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Characterization of $PSU(3, q)$ by its order and one special conjugacy class size

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ARTICLE INFO	ABSTRACT
Published Online: **19 April 2024** Corresponding Author: **Soleyman Askary**	Suppose that G be a finite group, and let $N(G)$ be the set of conjugacy class sizes of G. By Thompson's conjecture, if H is a finite non abelian simple group, G is a finite group with a trivial center, and $N(G) = N(H)$, then H and G are isomorphic. Chen et al. contributed interestingly to Thompsons conjecture under a weak condition. In this article, we investigate validity ofThompsons conjecture under a weak condition for the projective special unitary groups. This work implies that Thompsons conjecture holds for the $PSU(3, q)$, where q is prime power. **2020 Mathematics Subject Classification:** *20D08, 20D60.*

KEYWORDS: Conjugacy class size, Thompson's conjecture

INTRODUCTION

All groups considered in this paper are finite, and simple groups are non abelian. For convenience, we use g^G and $|g^G|$ to denote the conjugacy class of G containing g and the size of g^G, respectively. Denote by $N(G)=\{|g^G|: g\in G\}$. Suppose that $\pi(G)$ denote the set of primes dividing the order of G. For a group G, we construct the prime graph of G which is denoted by $\Gamma(G)$ as follows: the vertex set is $\pi(G)$ and two distinct primes p and q are joined by an edge if and only if G contains an element of order pq. Let $t(G)$ be the number of connected components of $\Gamma(G)$ and let $\pi_1, \pi_2, ..., \pi_{t(G)}$ be the connected components of $\Gamma(G)$. If $2\in\pi(G)$, then we always suppose that $2\in\pi_1(G)$. $|G|$ can be expressedas a product of co-prime positive integers $OC_i, i=1,2,...,t(G)$, where $\pi(OC_i)=\pi_i$. These OC_i's are called the order components of G and the set of order components of G will be denoted by $OC(G)$. Also wecall $OC_2,...,OC_{t(G)}$ the odd order components of G. Let n be a positive integer and p be a prime number. Then $|n|_p$ denotes the p - part of n.

In 1987, John Thompson posted the following conjecture concerning $N(G)$.

Thompson's conjecture (See [16], Question 12.38). Let G be a group with trivial central. If H is a simple group satisfying $N(G)=N(H)$, then $G\cong H$.

In [8], [9], Thompson's conjecture is verified for a few finite simple groups. In [11], Chen contributed to Thompson's conjecture under a week condition. The only used order and one or two special conjugacy class sizes of simple groups and characterized successfully sporadic simple groups, A_{10}, $PSL(4, 4)$, $PSL(2, p)$, $PSL(n, 2)$, ${}^2D_n(2)$, ${}^2D_{n+1}(2)$, $C_n(2)$, alternating group of degree p, $p + 1$, $p + 2$ and symmetric group of degree p, where p is prime number.

In this paper, we are going to characterize the projective special unitary group $PSU(3, q)$ by its order and one special conjugacy class length,where $q > 5$ is a prime power.

According to the classification theorem of finite simple groups and [12], [15], [19], we can list the order components of finite simple groups with disconnected prime graphs as in Tables 1-4 in [9]. All further unexplainednotation is standard and we refer to [12], for example.

1. **First section**

Definition 1. A Frobenius group is a transitive permutation group in which the stabilizer of any two points is trivial.

Definition 2. A group G is a 2-Frobenius group if there exists a normal series 1⊴ H⊴ K⊴ G such that K and G/H are Frobenius groups with kernels H and K/H, respectively.

Lemma 1. [7] Let G be a Frobenius group of even order with kernel K and complement H. Then t(G) = 2, the prime graph components of G are π(H) and π(K) and the following assertions hold:

(1) K is nilpotent;

(2) |K| ≡ 1 (mod |H|).

Lemma 2. [7] Let G be a 2-Frobenius group, i.e., G is a finite group andhas a normal series 1⊴ H⊴ K⊴ G such that K and G/H are Frobenius groups with kernels H and K/H, respectively. Then:

(a) t(G) = 2, $\pi_1 = \pi(G/K) \cup \pi(H)$ and $\pi_2 = \pi(K/H)$;

(b) G/K and K/H are cyclic, $|G/K| \,|\, |K/H-1|$ and $G/K \le Aut(K/H)$.

Lemma 3. [19] If G is a finite group such that t(G) ≥ 2, then G hasone of the following structures:

(a) G is a Frobenius group or 2-Frobenius group;

G has a normal series 1⊴ H⊴ K⊴ G such that π(H)∪ π(G/K) ⊆ π_1 and K/H is a nonabelian simple group. In particular, H is nilpotent, G/K ≲ Out(K/H) and the odd order components of G are the odd order components of K/H.

