \left(\frac{G_r\beta_1}{4\alpha}\right)^3\left(\frac{h^6 r^7}{168} - \frac{r^{13}}{2496} + \frac{h^4 r^9}{144} - \frac{h^2 r^{11}}{352}\right) + \frac{{G_r}^2{\beta_1}^2}{32\alpha^3}\left(\frac{\partial p_0}{\partial z}\right)\left(\frac{h^4 r^7}{28} + \frac{r^{11}}{176} - \frac{h^2 r^9}{36}\right) + \frac{1}{16\alpha^3}G_r\beta_1\left(\frac{\partial p_0}{\partial z}\right)^2\left(\frac{h^2 r^8}{112} + \frac{r^{10}}{360}\right)\right] - \frac{h^3}{6\alpha}\left(\frac{\partial p_1}{\partial z}\right) - \frac{1}{2\alpha}\left[\frac{h^7}{168\alpha^3}\left(\frac{\partial p_0}{\partial z}\right)^3 + \left(\frac{G_r\beta_1}{4\alpha}\right)^3\left(\frac{38418h^{13}}{576576}\right) + \frac{{G_r}^2{\beta_1}^2}{32\alpha^3}\left(\frac{\partial p_0}{\partial z}\right)\left(\frac{151h^{11}}{110088}\right) + \frac{1}{16\alpha^3}G_r\beta_1\left(\frac{\partial p_0}{\partial z}\right)^2\left(\frac{31h^{10}}{5040}\right)\right] \tag{25}$$

The volume flow rate q_0 in the moving coordinate system is given by

$$q_0 = \frac{1}{6\alpha}\left(\frac{-3h^4}{4}\right)\frac{\partial p_0}{\partial z} + \frac{G_r\beta_1}{4\alpha}\left[\frac{7h^6}{60} - \frac{h^6}{24} + \frac{h^6}{120}\right] - h \tag{26}$$

$$q_1 = \frac{h^4}{24\alpha}\left(\frac{\partial p_1}{\partial z}\right) - \frac{1}{2\alpha}\left[\frac{h^8}{1344\alpha^3}\left(\frac{\partial p_0}{\partial z}\right)^3 + \left(\frac{G_r\beta_1}{4\alpha}\right)^3\left(\frac{h^{14}}{1344} - \frac{h^{14}}{34944} + \frac{h^{14}}{1440} - \frac{h^{14}}{4224}\right) + \frac{{G_r}^2{\beta_1}^2}{32\alpha^3}\left(\frac{\partial p_0}{\partial z}\right)\left(\frac{h^{12}}{224} + \frac{h^{12}}{2112} - \frac{h^{12}}{360}\right) + \frac{1}{16\alpha^3}G_r\beta_1\left(\frac{\partial p_0}{\partial z}\right)^2\left(\frac{h^{11}}{1008} + \frac{h^{11}}{3960}\right)\right] - \frac{h^4}{6\alpha}\left(\frac{\partial p_1}{\partial z}\right) - \frac{1}{2\alpha}\left[\frac{h^7}{168\alpha^3}\left(\frac{\partial p_0}{\partial z}\right)^3 + \left(\frac{G_r\beta_1}{4\alpha}\right)^3\left(\frac{38418h^{14}}{576576}\right) + \frac{{G_r}^2{\beta_1}^2}{32\alpha^3}\left(\frac{\partial p_0}{\partial z}\right)\left(\frac{151h^{12}}{110088}\right) + \frac{1}{16\alpha^3}G_r\beta_1\left(\frac{\partial p_0}{\partial z}\right)^2\left(\frac{31h^{10}}{5040}\right)\right] \tag{27}$$

