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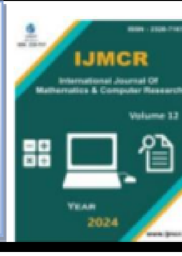

EXISTENCE AND CONTINUOUS DEPENDENCE RESULTS FOR VOLTERRA-FREDHOLM INCLUSION

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Abstract. The solvability of a state-dependent Volterra-Fredholm integro-differential inclusion (V-FIDI) with a nonlocal condition is explored, along with the uniqueness and existence of its solutions. We first prove the uniqueness and existence properties of the solution under certain assumptions about the given data. Lastly, a sample that highlights the primary finding is provided.

1. Introduction

Since many mathematical models are used to formulate various phenomena, such as mechanics, physics, chemical kinetics, astronomy, biology, economics, potential theory, and electrostatistics, which are modelled using integro-differential equations, differential equations and integral equations have recently become extremely important in several branches of science and engineering [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 20, 25, 28].

In mathematical physics, nonlocal issues are those in which the values of the desired function at different locations along the boundary are connected, in contrast to typical boundary value requirements. In the differential and integral equations with diverging arguments found in the present literature, the argument's deviation often only affects the time itself, as seen in [1, 2, 18, 19, 21, 22, 24]. Alternatively, there is another instance, both in theory and in practise, where the deviating arguments rely on the time $\wp$ as well as the state variable Φ. Recently, a number of papers have been published, for instance [3, 4, 5, 26, 29], that focus on these differential equations.

The differential equation's unique solution was examined in the earliest studies of this family of functional equations with self-reference, by Eder [6].

$$\Psi'(\wp) = \Psi(\Psi(\wp)), \quad \Psi(0) = \Psi_0, \quad \wp \in \eta \subset \mathbb{R}$$

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Wang [27] investigated the equation's maximal strong solution as well as its strong solution.

$$\Psi'(\wp) = B(\Psi(\Psi(\wp)))$$

where $B : \mathbb{R} \to \mathbb{R}$ is monotone and continuous and , and $B(0) = 0$.

Buica [3] investigated the functional differential equation's continuous dependence and existence of solutions on Ψ_0:

$$\Psi'(\wp) = B(\wp, \Psi(\Psi(\wp))), \quad \Psi(\wp_0) = \Psi_0, \quad \wp \in [a, d]$$

where $\wp_0, \Psi_0 \in [a, d]$ and $B \in C([a, d], [a, d])$.

The V-FIDI with self-dependence on a nonlinear operator is examined in this work.

$$\frac{\mathrm{d}\Phi}{\mathrm{d}\wp} \in \Psi\left(\wp, \int_0^{\Omega(\wp)} \Phi\ \xi(\hbar, \Phi(\hbar))\mathrm{d}\hbar, \int_0^{\eta} \Phi\ \vartheta(\hbar, \Phi(\hbar))\mathrm{d}\hbar\right) \text{ a.e. } \wp \in (0, \eta] \quad (1.1)$$

$$\Phi(0) + \sum_{\gamma=1}^{\Theta} \Delta_\gamma \Phi\left(\Omega\left(\rho_\gamma\right)\right) = \Phi_0, \quad \Delta_\gamma > 0, \quad \rho_\gamma \in (0, \eta). \quad (1.2)$$

We investigate the existence and uniqueness of the solution $\Phi \in AC[0, \eta]$ and demonstrate the continuous dependence on Φ_0 and ξ.

2. Preliminaries

Take into account these presumptions:

(a) $\Psi(\wp, \Phi)$ be nonempty, closed and convex $\forall(\wp, \Phi) \in [0, \eta] \times \mathbb{R}$.

(b) $\Psi(\wp, \Phi)$ be measurable in $\wp \in [0, \eta]$ for every $\Phi \in \mathbb{R}$.

(c) $\Psi(\wp, \Phi)$ be semi upper continuous in Φ for every $\wp \in [0, \eta]$.

(d) There exist a measurable bounded function $c : [0, \eta] \to \mathbb{R}$ and constant $\Upsilon > 0$, such that

$$\|\Psi(\wp, \Phi)\| = \sup\{|\Im| : \Im \in \Psi(\wp, \Phi)\} \leq |c(\wp)| + \Upsilon|\Phi|, \quad |c(\wp)| \leq M$$

We may infer that there exists a $\Im \in \Psi(\wp, \Phi)$ such that the following is met based on the assumptions (a)–(d):

(A1). $\Im : [0, \eta] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ fulfils the Caratheodory requirement:

- For each $\wp \in [0, \eta], \Im(\wp, ., .)$ is continuous;
- For each $\theta, \theta_1 \in \mathbb{R}, \Im(., \theta, \theta_1)$ is measurable;
- There exist a measurable bounded function $c(\wp)$ and a constant $\Upsilon, \Upsilon_1 > 0$, such that

$$|\Im(\wp, \theta, \theta_1)| \leq |c(\wp)| + \Upsilon|\theta| + \Upsilon_1|\theta_1|, \quad |c(\wp)| \leq M,$$

and the functional $\Im$ satisfies the Volterra-Fredholm equation

$$\begin{aligned}&\frac{d\Phi}{d\wp} \\ = \quad & \Im\left(\wp, \int_0^{\Omega(\wp)} \Phi\ \xi(\hbar, \Phi(\hbar))d\hbar, \int_0^{\eta} \Phi\ \vartheta(\hbar, \Phi(\hbar))d\hbar\right) \text{ a.e. } \wp \in (0, \eta],\end{aligned} \tag{2.1}$$