Lemma 4. [17] If n ≥ 6 is a natural number, then there are at least s(n)

prime numbers p_i such that $(n + 1)/2 < p_i < n$. Here

s(n) = 1, for 6 ≤ n≤ 13;

s(n) = 2, for 14 ≤ n ≤ 17;

s(n) = 3, for 18 ≤ n ≤ 37;

s(n) = 4, for 38 ≤ n ≤ 41;

s(n) = 5, for 42 ≤ n ≤ 47;

s(n) = 6, for n ≥ 48.

Lemma 5. [14]] Let M = PSU(3,q), where q > 5 and $OC_2 = \frac{q^2-q+1}{d}$ with d = (3,q+ 1).

(i) If p∈π_1(M) and p^α| |M|, then $p^\alpha-1$ not congruent to 0 modulo OC_2.

(ii) If p∈π_1(M), p^α| |M|, then p^α+ 1≡ 0 (mod (OC_2)) if and only if $p^\alpha = q^3$ or q = 7 and $p^\alpha = 2^7$, for every positive integer α.

(iii) If d = 3 and (OC_2-1) | |M|, then q = 8.

2. Second section

Main Theorem Let G be a group and $q = p^\alpha$ is prime power. Then$G \cong PSU(3,q)$ if and only if |G| = |PSU(3, q)| and G has one conjugacy class size |PSU(3, q)| / r, where r= $\frac{q^2-q+1}{(3,q+1)}$ be a prime number.

Proof. By [13], $PSU(3, q)$ has one conjugacy class size $|GU(3, q)|/q^3 + 1$.Since the necessity of the theorem is easy, we only need to prove the sufficiency.

By hypothesis, there exists an element g of order r in G such that

$C_G(g) = \langle g \rangle$ and $C_G(g)$ is a Sylow r-subgroup of G. By The Sylow theorem, we have that $C_G(h) = \langle h \rangle$ for any element h in G of order r.So, $\{r\}$ is a prime graph component of G and $t(G) \ge 2$. Therefore, r is the maximal prime divisor of $|G|$ and an odd order component of G.

Now, if t(G)=2, then $OC(G) = OC(PSU(3, q))$. By [14],

$G \cong PSU(3,q)$.

If $t(G) \ge 3$, then we will show that there is no such group. Since $t(G) \ge 3$, Lemma 1, and 2 show that G is neither a Frobenius group nor a 2- Frobenius group. By Lemma 3, G has normal series1⊴ H⊴ K⊴ G such that K/H is a non abelian simple group and r is an odd order component of K/H. Moreover, $t(K/H) \ge 3$.

According to the classification theorem of finite simple groups and the results in Tables 1-4 in [9], K/H is an alternating group, sporadic groupor simple group of Lie type, which consider in the following.

Step 1. *K/H* is not an sporadic simple group.

Proof. Suppose that *K/H* is an sporadic simple group. Since $q > 5$,

$r \ge 19$ and hence

$r \in \{19, 23, 29, 31, 37, 41, 43, 47, 59, 67, 71\}$.

Now, if $r = 19$ and $d = 3$, then $q = 8$, $|PSU(3, 8)| = 2^9.3^4.7.19$ and $K/H \in \{J_1, J_3, On, TH, BN\}$, so $5 \mid |K/H|$, which is a contradiction. If$d = 1$, then $q(q-1) = 18$, which is impossible. By the same method, we can consider the other possibilities for r.

Step 2. *K/H* can not be an alternating group A_m, where $m \ge 5$.

Proof. If $K/H = A_m$, then since $r \in \pi(K/H)$, $m \ge r$. Also, since$q > 5$ is a prime power, $r \ge 19$. Thus by Lemma 4, there exists a prime number $s \in \pi(A_m)$ such that $(r + 1)/2 < s < r$. Also, since $|K/H| \mid |G|, s \mid r.q^3(q + 1)^2(q-1)$, which is impossible.

Step 3. *K/H* is not a simple group of Lie type.

Proof. Assume that *K/H* is isomorphic to one of the finite simple groups:

Case 1. Let t(K/H) = 3. Then r ∈ {$OC_2(K/H)$, $OC_3(K/H)$}:

1.1. If $K/H \cong A_1(q')$, where 4 | q′, then the odd order components of K/H are q′ + 1 and q′ − 1. If q′ + 1 = r, then $q' = r - 1 = \frac{q^2-q+1}{d} - 1$ and hence, either $q' = q(q-1)$ or $q' = \frac{(q+1)(q-2)}{3}$, which are impossible. If $q'-1 = r$, then by Lemma 5, we get a contradiction.

