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Kiefer Bound for Doubly Truncated Distributions

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ABSTRACT

We consider uniform density on $(C_1(\theta), C_2(\theta))$. Identifying suitable prior densities we compute Kiefer bound on variance of unbiased estimator of the parametric function $\varphi = C_2(\theta) - C_1(\theta)$. Doubly censored sampling is taken into consideration. Further, the bounds are shown to be attained by variances of estimators based on the samples considered. Results are illustrated through examples. The bounds based on complete and censored samples are compared.

Key words: censored samples; ideal estimation equation; Kiefer bound; minimum; variance unbiased estimator; parametric function; truncated distribution; variance bound.

1. INTRODUCTION

Let $f(x;\theta)$ be the probability density function (pdf) of random variable(r.v.) $X \in \mathfrak{X}_{\boldsymbol{\theta}}$ .For each $\theta \in \Theta, \Theta$:the parameter space being an interval of the real line, let $\boldsymbol{\Theta}_{\boldsymbol{\theta}} = \{h;\ \theta + h \in \Theta\}$. For a fixed$\theta$, let G_1 and G_2 be any two probability measures $\in$ G defined on $\boldsymbol{\Theta}_{\boldsymbol{\theta}}$ such that $E_i(h)$ = expectation of h w.r.t. G_i, exists for i=1,2. Then,

$V_G = \frac{\int_{\Theta_\theta} f(x;\theta+h)d[G_1(h)-G_2(h)]}{f(x;\theta)}$ gives, for the variance of estimator T of θ, that

$$Var(T) \geq \sup_{G_1,G_2 \in G} K(G_1(h), G_2(h), \theta) = K(\theta) \tag{1.1}$$

where, $K(G_1, G_2, \theta) = \dfrac{\{E_1(h)-E_2(h)\}^2}{\int_{\mathfrak{X}} \left[\frac{\left\{\int_{\Theta_\theta} f(x;\theta+h)d[G_1(h)-G_2(h)]\right\}^2}{f(x;\theta)} \right] d\mu(x)}$.

The quantity $K(\theta)$ is called Kiefer bound on the variance of estimator of θ. It is introduced by Kiefer(1952). Kiefer bound is the sharpest lower bound of Cramér- Rao type. If it is computed successfully, it provides the value of variance of UMVU Estimator directly. Getting proper prior probability distributions G_1, G_2 is the most important part in computation of Kiefer bound. Kiefer computed it for variances of unbiased estimators of θ in uniform distribution on $(0,\theta)$ and exponential distribution with location parameter θ. Bartlett (1982) computed it for some more distributions. Bartlett has shown that defining

$$\Delta_1 f(x;\theta) = \int_{\Theta_\theta} f(x;\theta+h)G_1(h) - f(x;\theta)\ , \qquad (1.2)$$

if there exists an estimator T and probability distribution $G_1(h)$ such that an 'ideal estimation' equation

$$\frac{\Delta_1 f(t;\theta)}{f(t;\theta)\Delta_1\theta} = \frac{1}{Var(T)}(t-\theta) \qquad (1.3)$$

exists, then T is uniformly minimum variance unbiased estimator of θ with $Var(T) = K(\theta)$. We write this as T is UMVUKBE of θ. But it has not been possible to compute Kiefer bound in many situations. Jadhav, D.B. and Shanubhogue, A. (2014) , Shanubhogue, A. and Jadhav, D.B. (2014)a, (2014)b computed it for estimators of θ and functions of θ for left and right truncated distributions for complete and censored samples. In this paper we compute Kiefer bound on the variance of unbiased estimators of parameter or functions of the parameter when both the ends of the support of the distribution depend on the parameter. Let the support be $(C_1(\theta), C_2(\theta))$, $C_1(\theta) < C_2(\theta)$. The probability distributions on this support may or may not admit a single sufficient statistic. In Section 2 we consider the case when there is no single sufficient statistic. Here, $Z = X_{(1)}$ and $Y = X_{(n)}$ are jointly sufficient for θ. Based on Z, Y we deal with the estimation of $\varphi = C_2(\theta) - C_1(\theta)$ and provide attainable Kiefer bound on variance of its estimator. In the following sections we consider the situations admitting single sufficient statistic and prove that this statistic provides UMVUKBE of φ. In Section 3 we consider $C_1(\theta)$ to be monotone decreasing and $C_2(\theta)$ to be monotone increasing. In section 4 we consider $C_1(\theta)$ to be monotone increasing and $C_2(\theta)$ to be monotone decreasing. In both the cases single sufficient statistics provide UMVUKBEs of φ. These results are illustrated by suitable examples.