(A2). $\xi, \vartheta : [0, \eta] \times \mathbb{R} \to \mathbb{R}^+$ fulfils the Caratheodory requirement:

- For each $\wp \in [0, \eta], \xi(\wp, .), \vartheta(\wp, .)$ be continuous;
- For each $\Phi \in \mathbb{R}, \xi(., \Phi), \vartheta(., \Phi)$ be measurable;
- $|\xi(\wp, \Phi)| \leq 1,\ |\vartheta(\wp, \Phi)| \leq 1$.

(A3). $\Omega : [0, \eta] \to [0, \eta]$ is nondecreasing continuous, $\Omega(\wp) \leq \wp, \wp \in [0, \eta]$.

(A4). $\left(1 + 2\sum_{\gamma=1}^{\mathcal{K}} \Delta_\gamma\right)(\Upsilon + \Upsilon_1)\eta < 1$.

Remark 2.1. We may infer from (a) and (1.1) that each solution to (1.1) also solves (2.1).

Lemma 2.2. *If* ***(A1)-(A4)*** *satisfies, then the system* (1.1)-(1.2) *and the equation*

$$\begin{aligned}\Phi(\wp) &= \frac{1}{1 + \sum_{\gamma=1}^{\Theta} \Delta_\gamma} \\ &\times \left[\Phi_0 - \sum_{\gamma=1}^{\Theta} \Delta_\gamma \int_0^{\Omega(\rho_\gamma)} \Im\left(\hbar, \int_0^{\Omega(\hbar)} \Phi\ \xi(\theta, \Phi(\theta))d\theta, \int_0^{\eta} \Phi\ \vartheta(\theta, \Phi(\theta))d\theta\right) d\hbar\right] \\ &+ \int_0^{\wp} \Im\left(\hbar, \int_0^{\Omega(\hbar)} \Phi\ \xi(\theta, \Phi(\theta))d\theta, \int_0^{\eta} \Phi\ \vartheta(\theta, \Phi(\theta))d\theta\right) d\hbar \quad \textit{for } \wp \in [0, \eta]\end{aligned} \tag{2.2}$$

are equivalent.

Proof. Let the solution of the system (1.1)-(1.2) exists. Integrating (2.1) from 0 to $\wp$, we have

$$\Phi(\wp) = \Phi(0) + \int_0^{\wp} \Im\left(\hbar, \int_0^{\Omega(\hbar)} \Phi\ \xi(\theta, \Phi(\theta))d\theta, \int_0^{\eta} \Phi\ \vartheta(\theta, \Phi(\theta))d\theta\right) d\hbar \tag{2.3}$$

Using (1.2), we get

$$\begin{aligned}&\sum_{\gamma=1}^{\Theta} \Delta_\gamma \Phi\left(\Omega\left(\rho_\gamma\right)\right) \\ &= \Phi(0) \sum_{\gamma=1}^{\Theta} \Delta_\gamma + \sum_{\gamma=1}^{\Theta} \Delta_\gamma \int_0^{\Omega(\rho_\gamma)} \Im\left(\hbar, \int_0^{\Omega(\hbar)} \Phi\ \xi(\theta, \Phi(\theta))d\theta, \int_0^{\eta} \Phi\ \vartheta(\theta, \Phi(\theta))d\theta\right) d\hbar\end{aligned}$$

By $\sum_{\gamma=1}^{\Theta} \Delta_\gamma \Phi\left(\Omega\left(\rho_\gamma\right)\right) = \Phi_0 - \Phi(0)$, we get

$$\Phi_0 - \Phi(0) =$$
$$\Phi(0)\sum_{\gamma=1}^{\Theta}\Delta_\gamma + \sum_{\gamma=1}^{\Theta}\Delta_\gamma\int_0^{\Omega(\rho_\gamma)}\Im\left(\hbar, \int_0^{\Omega(\hbar)}\Phi\ \xi(\theta,\Phi(\theta))\mathrm{d}\theta, \int_0^{\eta}\Phi\ \vartheta(\theta,\Phi(\theta))\mathrm{d}\theta\right)\mathrm{d}\hbar,$$

which implies

$$\Phi(0) = \frac{1}{1+\sum_{\gamma=1}^{\Theta}\Delta_\gamma} \times\left[\Phi_0 - \sum_{\gamma=1}^{\Theta}\Delta_\gamma\int_0^{\Omega(\rho_\gamma)}\Im\left(\hbar, \int_0^{\Omega(\hbar)}\Phi\ \xi(\theta,\Phi(\theta))\mathrm{d}\theta, \int_0^{\eta}\Phi\ \vartheta(\theta,\Phi(\theta))\mathrm{d}\theta\right)\mathrm{d}\hbar\right] \tag{2.4}$$