1.2. If $K/H \cong A_1(q')$, where $4 \mid q'+1$, then $q' = r$ or $\frac{q'-1}{2} = r$. Now, if $\frac{q'-1}{2} = r$, then $q'-1 \equiv 0 (\mathrm{mod}\, r)$, which is a contradiction by Lemma 5 (i). If $q' = r$ and $d = 1$, then $q' = q^2 - q + 1$ and hence,

$$|K/H| = q(q^2-q+1)(q^2-q+2)(q-1)/2.$$

Since |K/H| | |G| and $\frac{q^2-q+2}{2}$ not divides |G|, we get a contradiction. If $d = 3$, then $(r-1) \,||\, |G|$ and hence, by Lemma 5(iii), we must have q = 8 and r =19. Hence q′=19. But 5 | |PSL(2,19)| and 5 not divides |PSU(3,8)|, which is a contradiction.

1.3. If $K/H \cong A_1(q')$, where $4 \mid q'-1$, then $q' = r$ or $\frac{q'+1}{2} = r$. Since the possibility $q' = r$ is discussed in former case, we suppose that $\frac{q'+1}{2} = r$. Then $q'+1 \equiv 0 (\mathrm{mod}\, r)$, By Lemma 5(ii), we can conclude that $q = 7$ or $q' = q^3$. Now, if $q = 7$, then $q' = 2^7$. Since 4 not divides $2^7 - 1$, we get a contradiction. If $q' = q^3$, then $|K/H| = |PSL(2,q')| = q^3(q^3-1)(q^3+1)/2$.

On the other hand, |K/H| | |G|, thus q^2+q+1 must be divide $(q+1)^2$, which is a contradiction.

1.4. If $K/H \cong^2 G_2(q')$ where $q' = 3^{2t+1} > 3$, then $q' - \sqrt{3q'} + 1 = \frac{q^2-q+1}{d}$ or $q' + \sqrt{3q'} + 1 = \frac{q^2-q+1}{d}$. Let $(3,q) = 1$. Assume that $q' - \sqrt{3q'} + 1 = \frac{q^2-q+1}{d}$ and $d = 1$, thus $3^{t+1}(3^t+1) = q(q-1)$. Now, since $(3,q) = 1$, 3 dos not divide q and hence $|q-1|_3 = 3^{t+1}$. Thus $|G|_3 = |q-1|_3 = 3^{t+1}$. On the other hand, $3^{3(2t+1)} = |K/H|_3 \le |G|_3 = 3^{t+1}$, which is a contradiction.

If $d = 3$, then $q' + \sqrt{3q'} + 1 = \frac{q^2-q+1}{3}$ and hence, $3^{t+2}(3^t+1) = (q+1)(q-2)$. Thus either $3^{t+1} \mid (q+1)$ and $(q-2) \mid 3(3^t+1)$ or $3^{t+1} \mid (q-2)$ and $(q+1) \mid 3(3^t+1)$. This forces $(q+1) = 3^{t+1}$ and $(q-2) = 3(3^t+1)$. This guarantees that $|G|_3 = 3^{2t+2}$. Also, $3^{3(2t+1)} = |K/H|_3 \le |G|_3 = 3^{2t+2}$, which is a contradiction. If $q' - \sqrt{3q'} + 1 = \frac{q^2-q+1}{d}$, then similar to the above we get a contradiction. Assume that $(3,q) \neq 1$. So $d = 1$ and $q' \pm \sqrt{3q'} + 1 = q^2 + q + 1$, this forces $q = 3^{t+1}$ and $q - 1 = 3^t \pm 1$, which is a contradiction.

1.5. If $K/H \cong^2 D_s(3)$, where $s = 2^t + 1 \ge 5$, then $\frac{3^s+1}{4} = \frac{q^2-q+1}{d}$ or $\frac{3^{s-1}+1}{2} = \frac{q^2-q+1}{d}$. If $\frac{3^s+1}{4} = r$, then

$$3^{s(s-1)}(3^{s-1}-1)(3^{s-1}+1)\prod_{i=1}^{s-2}(3^{2i}-1) \mid q^3(q+1)^2(q-1)$$

On the other hand, $r^6 = \frac{(3^s+1)^6}{4096} \le 3^{6s}$ and

$$3^{2s(s-1)-s} < 3^{s(s-1)}(3^{s-1}-1)(3^{s-1}+1)\prod_{i=1}^{s-2}(3^{2i}-1) \mid q^3(q+1)^2(q-1)$$

, which implies that $2s(s-1) < 7s$ and hence, $s < 5$, which is a contradiction. If $\frac{3^{s-1}+1}{2} = \frac{q^2-q+1}{d}$, then similar to the above, we get a contradiction.

1.6. If $K/H \cong^2 D_{s+1}(2)$, where $s = 2^n - 1$ and $n \ge 2$, then $2^s + 1 = \frac{q^2-q+1}{d}$ or $2^{s+1} + 1 = \frac{q^2-q+1}{d}$. If $2^s + 1 = r$, then $2^s = q(q-1)$ or $2^s = \frac{(q+1)(q-2)}{3}$, which is impossible. The same reasoning rules out the case when $2^{s+1} + 1 = r$.