Let the probability density function of the random variable be

$$f(x;\theta) = \frac{1}{C_2(\theta) - C_1(\theta)}\ , C_1(\theta) < x < C_2(\theta) \qquad (1.4)$$

Then the distribution function is given by

$$F(x) = \int_{C_1(\theta)}^{x} \frac{1}{C_2(\theta)-C_1(\theta)}du$$

$$= \frac{x - C_1(\theta)}{C_2(\theta)-C_1(\theta)}$$

and the survival function is given by

$$\bar{F}(t) = P(X > t)$$

$$= \frac{C_2(\theta) - t}{C_2(\theta)-C_1(\theta)}\ .$$

2. ESTIMATION WHEN $(X_{(1)}, X_{(n)})$ IS JOINTLY SUFFICIENT

Let $X_1, X_2, \ldots, X_n; n \geq 1$ be independent observations from (1.4). Clearly, the minimum $Z = X_{(1)}$ and maximum $Y = X_{(n)}$ are jointly sufficient for θ. Now, based on Z and Y, we have the following

Theorem 2.1

If r.v. X has the pdf (1.4) and $\varphi = C_2(\theta) - C_1(\theta)$. Then, φ has UMVUKBE $\hat{\varphi} = \left(\frac{n+1}{n-1}\right)\left(X_{(n)} - X_{(1)}\right)$, with Kiefer bound $K(\varphi) = \frac{2\varphi^2}{(n-1)(n+2)}$.

Proof

The joint pdf of (Z,Y) where $Z = X_{(1)}$ and $Y = X_{(n)}$ is

$$f\big(z,y;C_1(\theta),C_2(\theta)\big) = \frac{n(n-1)(y-z)^{n-2}}{[C_2(\theta)-C_1(\theta)]^n},\ C_1(\theta) < z < y < C_2(\theta) \tag{2.1}$$

The pdf of $U = Y - Z$ is

$$g(u;\varphi) = \frac{n(n-1)u^{n-2}(\varphi-u)}{\varphi^n}, \qquad 0 < u < \varphi \tag{2.2}$$

where $\varphi = C_2(\theta) - C_1(\theta)$.

The set of possible values of φ is $\Phi = (0,\infty)$. $\Phi_\varphi = \{h; (\varphi+h) \in \Phi\} = (-\varphi,\infty)$. On $(-\varphi,0)$, let us define the prior probability distribution

$$dG_1(h) = \frac{(n+1)(\varphi+h)^n}{\varphi^{n+1}}, -\varphi < h < 0$$

Then, it can be shown that $E_1(h) = \frac{-\varphi}{n+2} = \Delta_1\varphi$.