Using (2.3)-(2.4), we have

$$\begin{aligned}\Phi(\wp) =&\frac{1}{1+\sum_{\gamma=1}^{\Theta}\Delta_\gamma}\\ &\left[\Phi_0 - \sum_{\gamma=1}^{\Theta}\Delta_\gamma\int_0^{\Omega(\rho_\gamma)}\Im\left(\hbar, \int_0^{\Omega(\hbar)}\Phi\ \xi(\theta,\Phi(\theta))\mathrm{d}\theta, \int_0^{\eta}\Phi\ \vartheta(\theta,\Phi(\theta))\mathrm{d}\theta\right)\mathrm{d}\hbar\right]\\ &+\int_0^{\wp}\Im\left(\hbar, \int_0^{\Omega(\hbar)}\Phi\ \xi(\theta,\Phi(\theta))\mathrm{d}\theta, \int_0^{\eta}\Phi\ \vartheta(\theta,\Phi(\theta))\mathrm{d}\theta\right)\mathrm{d}\hbar.\end{aligned}$$

Now, assume that $\Phi \in C[0,\eta]$ is a solution of (2.2). Now differentiate (2.2), we get

$$\begin{aligned}\frac{\mathrm{d}\Phi}{\mathrm{d}\wp} = \frac{\mathrm{d}}{\mathrm{d}\wp}\Big\{&\frac{1}{1+\sum_{\gamma=1}^{\Theta}\Delta_\gamma}\\ &\left[\Phi_0 - \sum_{\gamma=1}^{\Theta}\Delta_\gamma\int_0^{\Omega(\rho_\gamma)}\Im\left(\hbar, \int_0^{\Omega(\hbar)}\Phi\ \xi(\theta,\Phi(\theta))\mathrm{d}\theta, \int_0^{\eta}\Phi\ \vartheta(\theta,\Phi(\theta))\mathrm{d}\theta\right)\mathrm{d}\hbar\right]\\ &+\int_0^{\wp}\Im\left(\hbar, \int_0^{\Omega(\hbar)}\Phi\ \xi(\theta,\Phi(\theta))\mathrm{d}\theta, \int_0^{\eta}\Phi\ \vartheta(\theta,\Phi(\theta))\mathrm{d}\theta\right)\mathrm{d}\hbar\Big\}\\ =&\Im\left(\wp, \int_0^{\Omega(\wp)}\Phi\ \xi(\theta,\Phi(\theta))\mathrm{d}\theta, \int_0^{\eta}\Phi\ \vartheta(\theta,\Phi(\theta))\mathrm{d}\theta\right).\end{aligned}$$

From (2.2), we have

$$\begin{aligned}\Phi\left(\rho_\gamma\right)= & \frac{1}{1+\sum_{\gamma=1}^{\Theta} \Delta_\gamma} \\ & \times\left[\Phi_0-\sum_{\gamma=1}^{\Theta} \Delta_\gamma \int_0^{\Omega\left(\rho_\gamma\right)} \Im\left(\hbar, \int_0^{\Omega(\hbar)} \Phi\ \xi(\theta, \Phi(\theta)) \mathrm{d} \theta, \int_0^\eta \Phi\ \vartheta(\theta, \Phi(\theta)) \mathrm{d} \theta\right) \mathrm{d} \hbar\right] \\ & +\int_0^{\rho_\gamma} \Im\left(\hbar, \int_0^{\Omega(\hbar)} \Phi\ \xi(\theta, \Phi(\theta)) \mathrm{d} \theta, \int_0^\eta \Phi\ \vartheta(\theta, \Phi(\theta)) \mathrm{d} \theta\right) \mathrm{d} \hbar, \qquad (2.5)\\ \Phi(0)= & \frac{1}{1+\sum_{\gamma=1}^{\Theta} \Delta_\gamma} \\ & \times\left[\Phi_0-\sum_{\gamma=1}^{\Theta} \Delta_\gamma \int_0^{\Omega\left(\rho_\gamma\right)} \Im\left(\hbar, \int_0^{\Omega(\hbar)} \Phi\ \xi(\theta, \Phi(\theta)) \mathrm{d} \theta, \int_0^\eta \Phi\ \vartheta(\theta, \Phi(\theta)) \mathrm{d} \theta\right) \mathrm{d} \hbar\right],\end{aligned}$$

and

$$\sum_{\gamma=1}^{\Theta} \Delta_\gamma \Phi\left(\Omega\left(\rho_\gamma\right)\right)=\frac{1}{1+\sum_{\gamma=1}^{\Theta} \Delta_\gamma} \sum_{\gamma=1}^{\Theta} \Delta_\gamma \qquad (2.6)$$

$$\times\left[\Phi_0-\sum_{\gamma=1}^{\Theta} \Delta_\gamma \int_0^{\Omega\left(\rho_\gamma\right)} \Im\left(\hbar, \int_0^{\Omega(\hbar)} \Phi\ \xi(\theta, \Phi(\theta)) \mathrm{d} \theta, \int_0^\eta \Phi\ \vartheta(\theta, \Phi(\theta)) \mathrm{d} \theta\right) \mathrm{d} \hbar\right]$$

$$+\sum_{\gamma=1}^{\Theta} \Delta_\gamma \int_0^{\Omega\left(\rho_\gamma\right)} \Im\left(\hbar, \int_0^{\Omega(\hbar)} \Phi\ \xi(\theta, \Phi(\theta)) \mathrm{d} \theta, \int_0^\eta \Phi\ \vartheta(\theta, \Phi(\theta)) \mathrm{d} \theta\right) \mathrm{d} \hbar .$$