From Eq. (26), (27) we have

$$\frac{\partial p_0}{\partial z} = \left[G_r\beta_1\frac{h^2}{6} - \frac{8q_0\alpha}{h^4} - \frac{8\alpha}{h^3}\right] \tag{28}$$

$$\frac{\partial p_1}{\partial z} = \frac{-8\alpha q_1}{h^4} - 4\left[\frac{h^4}{1344\alpha^3}\left(\frac{\partial p_0}{\partial z}\right)^3 + \left(\frac{10768h^{10}}{5765760}\right)\left(\frac{G_r\beta_1}{4\alpha}\right)^3 + \frac{{G_r}^2{\beta_1}^2}{32\alpha^3}\left(\frac{\partial p_0}{\partial z}\right)\left(\frac{479}{221760}h^8\right) + \frac{1}{16\alpha^3}G_r\beta_1\left(\frac{\partial p_0}{\partial z}\right)^2\left(\frac{69}{55440}h^7\right)\right] + 4\left[\frac{h^3}{168\alpha^3}\left(\frac{\partial p_0}{\partial z}\right)^3 + \left(\frac{38418h^{10}}{576576}\right)\left(\frac{G_r\beta_1}{4\alpha}\right)^3 + \frac{{G_r}^2{\beta_1}^2}{32\alpha^3}\left(\frac{\partial p_0}{\partial z}\right)\left(\frac{161}{110088}h^8\right) + \frac{1}{16\alpha^3}G_r\beta_1\left(\frac{\partial p_0}{\partial z}\right)^2\left(\frac{31}{5040}h^7\right)\right] \quad (29)$$

$$\frac{\partial p}{\partial z} = \frac{\partial p_0}{\partial z} + \epsilon\frac{\partial p_1}{\partial z} \quad (30)$$

From Eq. (28), (29) putting the values in (30),

$$\frac{\partial p}{\partial z} = \frac{-8\alpha q}{h^4} + \frac{G_r\beta_1 h^2}{6} - \frac{8\alpha}{h^3} - 4\epsilon\left[\frac{h^4}{1344\alpha^3}\left(\frac{\partial p_0}{\partial z}\right)^3 + \left(\frac{10768h^{10}}{5765760}\right)\left(\frac{G_r\beta_1}{4\alpha}\right)^3 + \frac{{G_r}^2{\beta_1}^2}{32\alpha^3}\left(\frac{\partial p_0}{\partial z}\right)\left(\frac{479}{221760}h^8\right) + \frac{1}{16\alpha^3}G_r\beta_1\left(\frac{\partial p_0}{\partial z}\right)^2\left(\frac{69}{55440}h^7\right)\right] + 4\epsilon\left[\frac{h^3}{168\alpha^3}\left(\frac{\partial p_0}{\partial z}\right)^3 + \left(\frac{38418h^{10}}{576576}\right)\left(\frac{G_r\beta_1}{4\alpha}\right)^3 + \frac{{G_r}^2{\beta_1}^2}{32\alpha^3}\left(\frac{\partial p_0}{\partial z}\right)\left(\frac{161}{110088}h^8\right) + \frac{1}{16\alpha^3}G_r\beta_1\left(\frac{\partial p_0}{\partial z}\right)^2\left(\frac{31}{5040}h^7\right)\right] \quad (31)$$

Numerical Results and discussion:

In order to see the quantitative effect to various emerging parameters involving in the results on the pumping characteristics we use MATLAB package. Fig. 2 shows that the variation of velocity with respect to the radius for different values of ε. It shows that the velocity increases for increasing values of radius, and decreases for increasing values of ε. This was also reported by [2]Duarte et al. Fig. 3 shows that the variation of velocity with respect to the radius for different values of G_r. It shows that the velocity increases for the increasing values of radius r. And it is also seems that the velocity is increases for the increasing values of G_r. This similar result was pointed out [12]Srivastava.