If φ is incremented to $\varphi + h$,we have, $0 < u < \varphi + h$. Therefore, $u - \varphi < h < 0$ and then we have

$$\begin{aligned}
\Delta_1 g(u;\varphi) &= \int_{u-\varphi}^{0} g(u;\varphi+h)\,dG_1(h) - g(u;\varphi)\\
&= \int_{u-\varphi}^{0} \frac{n(n-1)u^{n-2}(\varphi+h-u)(n+1)(\varphi+h)^n dh}{(\varphi+h)^n\varphi^{n+1}} - \frac{n(n-1)u^{n-2}(\varphi-u)}{\varphi^n}.\\
&= \frac{n(n-1)(n+1)u^{n-2}}{\varphi^{n+1}}\int_{u-\varphi}^{0}(\varphi+h-u)dh - \frac{n(n-1)u^{n-2}(\varphi-u)}{\varphi^n}\\
&= \frac{n(n-1)(n+1)u^{n-2}(\varphi-u)^2}{\varphi^{n+1}\,2} - \frac{n(n-1)u^{n-2}(\varphi-u)}{\varphi^n}\\
&= \frac{n(n-1)u^{n-2}(\varphi-u)}{2\varphi^{n+1}}[(n+1)(\varphi-u)-2\varphi]
\end{aligned}$$

$$\frac{\Delta_1 g(u;\varphi)}{g(u;\varphi)\Delta_1\varphi} = \frac{(n+2)}{2\varphi^2}[(n+1)u-(n-1)\varphi]$$

$$= \frac{(n-1)(n+2)}{2\varphi^2}\left[\left(\frac{n+1}{n-1}\right)u - \varphi\right] \tag{2.3}$$

From (1.3) and (2.3), it is clear that φ has UMVUKBE $\hat{\varphi} = \left(\frac{n+1}{n-1}\right)U$ with Kiefer bound

$$K(\varphi) = \frac{2\varphi^2}{(n-1)(n+2)}$$

Example 2.1

Consider the density,

$$f(x;\theta)=\frac{1}{\theta-\theta^2}, \qquad \theta^2<x<\theta, 0<\theta<1.$$

Both θ^2 and θ are monotone increasing functions of θ.Therefore, here there is no single sufficient statistic. $Z=X_{(1)}$ and $Y=X_{(n)}$ are jointly sufficient for θ. The probability density function of range $U=X_{(n)}-X_{(1)}$ is given by

$$g(u;\varphi)=\frac{n(n-1)u^{n-2}(\varphi-u)}{\varphi^n}, \quad 0<u<\varphi; \quad \varphi=\theta-\theta^2.$$

Then, $\hat{\varphi}=\left(\frac{n+1}{n-1}\right)U$ is UMVUKBE of $\varphi=\theta-\theta^2$ with

$$\begin{aligned} K(\varphi) &= \frac{2\varphi^2}{(n-1)(n+2)} \\ &= \frac{2\theta^2(1-\theta)^2}{(n-1)(n+2)} \end{aligned}$$

3. THE RANGE $(C_1(\theta), C_2(\theta))$ IS MONOTONIC TYPE I

In this section, we consider the situation when this family of distributions admits a single sufficient statistic. Here, $C_1(\theta), C_2(\theta)$ are monotonic type I which means that $C_1(\theta)$ is monotone decreasing and $C_2(\theta)$ is monotone increasing. Huzurbazar (1976) has proved that in such a situation

$$T=max\{C_1^{-1}(X_{(1)}),\ C_2^{-1}(X_{(n)})\ \}$$

Is sufficient for θ. Using the pdf of T we can find unbiased estimator of θ and Kiefer bound on its variance. We illustrate this by the following

Example 3.1

Let $X_1, X_2, \dots, X_n$ be a random sample on X having probability density function,

$$f(x;\theta)=\frac{1}{2\theta}, \qquad -\theta<x<\theta.$$

Let $Z=X_{(1)}$ and $Y=X_{(n)}$ be the minimum and maximum observations in the sample respectively. $Max\{-Z=-X_{(1)}, Y=X_{(n)}\}$ is complete sufficient statistic for θ. The joint pdf of (Z,Y) is

$$f_{Z,Y}(z,y) \;=\frac{n(n-1)(y-z)^{n-2}}{(2\theta)^n}, \quad 0<(y-z)<\theta$$

To find probability distribution of $T=max\{-Z,Y\}$, note that, the range of T is $(0,\theta)$. The distribution function of T ,namely, $F(t)=0, if\ t\le 0$ and $F(t)=1, if\ t>\theta$. If $0<t\le\theta$,