1.7. If $K/H \cong F_4(q')$, where q' is even, then $q'^4+1=\frac{q^2-q+1}{d}$ or $q'^4-q'^2+1=\frac{q^2-q+1}{d}$.

If $q'^4+1=r$, then $q'^{24}(q'^6-1)^2(q'^4-1)^2(q'^4-q'^2+1) \mid q^3(q+1)^2(q-1)$. Also, $r^6=(q'^4+1)^6 < (q'^5)^6 = q'^{30}$ and $q'^{24}(q'^6-1)^2(q'^4-1)^2(q'^4-q'^2+1) \leq q^3(q+1)^2(q-1) < r^6 < q'^{30}$, whichis a contradiction. If $q'^4-q'^2+1=r$, then similar to the above, we get a contradiction. By the same method, we can prove that K/H cannot be a simple group $F_4(q')$, where q' is odd.

1.8. If $K/H \cong E_7(2)$, then $r \in \{73, 127\}$. Therefore, $r=73$ and $q=9$.

So $|PSU(3,9)| = 2^{10}.3^9.5^2.73$. On the other hand, $13 \mid |E_7(2)|$, which isa contradiction. By the same method, we can prove that K/H cannot be a simple group $E_7(3)$.

1.9. If $K/H \cong A_2(2), A_2(4)$ or ${}^2A_5(2)$, then since $q>5$ is a prime power,

we get a contradiction.

1.10. If $K/H \cong {}^2F_4(q')$, where $q'=2^{2t+1} \geq 2$, then $r=q'^2 \pm \sqrt{2q'^3}+q' \pm \sqrt{2q'}+1$. In both cases, we can see at once that $|K/H| > |G|$, which

is a contradiction.

Case 2. Let $t(K/H) \in \{4,5\}$. Then

r ∈ {OC₂(K/H), OC₃(K/H), OC₄(K/H), OC₅(K/H)},

as follows:

2.1. If $K/H \cong {}^2B_2(q')$, where $q'=2^{2t+1}$ and $t \geq 1$, then $r \in \{q'-1, q' \pm \sqrt{2q'}+1\}$. If $q'-1=r$, then by Lemma 5(i), we get a contradiction. Assume that $q'-\sqrt{2q'}+1=r$. Then $q'^2-1 \equiv 0 \pmod r$. Therefore, $q'^2=2^7$ or $q'^2=q^3$. Now, since 2^7 is not a square, we get a contradiction. If $q'^2=q^3$, then q is even. Let $d=1$. Thus $2^{2t+1}-2^{t+1}=q(q-1)$ and since $(q,q-1)=1$, we conclude that $q=2^{t+1}$ and $r<q'^9$, which is a contradiction. . If $d=3$ and we assume that $q=2^m$, then $2(3.2^{2t}-3.2^t+1)=2^m(2^m-1)$, which implies that $m=1$. But $q>5$ and hence, we get a contradiction. For $q'+\sqrt{2q'}+1=r$, similar to the above we get a contradiction.

2.2. If $K/H \cong A_2(4)$, then $\frac{q^2-q+1}{d} \in \{5,7,9\}$. Since $q>5$ is a prime power, we get a contradiction.

2.3. If $K/H \cong {}^2E_6(2)$, then $\frac{q^2-q+1}{d} \in \{13,17,19\}$.

Let $\frac{q^2-q+1}{d}=19$ and $d=3$, then $q(q-1)=56$, $q=8$ and $|PSU(3,8)| = 2^9.3^4.7.19$. On the other hand $13 \mid |{}^2E_6(2)|$, which is a contradiction. For $r \in \{13, 17\}$, similar to the above we get a contradiction.

2.4. If $K/H \cong E_8(q')$, then

$r \in \{q'^8-q'^7+q'^5-q'^4+q'^3-q'+1, q'^8+q'^7-q'^5-q'^4-q'^3+$ $q'^8-q'^6+q'^4-q'^2+1, q'^8-q'^4+1\}$.

If $q'^8-q'^7+q'^5-q'^4+q'^3-q'+1=r$, then $r<q'^9$. On the other hand, $r^6<q'^{54}$ and $|G|<r^6$. Since $q'^{120} \mid |K/H|$ and $|K/H| \mid |G|$, we get a contradiction. For other cases, similarly we get a contradiction.

3. Third section

Corollary 1. Let $q=p^\alpha$ be a prime power. Then Thompson's conjecture holds for the simple groups $PSU(3,q)$, where $\frac{q^2-q+1}{(3,q+1)}$ is prime number.

Proof. Let G be a group with trivial center and $N(G)=N(PSU(3,q))$. Then it is proved in [6], that $|G|=|PSU(3,q)|$. Hence, the corollary follows from the main theorem.

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