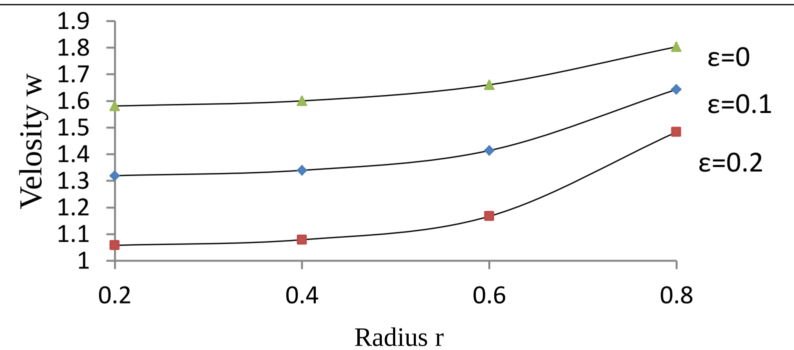

Fig. 2 Variation of velosity with radius for different values of ε

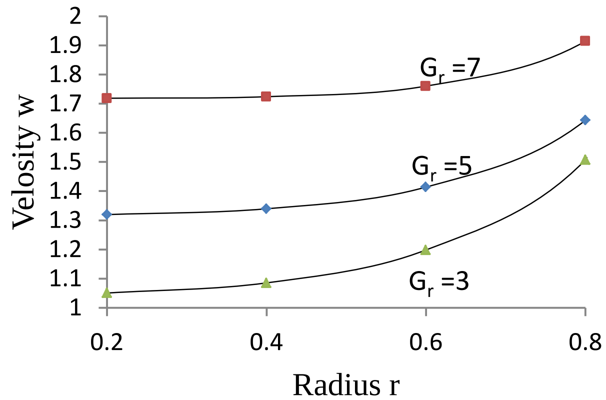

Fig. 3 Variation of velosity with radius for different values of G_r

Fig. 4 depicts the variation of pressure gradient with volume flow rate for different values of ε. Fig. gives that the pressure gradient decreases when the flow rate is increases. It is also shown in the figure that the pressure gradient is increases for the increasing values of ε. The present results are therefore consistent with the observation of [9]Hemadri Reddy.

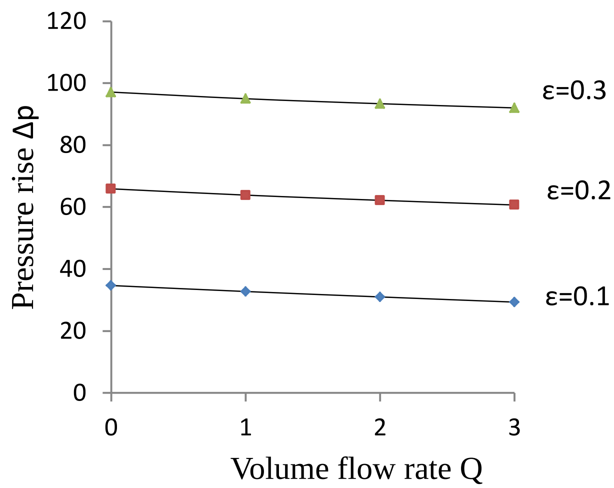

Fig. 4 Variation of volume flow rate with pressure rise for different ε

Fig. 5 gives the variation of pressure gradient with respect to volume flow rate for different values of Grashof number G_r. from the figure it can be shown that the pressure gradient is decreases for the increasing values of flow rate. It can be also shown that the pressure gradient is increases for the increasing values of G_r. This result is consistent with result of [11]Jothi.