From (2.5) and (2.6), we get

$$\Phi(0)+\sum_{\gamma=1}^{\Theta} \Delta_\gamma \Phi\left(\Omega\left(\rho_\gamma\right)\right)=\frac{1}{1+\sum_{\gamma=1}^{\Theta} \Delta_\gamma}\left(1+\sum_{\gamma=1}^{\Theta} \Delta_\gamma\right)\left[\Phi_0-\sum_{\gamma=1}^{\Theta} \Delta_\gamma\right.$$

$$\left.\int_0^{\Omega\left(\rho_\gamma\right)} \Im\left(\hbar, \int_0^{\Omega(\hbar)} \Phi\ \xi(\theta, \Phi(\theta)) \mathrm{d} \theta, \int_0^\eta \Phi\ \vartheta(\theta, \Phi(\theta)) \mathrm{d} \theta\right) \mathrm{d} \hbar\right]$$

$$+\sum_{\gamma=1}^{\Theta} \Delta_\gamma \int_0^{\Omega\left(\rho_\gamma\right)} \Im\left(\hbar, \int_0^{\Omega(\hbar)} \Phi\ \xi(\theta, \Phi(\theta)) \mathrm{d} \theta, \int_0^\eta \Phi\ \vartheta(\theta, \Phi(\theta)) \mathrm{d} \theta\right) \mathrm{d} \hbar .$$

Then

$$\Phi(0)+\sum_{\gamma=1}^{\Theta} \Delta_\gamma \Phi\left(\Omega\left(\rho_\gamma\right)\right)=\Phi_0$$

The proof is finished now.

3. Results

Hare, we prove that there is at least one solution to system (1.1)-(1.2) using Schauder's argument.

Theorem 3.1. *Let the hypotheses **(A1)-(A4)** be hold, then the system* (1.1)-(1.2) *has at least one solution* $\Phi \in AC[0, \eta]$.

Proof. We first establish P_L, create the operator A related to Eq. (2.2) by

$$A\Phi(\wp) = \frac{1}{1+\sum_{\gamma=1}^{\Theta}\Delta_\gamma} \times \left[\Phi_0 - \sum_{\gamma=1}^{\Theta}\Delta_\gamma \int_0^{\Omega(\rho_\gamma)} \Im\left(\hbar, \int_0^{\Omega(\hbar)} \Phi\ \xi(\theta,\Phi(\theta))\mathrm{d}\theta, \int_0^{\eta} \Phi\ \vartheta(\theta,\Phi(\theta))\mathrm{d}\theta\right)\mathrm{d}\hbar\right] + \int_0^{\wp} \Im\left(\hbar, \int_0^{\Omega(\hbar)} \Phi\ \xi(\theta,\Phi(\theta))\mathrm{d}\theta, \int_0^{\eta} \Phi\ \vartheta(\theta,\Phi(\theta))\mathrm{d}\theta\right)\mathrm{d}\hbar.$$

Now $P_L = \{\Phi \in C[0,\eta] : |\Phi(\wp) - \Phi(\hbar)| \le L|\wp - \hbar|, \forall \wp, \hbar \in [0,\eta]\}$, where

$$L = \frac{(\Upsilon + \Upsilon_1)|\Phi_0| + \left(1+\sum_{\gamma=1}^{\mathcal{K}}\Delta_\gamma\right)M}{\left(1+\sum_{\gamma=1}^{K}\Delta_\gamma\right) - \left(1+2\sum_{\gamma=1}^{\Theta}\Delta_\gamma\right)(\Upsilon+\Upsilon_1)\eta}$$

Hence, we get for $\Phi \in P$,

$$|A\Phi(\wp)| = \Big| \frac{1}{1+\sum_{\gamma=1}^{\Theta}\Delta_\gamma} \times \left[\Phi_0 - \sum_{\gamma=1}^{\Theta}\Delta_\gamma \int_0^{\Omega(\rho_\gamma)} \Im\left(\hbar, \int_0^{\Omega(\hbar)} \Phi\ \xi(\theta,\Phi(\theta))\mathrm{d}\theta, \int_0^{\eta} \Phi\ \vartheta(\theta,\Phi(\theta))\mathrm{d}\theta\right)\mathrm{d}\hbar\right] + \int_0^{\wp} \Im\left(\hbar, \int_0^{\Omega(\hbar)} \Phi\ \xi(\theta,\Phi(\theta))\mathrm{d}\theta, \int_0^{\eta} \Phi\ \vartheta(\theta,\Phi(\theta))\mathrm{d}\theta\right)\mathrm{d}\hbar \Big|$$