$$\begin{aligned} F(t) &= P[-Z\le t, Y\le t] \\ &= P[Z>-t, Y\le t] \\ &= \int_{-t}^{t}\int_{-t}^{y}\frac{n(n-1)(y-z)^{n-2}}{(2\theta)^n}dzdy \end{aligned}$$

$$= \int_{-t}^{t} \frac{n(n-1)}{(2\theta)^n} \left\{ \left[\frac{(y-z)^{n-1}}{-(n-1)} \right]_{-t}^{y} \right\} dy$$

$$= \frac{n}{(2\theta)^n} \int_{-t}^{t} (y+t)^{n-1} dy$$

$$= \frac{n}{(2\theta)^n} \left[\frac{(y+t)^n}{n} \right]_{-t}^{t}$$

$$= \left(\frac{t}{\theta} \right)^n$$

Therefore, the pdf of T is given by,

$$f(t;\theta) = \frac{d}{dt} \left(\frac{t}{\theta} \right)^n$$

$$= \frac{nt^{n-1}}{\theta^n} \quad , 0 < t < \theta$$

Using the results(Theorem4.1) in Jadhav and Shanubhogue (2014), $\frac{(n+r)}{n} t^r$ gives UMVUKBE of θ^r with $K(\theta^r) = \frac{r^2\theta^{2r}}{[n(n+2r)]}, r > -\frac{n}{2}$. In particular, $\frac{(n+1)}{n} t$ is UMVUKBE of θ with $K(\theta) = \frac{\theta^2}{n(n+2)}$:the result proved by Kiefer(1952) in his Example 1.

4. THE RANGE $(C_1(\theta), C_2(\theta))$ IS MONOTONIC TYPE II

In this section we consider the situation when this family of distributions admits a single sufficient statistic. Here, $(C_1(\theta), C_2(\theta))$ are monotonic of type II which means that $C_1(\theta)$ is monotone increasing and $C_2(\theta)$ is monotone decreasing. Huzurbazar (1976) has proved that in such a situation $T = min\{C_2^{-1}(X_{(1)}), C_2^{-1}(X_{(n)})\}$ to be modified is sufficient for θ. Using the pdf of T we can find unbiased estimator of θ and Kiefer bound on its variance. We illustrate this by the following

Example 4.1

Let X be a r.v. having uniform probability distribution on $(\theta, 1/\theta)$. Then,

$$f(x;\theta) = \frac{1}{\left(\frac{1}{\theta} - \theta\right)}, \theta < x < \frac{1}{\theta}$$

$$= \frac{\theta}{1-\theta^2} I_{\left(\theta, \frac{1}{\theta}\right)}(x)$$

$$E(X^r) = \int_{\frac{1}{\theta}}^{\theta} x^r \frac{\theta}{1-\theta^2} dx$$

$$= \frac{\theta}{1-\theta^2} \frac{\left[\left(\frac{1}{\theta}\right)^{r+1} - \theta^{r+1}\right]}{r+1}$$

$$= \frac{1-\theta^{2(r+1)}}{(r+1)\theta^r(1-\theta^2)}$$

Using this we get,

$$\mu_1' = \frac{1-\theta^4}{2\theta(1-\theta^2)} \quad , \mu_2' = \frac{1-\theta^6}{3\theta^2(1-\theta^2)}$$

$$\mu_3' = \frac{1-\theta^8}{4\theta^3(1-\theta^2)} \quad , \mu_4' = \frac{1-\theta^{10}}{5\theta^4(1-\theta^2)}$$

$$\mu_2 = \frac{1+\theta^8-4\theta^2+6\theta^4-4\theta^6}{12\theta^2(1-\theta^2)^2}$$