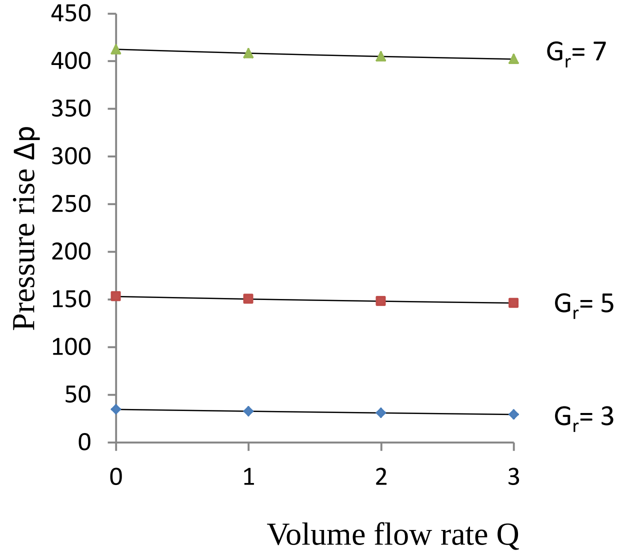

Fig. 5 Variation of Volume flow rate with pressure rise for different G_r

Fig. 6 – 7 gives the results for variation of pressure gradient with volume flow rate. Fig. 6 shows that the pressure gradient is decreases for the increasing values of volume flow rate, and increases for the increasing values of the amplitude ratio φ. The present results are therefore consistent with the observation of [8]Gangavathi. Fig. 7 shows the pressure gradient is decreases for the increasing values of volume flow rate, and increases for the increasing values of the α. The result is consisting with the result of [5]Lukashev and [13]Wang.

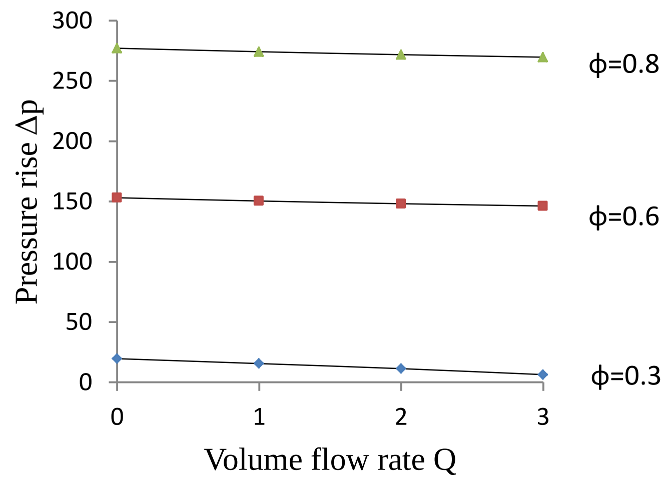

Fig. 6 Variation of Volume flow rate with pressure rise for different ϕ

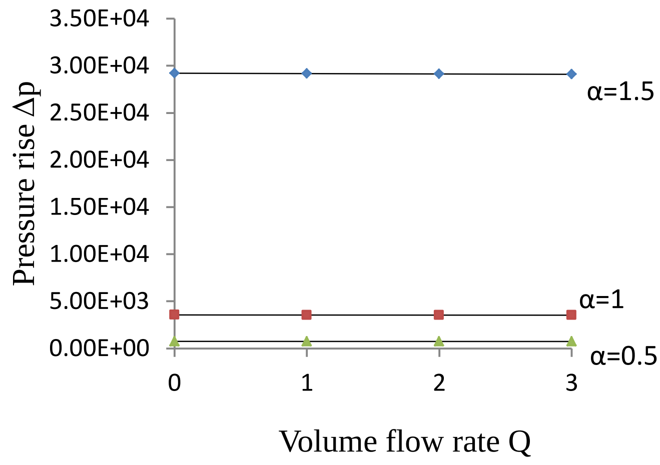

Fig. 7 Variation of Volume flow rate with pressure rise for different α

Conclusion:

This paper presents the peristaltic blood flow of prandtl fluid flows in the vertical artery. Series solution of axial velocity and pressure gradient are given by using regular perturbation technique when prandtl number is small. It is observe that, the pressure gradient increases with increasing values of ε, ϕ, and α. Also it is observed that, the pumping is more for prandtl fluid than that of Newtonian fluid.

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