$$\le \frac{1}{1+\sum_{\gamma=1}^{\Theta}\Delta_\gamma}\left[\left|\Phi_0\right| + \sum_{\gamma=1}^{\Theta}\Delta_\gamma \int_0^{\Omega(\rho_\gamma)} \times \left|\Im\left(\hbar, \int_0^{\Omega(\hbar)} \Phi\ \xi(\theta,\Phi(\theta))\mathrm{d}\theta, \int_0^{\eta} \Phi\ \vartheta(\theta,\Phi(\theta))\mathrm{d}\theta\right)\right|\mathrm{d}\hbar\right] + \int_0^{\wp} \left|\Im\left(\hbar, \int_0^{\Omega(\hbar)} \Phi\ \xi(\theta,\Phi(\theta))\mathrm{d}\theta, \int_0^{\eta} \Phi\ \vartheta(\theta,\Phi(\theta))\mathrm{d}\theta\right)\right|\mathrm{d}\hbar,$$

then

$$
\begin{aligned}
&|A\Phi(\wp)| \\
\leq & \frac{1}{1+\sum_{\gamma=1}^{\Theta}\Delta_\gamma}\Big[|\Phi_0| + \sum_{\gamma=1}^{\Theta}\Delta_\gamma\int_0^{\Omega(\rho_\gamma)}\Big(|c(\hbar)| + \Upsilon\left|\int_0^{\Omega(\hbar)}\Phi\ \xi(\theta,\Phi(\theta))\mathrm{d}\theta\right| \\
&+\Upsilon_1\Big|\int_0^{\eta}\Phi\ \vartheta(\theta,\Phi(\theta))\mathrm{d}\theta\Big|\Big)\mathrm{d}\hbar\Big] \\
&+\int_0^{\wp}\left(|c(\hbar)| + \Upsilon\left|\int_0^{\Omega(\hbar)}\Phi\ \xi(\theta,\Phi(\theta))\mathrm{d}\theta\right| + \Upsilon_1\left|\int_0^{\eta}\Phi\ \vartheta(\theta,\Phi(\theta))\mathrm{d}\theta\right|\right)\mathrm{d}\hbar \\
\leq & \frac{1}{1+\sum_{\gamma=1}^{\Theta}\Delta_\gamma}\left[|\Phi_0| + \sum_{\gamma=1}^{\Theta}\Delta_\gamma\int_0^{\Omega(\rho_\gamma)}(|c(\hbar)|\right. \\
&+\Upsilon\left|\int_0^{\Omega(\hbar)}\Phi\ \xi(\theta,\Phi(\theta))\mathrm{d}\theta - \Phi(0)\right| + \Upsilon|\Phi(0)| \\
&\left.+\Upsilon_1\left|\int_0^{\eta}\Phi\ \vartheta(\theta,\Phi(\theta))\mathrm{d}\theta - \Phi(0)\right| + \Upsilon_1|\Phi(0)|\Big)\mathrm{d}\hbar\right] \\
&+\int_0^{\wp}\Big(|c(\hbar)| + \Upsilon\left|\int_0^{\Omega(\hbar)}\Phi\ \xi(\theta,\Phi(\theta))\mathrm{d}\theta - \Phi(0)\right| + \Upsilon|\Phi(0)| \\
&+\Upsilon_1\left|\int_0^{\eta}\Phi\ \vartheta(\theta,\Phi(\theta))\mathrm{d}\theta - \Phi(0)\right| + \Upsilon_1|\Phi(0)|\Big)\mathrm{d}\hbar. \\
\leq & \frac{1}{1+\sum_{\gamma=1}^{\Theta}\Delta_\gamma}\Big[|\Phi_0| + \sum_{\gamma=1}^{\Theta}\Delta_\gamma\int_0^{\Omega(\rho_\gamma)}\Big(|c(\hbar)| + \Upsilon L\int_0^{\Omega(\hbar)}|\xi(\theta,\Phi(\theta))|\mathrm{d}\theta + \Upsilon|\Phi(0)| \\
&+\Upsilon_1 L\int_0^{\eta}|\vartheta(\theta,\Phi(\theta))|\mathrm{d}\theta + \Upsilon_1|\Phi(0)|\Big)\mathrm{d}\hbar\Big] \\
&+\int_0^{\wp}\Big(|c(\hbar)| + \Upsilon L\int_0^{\Omega(\hbar)}|\xi(\theta,\Phi(\theta))|\mathrm{d}\theta + \Upsilon|\Phi(0)| \\
&+\Upsilon_1 L\int_0^{\eta}|\vartheta(\theta,\Phi(\theta))|\mathrm{d}\theta + \Upsilon_1|\Phi(0)|\Big)\mathrm{d}\hbar \\
\leq & \frac{1}{1+\sum_{\gamma=1}^{\Theta}\Delta_\gamma}|\Phi_0| + \frac{\sum_{\gamma=1}^{\Theta}\Delta_\gamma}{1+\sum_{\gamma=1}^{\Theta}\Delta_\gamma}\left(\eta M + (\Upsilon+\Upsilon_1)L\eta^2 + (\Upsilon+\Upsilon_1)\eta|\Phi(0)|\right) \\
&+\eta M + (\Upsilon+\Upsilon_1)L\eta^2 + (\Upsilon+\Upsilon_1)\eta|\Phi(0)| \\
= & \frac{1}{1+\sum_{\gamma=1}^{\Theta}\Delta_\gamma}|\Phi_0| + \left(\frac{\sum_{\gamma=1}^{\mathcal{K}}\Delta_\gamma}{1+\sum_{\gamma=1}^{K}\Delta_\gamma} + 1\right)\left(\eta M + (\Upsilon+\Upsilon_1)L\eta^2 + (\Upsilon+\Upsilon_1)\eta|\Phi(0)|\right).
\end{aligned} \tag{3.1}
$$