Also

$$F(x) \quad = \frac{\theta(x-\theta)}{1-\theta^2}, \theta < x < \frac{1}{\theta} \quad ,$$

$$1-F(x) \quad = \frac{1-\theta^2-\theta x+\theta^2}{1-\theta^2}$$

$$= \frac{1-\theta x}{1-\theta^2}$$

Let $X_1, X_2, \ldots, X_n$ be a random sample on X. Then,

$$C_1(\theta) \quad = \theta \le X_{(1)}, \ldots, X_{(n)} \le C_2(\theta) = \frac{1}{\theta}$$

Therefore, $\theta \le X_{(1)}$ *and* $X_{(n)} \le \frac{1}{\theta}$ which is equivalent to $\theta \le X_{(1)}$ *and* $\theta \le \frac{1}{X_{(n)}}$.

Therefore,

$$T \quad = min\left\{X_{(1)}, \frac{1}{X_{(n)}}\right\}$$

is a single sufficient statistic for θ . T $\in (\theta, 1)$. The pdf of T is given by,

$$f_T(t;\theta) \quad = \frac{n\theta^n}{(1-\theta^2)^n}\left(\frac{1-t^2}{t}\right)^{n-1}\left(\frac{1+t^2}{t^2}\right) \quad , \quad \theta < t < 1$$

Let

$$u = \frac{1-t^2}{t} = \frac{1}{t} - t \, , \quad \theta < t < 1 \Rightarrow 0 \le u \le \frac{(1-\theta^2)}{\theta}, du = -\left(\frac{1+t^2}{t^2}\right)dt$$

Therefore, the pdf of u is given by

$$g_U(u;\theta) \quad = \frac{n}{\left(\frac{(1-\theta^2)}{\theta}\right)^n} u^{n-1} , \quad 0 \le u \le \frac{(1-\theta^2)}{\theta}$$

Therefore, using results from uniform distribution, we have, $\frac{n+1}{n}U$ is UMVUE of $\frac{(1-\theta^2)}{\theta}$ with its variance $\frac{\left[\frac{(1-\theta^2)}{\theta}\right]^2}{n(n+2)} = K(\theta)$ Further,$\left(\frac{1-T^2}{T}\right)^r$ is UMVUE of $\left[\frac{(1-\theta^2)}{\theta}\right]^r$ with variance= $\frac{r^2\left[\frac{(1-\theta^2)}{\theta}\right]^2}{n(n+2r)} = K(\theta)$

$$f_T(t;\theta) = \frac{n}{\left(\frac{1-\theta^2}{\theta}\right)^n}\left(\frac{1-t^2}{t}\right)^{n-1}\left(\frac{1+t^2}{t^2}\right), \quad \theta < t < 1$$

$$= \frac{n}{(Q(\theta))^n}(Q(t))^{n-1}, \quad \theta < t < 1$$

Here, $Q(\theta) = \frac{(1-\theta^2)}{\theta}$, $\quad Q(t) = \frac{1-t^2}{t}$, $q(t) = \frac{1+t^2}{t^2}$.

This is the general form of pdf of sufficient statistic in case of left truncated one parameter family of distributions.

Example 4.2

Consider the density given Huzurbazar(1976)

$$f(x;\theta) = \frac{1}{5\theta}, \qquad -2\theta < x < 3\theta, \theta > 0.$$

The single complete sufficient statistic for θ is

$$T = max\left(-\frac{1}{2}X_{(1)}, \frac{1}{3}X_{(n)}\right)$$

The pdf of T is given by,

$$f(t;\theta) = \frac{nt^{n-1}}{\theta^n}, \ 0 < t < \theta.$$

Then, $\frac{(n+1)}{n}T$ is UMVUKBE of θ with variance = $\frac{\theta^2}{n(n+2)} = K(\theta)$.