But

$$
\begin{aligned}
&|\Phi(0)| \\
&= \left|\frac{1}{1+\sum_{\gamma=1}^{\Theta}\Delta_\gamma}\left[\Phi_0 - \sum_{\gamma=1}^{\Theta}\Delta_\gamma \int_0^{\Omega(\rho_\gamma)} \Im\left(\hbar, \int_0^{\Omega(\hbar)} \Phi\ \xi(\theta,\Phi(\theta))\mathrm{d}\theta, \int_0^{\eta} \Phi\ \vartheta(\theta,\Phi(\theta))\mathrm{d}\theta\right)\mathrm{d}\hbar\right]\right| \\
&\le \frac{1}{1+\sum_{\gamma=1}^{\Theta}\Delta_\gamma}\left[|\Phi_0| + \sum_{\gamma=1}^{\Theta}\Delta_\gamma \int_0^{\Omega(\rho_\gamma)} \left|\Im\left(\hbar, \int_0^{\Omega(\hbar)} \Phi\ \xi(\theta,\Phi(\theta))\mathrm{d}\theta, \int_0^{\eta} \Phi\ \vartheta(\theta,\Phi(\theta))\mathrm{d}\theta\right)\right|\mathrm{d}\hbar\right]
\end{aligned}
$$

then

$$
\begin{aligned}
&|\Phi(0)| \\
&\le \frac{1}{1+\sum_{\gamma=1}^{\Theta}\Delta_\gamma}\left[|\Phi_0| + \sum_{\gamma=1}^{\Theta}\Delta_\gamma \int_0^{\Omega(\rho_\gamma)} (|c(\hbar)|\right. \\
&\quad + \Upsilon\left|\int_0^{\Omega(\hbar)} \Phi\ \xi(\theta,\Phi(\theta))\mathrm{d}\theta - \Phi(0)\right| + \Upsilon|\Phi(0)| \\
&\quad \left. + \Upsilon_1\left|\int_0^{\eta} \Phi\ \vartheta(\theta,\Phi(\theta))\mathrm{d}\theta - \Phi(0)\right| + \Upsilon_1|\Phi(0)|\Big)\mathrm{d}\hbar\right] \\
&\le \frac{1}{1+\sum_{\gamma=1}^{\Theta}\Delta_\gamma}\Big[|\Phi_0| + \sum_{\gamma=1}^{\Theta}\Delta_\gamma \int_0^{\Omega(\rho_\gamma)} \Big(|c(\hbar)| + \Upsilon L\int_0^{\Omega(\hbar)} |\xi(\theta,\Phi(\theta))|\mathrm{d}\theta + \Upsilon|\Phi(0)| \\
&\quad + \Upsilon_1 L\int_0^{\eta} |\vartheta(\theta,\Phi(\theta))|\mathrm{d}\theta + \Upsilon_1|\Phi(0)|\Big)\mathrm{d}\hbar\Big] \\
&\le \frac{1}{1+\sum_{\gamma=1}^{\Theta}\Delta_\gamma}\left[|\Phi_0| + \sum_{\gamma=1}^{\Theta}\Delta_\gamma\left(\eta M + (\Upsilon+\Upsilon_1)L\eta^2 + (\Upsilon+\Upsilon_1)\eta|\Phi(0)|\right)\right].
\end{aligned}
$$

Then

$$
|\Phi(0)| \le \frac{|\Phi_0| + \left(\eta M + (\Upsilon+\Upsilon_1)L\eta^2\right)\sum_{\gamma=1}^{\Theta}\Delta_\gamma}{\left(1+\sum_{\gamma=1}^{K}\Delta_\gamma\right) - (\Upsilon+\Upsilon_1)\eta\sum_{\gamma=1}^{K}\Delta_\gamma} \tag{3.2}
$$

From (3.1)-(3.2), we have

$$
\begin{aligned}
&|A\Phi(\wp)| \\
\leq&\frac{1}{1+\sum_{\gamma=1}^{K}\Delta_\gamma}|\Phi_0|+\left(\frac{\sum_{\gamma=1}^{\mathcal{K}}\Delta_\gamma}{1+\sum_{\gamma=1}^{\mathcal{K}}\Delta_\gamma}+1\right)\\
&\times\left(\eta M+(\Upsilon+\Upsilon_1)L\eta^2+(\Upsilon+\Upsilon_1)\eta|\Phi(0)|\right)\\
\leq&\frac{1}{1+\sum_{\gamma=1}^{\mathcal{N}}\Delta_\gamma}|\Phi_0|+\left(\frac{\sum_{\gamma=1}^{\mathcal{K}}\Delta_\gamma}{1+\sum_{\gamma=1}^{\mathcal{K}}\Delta_\gamma}+1\right)\\
&\times\left(\eta M+(\Upsilon+\Upsilon_1)L\eta^2+(\Upsilon+\Upsilon_1)\eta\frac{|\Phi_0|+\left(\eta M+(\Upsilon+\Upsilon_1)L\eta^2\right)\sum_{\gamma=1}^{\mathcal{K}}\Delta_\gamma}{\left(1+\sum_{\gamma=1}^{\mathcal{K}}\Delta_\gamma\right)-(\Upsilon+\Upsilon_1)\eta\sum_{\gamma=1}^{\mathcal{K}}\Delta_\gamma}\right)\\
=&\frac{|\Phi_0|+\left(\eta M+(\Upsilon+\Upsilon_1)L\eta^2\right)\sum_{\gamma=1}^{\mathcal{K}}\Delta_\gamma}{\left(1+\sum_{\gamma=1}^{\mathcal{K}}\Delta_\gamma\right)-(\Upsilon+\Upsilon_1)\eta\sum_{\gamma=1}^{\mathcal{K}}\Delta_\gamma}+L\eta.
\end{aligned}
$$