5. KIEFER BOUND FROM DOUBLY CENSORED SAMPLE FROM DISTRIBUTIONS ON $(\mathbf{C_1(\theta), C_2(\theta)})$

In this sort of probability distributions both the ends of the support of the distribution depend on the parameter. Therefore, it is suitable, natural and rational to consider doubly censored samples for effective inference when dealing with life testing experiments. This gives us the following.

Theorem 5.1

Doubly censored sample on variable having probability density function (pdf)

$$f(x,\theta) = \frac{1}{[C_2(\theta) - C_1(\theta)]}, \quad C_1(\theta) < x < C_2(\theta) \qquad , \qquad (5.1)$$

provides UMVUKBE of function $\varphi = [C_2(\theta) - C_1(\theta)]$.

Proof

Let the pdf of the variable be as in (5.1). In this case,

$$F(x) = \frac{x - C_1(\theta)}{\varphi},$$

$$1 - F(x) = \frac{[\, C_2(\theta) - x\,]}{\varphi}.$$

For convenience, let us write, $X_{(r)} = Z$ and $X_{(S)} = Y$, $r < s, w_{rs} = y - z$. Then the joint density is

$$f_{rs}(z,y) = \frac{n!}{(r-1)!\,(s-r-1)!\,(n-s)!}\left[\frac{z - C_1(\theta)}{\varphi\,]}\right]^{r-1}\left[\frac{y-z}{\varphi}\right]^{s-r-1}\left\{\frac{[\, C_2(\theta) - y]}{\varphi}\right\}^{n-s}\frac{1}{\varphi^2}$$

$$f_{rs}(z,w_{rs}) = \frac{n!}{(r-1)!\,(s-r-1)!\,(n-s)!}\frac{[y-z]^{s-r-1}[C_2(\theta) - y]^{n-s}[z - C_1(\theta)]^{r-1}}{\varphi^n}$$

$$= \frac{n!}{(r-1)!\,(s-r-1)!\,(n-s)!}\frac{[w_{rs}]^{s-r-1}[C_2(\theta) - Wrs - z]^{n-s}[z - C_1(\theta)]^{r-1}}{\varphi^n}$$

$$= \frac{n!\;\; [w_{rs}]^{s-r-1}[\varphi - Wrs]^{n-s+r}[C_2(\theta) - Wrs - z]^{n-s}[z - C_1(\theta)]^{r-1}}{(r-1)!\,(s-r-1)!\,(n-s)!\,\varphi^n[\varphi - Wrs]^{n-s+r}}$$

$$f(w_{rs};\varphi) = \frac{n!\,[w_{rs}]^{s-r-1}[\varphi - w_{rs}]^{n-s+r}}{(s-r-1)!\,(n-s+r+1)!\,\varphi^n\beta(r,n-s+1)}\int_{C_1(\theta)}^{C_2(\theta)}\frac{[z - C_1(\theta)]^{r-1}[C_2(\theta) - Wrs - z]^{n-s}dz}{[\varphi - Wrs]^{n-s+r}}$$

But,

$$\frac{1}{\beta(r,n-s+1)}\int_{C_1(\theta)}^{C_2(\theta)}\frac{[z - C_1(\theta)]^{r-1}[C_2(\theta) - Wrs - z]^{n-s}dz}{[\varphi - Wrs]^{n-s+r}} = 1$$

being the β density in the form

$$g(y;p,q) = \frac{1}{\beta(p,q)}\frac{(y-a)^{p-1}(b-y)^{q-1}}{(b-a)^{p+q-1}}, \quad a \leq y \leq b.$$

Therefore, the pdf of w_{rs} is given by

$$f(w_{rs};\varphi) = \frac{n!\,[w_{rs}]^{s-r-1}[\varphi - w_{rs}]^{n-s+r}}{(s-r-1)!\,(n-s+r+1)!\,\varphi^n\beta(r,n-s+1)} \quad (5.2)$$

The parameter space $\Phi = \{\varphi; f(w_{rs};\varphi) > 0\} = (0,\infty)$. For each fixed $\varphi \in \Phi$, let $\Phi_\varphi = \{\square; (\varphi + \square) \in \Phi\} = (-\varphi,\infty)$. On $(-\varphi, 0)$ let us define the prior probability distribution

$$G_1(h) = \frac{(n+1)(\varphi + h)^n I_{\{(-\varphi,0)\}}(h)}{\varphi^{n+1}},$$

Note that, $E_1(\square) = \frac{-\varphi}{n+2} = \Delta_1\varphi$.

If $\varphi$is increased to $\varphi + \square$, $0 < w_{rs} < \varphi + \square \Rightarrow w_{rs} - \varphi < \square$.

$$\Delta_1 f(w_{rs};\varphi) = \int_{w_{rs}-\varphi}^{0} f(w_{rs};\varphi+\square)dG_1(\square) - f(w_{rs};\varphi)$$

$$= \int_{w_{rs}-\varphi}^{0} \frac{[\mathrm{w_{rs}}]^{\mathrm{s-r-1}}[\varphi+\mathrm{h}-\mathrm{w_{rs}}]^{\mathrm{n-s+r}}}{\beta(s-r,n-s+r+1)(\varphi+\mathrm{h})^{\mathrm{n}}} \frac{(\mathrm{n}+1)(\varphi+\mathrm{h})^{\mathrm{n}}\mathrm{I}_{\{(-\varphi,0)\}}(\mathrm{h})\mathrm{dh}}{\varphi^{\mathrm{n}+1}} - f(w_{rs};\varphi)$$

$$= \frac{[\mathrm{w_{rs}}]^{\mathrm{s-r-1}}(\mathrm{n}+1)}{\beta(s-r,n-s+r+1)\varphi^{\mathrm{n}+1}} \int_{w_{rs}-\varphi}^{0} [\varphi+\mathrm{h}-\mathrm{w_{rs}}]^{\mathrm{n-s+r}}\, d\square - f(w_{rs};\varphi)$$

$$= \frac{[\mathrm{w_{rs}}]^{\mathrm{s-r-1}}(\mathrm{n}+1)}{\beta(s-r,n-s+r+1)\varphi^{\mathrm{n}+1}} \left[\frac{[\varphi+\mathrm{h}-\mathrm{w_{rs}}]^{\mathrm{n-s+r+1}}}{\mathrm{n-s+r+1}}\right]_{w_{rs}-\varphi}^{0} - \frac{[\mathrm{w_{rs}}]^{\mathrm{s-r-1}}[\varphi-\mathrm{w_{rs}}]^{\mathrm{n-s+r}}}{\beta(s-r,n-s+r+1)\varphi^{\mathrm{n}}}$$

$$= \frac{[\mathrm{w_{rs}}]^{\mathrm{s-r-1}}[\varphi-\mathrm{w_{rs}}]^{\mathrm{n-s+r}}}{\beta(s-r,n-s+r+1)\varphi^{\mathrm{n}}} \left[\frac{(n+1)(\varphi-\mathrm{w_{rs}})}{\varphi(n-s+r+1)} - 1\right]$$

Therefore,

$$\frac{\Delta_1 f(w_{rs};\varphi)}{f(w_{rs};\varphi)\Delta_1\varphi} = \left[\frac{(n+1)(\varphi-\mathrm{w_{rs}})}{\varphi(n-s+r+1)} - 1\right]\frac{n+2}{(-\varphi)}$$

$$= \frac{[(n+1)(\varphi-\mathrm{w_{rs}}) - \varphi(n-s+r+1)](n+2)}{\varphi(n-s+r+1)(-\varphi)}$$

$$= \frac{-(n+2)[n\varphi+\varphi-(n+1)\mathrm{w_{rs}} - n\varphi + (s-r)\varphi - \varphi]}{\varphi^2(n-s+r+1)}$$

$$= \frac{(n+2)[(n+1)\mathrm{w_{rs}} - (\mathrm{s-r})\varphi]}{\varphi^2(n-s+r+1)}$$

$$= \frac{(s-r)(n+2)}{\varphi^2(n-s+r+1)}\left[\frac{(n+1)}{(s-r)}\mathrm{w_{rs}} - \varphi\right] \tag{5.3}$$