Now, let $\wp_1,\wp_2\in(0,\eta]$ such that $|\wp_2-\wp_1|<\delta$, then

$$
\begin{aligned}
&|A\Phi(\wp_2)-A\Phi(\wp_1)|\\
&\leq\int_{\wp_1}^{\wp_2}\left|\Im\left(\hbar,\int_0^{\Omega(\hbar)}\Phi\ \xi(\theta,\Phi(\theta))\mathrm{d}\theta,\int_0^{\eta}\Phi\ \vartheta(\theta,\Phi(\theta))\mathrm{d}\theta\right)\right|\mathrm{d}\hbar\\
&\leq\int_{\wp_1}^{\wp_2}\left(|c(\hbar)|+\Upsilon\left|\int_0^{\Omega(\hbar)}\Phi\ \xi(\theta,\Phi(\theta))\mathrm{d}\theta-\Phi(0)\right|+\Upsilon|\Phi(0)|\right.\\
&\left.+\Upsilon_1\left|\int_0^{\eta}\Phi\ \vartheta(\theta,\Phi(\theta))\mathrm{d}\theta-\Phi(0)\right|+\Upsilon_1|\Phi(0)|\right)\mathrm{d}\hbar \qquad (3.3)\\
&\leq\int_{\wp_1}^{\wp_2}\left(|c(\hbar)|+\Upsilon L\int_0^{\Omega(\hbar)}|\xi(\theta,\Phi(\theta))|\mathrm{d}\theta+\Upsilon|\Phi(0)|\right.\\
&\left.+\Upsilon_1 L\int_0^{\eta}|\vartheta(\theta,\Phi(\theta))|\mathrm{d}\theta+\Upsilon_1|\Phi(0)|\right)\mathrm{d}\hbar\\
&\leq(\wp_2-\wp_1)M+(\wp_2-\wp_1)(\Upsilon+\Upsilon_1)L\eta+(\wp_2-\wp_1)(\Upsilon+\Upsilon_1)|\Phi(0)|\\
&=(\wp_2-\wp_1)(M+(\Upsilon+\Upsilon_1)L\eta+(\Upsilon+\Upsilon_1)|\Phi(0)|).
\end{aligned}
$$

From (3.2)-(3.3), we get

$$
\begin{aligned}
&|A\Phi(\wp_2)-A\Phi(\wp_1)|\\
\leq&(\wp_2-\wp_1)\left(M+(\Upsilon+\Upsilon_1)L\eta\right.\\
&\left.+(\Upsilon+\Upsilon_1)\left(\frac{|\Phi_0|+\left(\eta M+(\Upsilon+\Upsilon_1)L\eta^2\right)\sum_{\gamma=1}^{\mathcal{K}}\Delta_\gamma}{\left(1+\sum_{\gamma=1}^{\mathcal{K}}\Delta_\gamma\right)-(\Upsilon+\Upsilon_1)\eta\sum_{\gamma=1}^{\mathcal{K}}\Delta_\gamma}\right)\right)
\end{aligned}
$$

$$\leq (\wp_2 - \wp_1) \frac{(\Upsilon + \Upsilon_1) |\Phi_0| + \left(1 + \sum_{\gamma=1}^{\mathcal{K}} \Delta_\gamma\right) (M + (\Upsilon + \Upsilon_1) L\eta)}{\left(1 + \sum_{\gamma=1}^{\mathcal{K}} \Delta_\gamma\right) - (\Upsilon + \Upsilon_1)\eta \sum_{\gamma=1}^{\mathcal{K}} \Delta_\gamma} = (\wp_2 - \wp_1) L.$$

This shows that $A : P_L \to P_L$; the class of functions $\{A\Phi\}$ is uniformly equi-continuous and bounded in P_L.