Equation (5.3) is the ideal estimation equation. Therefore, $\frac{(n+1)}{(s-r)}\mathrm{W_{rs}}$ is UMVUKBE of φ with Kiefer bound $K(\varphi) = \frac{\varphi^2(n-s+r+1)}{(s-r)(n+2)}$.

<u>Remark 5.1.</u>

In this situation, for complete sample $\hat{\varphi} = \left(\frac{n+1}{n-1}\right)U$ is UMVUKBE with $K(\varphi) = \frac{2\varphi^2}{(n-1)(n+2)}$ while for doubly censored sample, $\hat{\varphi} = \frac{(n+1)}{(s-r)}\mathrm{w_{rs}}$ is UMVUKBE with $K(\varphi) = \frac{\varphi^2(n-s+r+1)}{(s-r)(n+2)}$

<u>Remark 5.2.</u>

To compare the results based on complete and censored samples, consider,

$$\frac{\text{Variance of UMVUE based on complete sample}}{\text{Variance of UMVUE based on censored sample}} = \frac{2\varphi^2}{(n-1)(n+2)}\frac{(s-r)(n+2)}{\varphi^2(n-s+r+1)}$$

$$= \frac{2(s-r)}{(n-1)(n-s+r+1)} < 1.$$

Remark 5.3.

If s=n and r=1, the results of complete and doubly censored samples coincide.

Example 5.1.

Consider a doubly censored sample from variable having pdf

$$f(x;\theta) \qquad = \frac{1}{2\theta}, \qquad -\theta < x < \theta \tag{5.4}$$

Then,

$$F(x) = \frac{x+\theta}{2\theta}$$

$$1 - F(x) = \frac{(\theta - x)}{2\theta}$$

Let $\varphi = 2\theta$. Then, $\left(\frac{n+1}{s-r}\right) W_{rs}$ is UMVUKBE of φ with Kiefer bound $K(\varphi) = \frac{\varphi^2(n-s+r+1)}{[n+2\,](s-r)} = \frac{4\theta^2(n-s+r+1)}{(n+2)(s-r)}$.

If s= n, r=1, the results reduce to the results of complete sample. That is, if, s= n, r=1,

$$K(\theta) = \frac{(2\theta)^2(n-s+r+1)}{[n+2\,](s-r)}$$

$$= \frac{2(4)\theta^2}{(n-1)(n+2).}$$

For comparison, let us consider,

$$\frac{\text{Kiefer bound from doubly censored sample}}{\text{Kiefer bound from complete sample}} = \frac{4\theta^2(n-s+r+1)}{[n+2\,](s-r)}\frac{(n-1)(n+2)}{8\theta^2}$$

$$= \frac{(n-1)(n-s+r+1)}{2(s-r)} \tag{5.5}$$

Example 5.2.

Let us consider,

$$f(x;\varphi) = \frac{1}{\varphi}, \theta < x < \frac{1}{\theta}, \quad \varphi = \frac{(1-\theta^2)}{\theta} \tag{5.6}$$

Then

$$E(W_{rs}) = \left(\frac{s-r}{n+1}\right)\varphi$$

and $$Var(w_{rs}) = \frac{(s-r)(n-s+r+1)\varphi^2}{(n+1)^2(n+2)}$$

Then it follows that $\left(\frac{n+1}{s-r}\right) W_{rs}$ is UMVUKBE of φ with Kiefer bound $K(\varphi) = \frac{\varphi^2(n-s+r+1)}{[n+2\,](s-r)}$.

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