Let $\Phi_n \in P_L, \Phi_n \to \Phi$ $(n \to \infty)$, then from hypotheses **(A1)-(A2)**, we get $\Im(\wp, \Phi_n(\wp), \Phi_n(\wp)) \to \Im(\wp, \Phi(\wp), \Phi(\wp))$, $\xi(\wp, \Phi_n(\wp)) \to \xi(\wp, \Phi(\wp))$ and $\vartheta(\wp, \Phi_n(\wp)) \to \vartheta(\wp, \Phi(\wp))$ as $n \to \infty$. Also

$$\begin{aligned}
&\lim_{n\to\infty} A\Phi_n(\wp) \\
&= \lim_{n\to\infty} \Big(\frac{1}{1 + \sum_{\gamma=1}^{\Theta} \Delta_\gamma} \Big[\Phi_0 - \sum_{\gamma=1}^{\Theta} \Delta_\gamma \int_0^{\Omega(\rho_\gamma)} \\
&\quad \times \Im\left(\hbar, \int_0^{\Omega(\hbar)} \Phi_n\ \xi(\theta, \Phi_n(\theta))\, \mathrm{d}\theta, \int_0^{\eta} \Phi_n\ \vartheta(\theta, \Phi_n(\theta))\, \mathrm{d}\theta\right) \mathrm{d}\hbar \Big] \\
&\quad + \int_0^{\wp} \Im\left(\hbar, \int_0^{\Omega(\hbar)} \Phi_n\ \xi(\theta, \Phi_n(\theta))\, \mathrm{d}\theta, \int_0^{\eta} \Phi_n\ \vartheta(\theta, \Phi_n(\theta))\, \mathrm{d}\theta\right) \mathrm{d}\hbar \Big).
\end{aligned} \tag{3.4}$$

Now

$$\begin{aligned}
&\left| \int_0^{\Omega(\hbar)} \Phi_n\ \xi(\theta, \Phi_n(\theta))\, \mathrm{d}\theta - \int_0^{\Omega(\hbar)} \Phi\ \xi(\theta, \Phi(\theta)) \mathrm{d}\theta \right| \\
\leq & \left| \int_0^{\Omega(\hbar)} \Phi_n\ \xi(\theta, \Phi_n(\theta))\, \mathrm{d}\theta - \int_0^{\Omega(\hbar)} \Phi_n\ \xi(\theta, \Phi(\theta)) \mathrm{d}\theta \right| \\
&+ \left| \int_0^{\Omega(\hbar)} \Phi_n\ \xi(\theta, \Phi(\theta)) \mathrm{d}\theta - \int_0^{\Omega(\hbar)} \Phi\ \xi(\theta, \Phi(\theta)) \mathrm{d}\theta \right| \\
\leq & L \int_0^{\Omega(\hbar)} |\xi(\theta, \Phi_n(\theta)) - \xi(\theta, \Phi(\theta))|\, \mathrm{d}\theta + \frac{\epsilon}{2} \\
\leq & \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
&\left|\int_0^{\eta} \Phi_n\ \vartheta\left(\theta, \Phi_n(\theta)\right) \mathrm{d}\theta-\int_0^{\eta} \Phi\ \vartheta(\theta, \Phi(\theta)) \mathrm{d}\theta\right| \\
\leq &\left|\int_0^{\eta} \Phi_n\ \vartheta\left(\theta, \Phi_n(\theta)\right) \mathrm{d}\theta-\int_0^{\eta} \Phi_n\ \vartheta(\theta, \Phi(\theta)) \mathrm{d}\theta\right| \\
&+\left|\int_0^{\eta} \Phi_n\ \vartheta(\theta, \Phi(\theta)) \mathrm{d}\theta-\int_0^{\eta} \Phi\ \vartheta(\theta, \Phi(\theta)) \mathrm{d}\theta\right| \\
\leq &L \int_0^{\eta}\left|\vartheta\left(\theta, \Phi_n(\theta)\right)-\vartheta(\theta, \Phi(\theta))\right| \mathrm{d}\theta+\frac{\epsilon}{2} \\
\leq &\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.
\end{aligned} \tag{3.6}$$

Using (3.4), (3.5) and (3.6) and The dominated convergence theorem of Lebesgue [23], we get

$$\begin{aligned}
&\lim_{n \rightarrow \infty} A \Phi_n(\wp) \\
=&\frac{1}{1+\sum_{\gamma=1}^{\Theta} \Delta_\gamma} \\
&\times\left[\Phi_0-\sum_{\gamma=1}^{\Theta} \Delta_\gamma \int_0^{\Omega(\rho_\gamma)} \lim_{n \rightarrow \infty} \Im\left(\hbar, \int_0^{\Omega(\hbar)} \Phi_n\ \xi\left(\theta, \Phi_n(\theta)\right) \mathrm{d}\theta \int_0^{\eta} \Phi_n\ \vartheta\left(\theta, \Phi_n(\theta)\right) \mathrm{d}\theta\right) \mathrm{d}\hbar\right] \\
&+\int_0^{\wp} \lim_{n \rightarrow \infty} \Im\left(\hbar, \int_0^{\Omega(\hbar)} \Phi_n\ \xi\left(\theta, \Phi_n(\theta)\right) \mathrm{d}\theta, \int_0^{\eta} \Phi_n\ \vartheta\left(\theta, \Phi_n(\theta)\right) \mathrm{d}\theta\right) \mathrm{d}\hbar \\
=&A \Phi(\wp).
\end{aligned}$$

Then $A\Phi_n \rightarrow A\Phi$ as $n \rightarrow \infty$, the operator A is continuous.

Therefore, by the Schauder's theorem [23], for Eq. (2.2), there exists at least one solution $\Phi \in C[0, \eta]$; consequently, for the system (1.1)-(1.2), there exists at least one solution $\Phi \in AC[0, \eta]$.

Remark 3.2. We may infer from these hypotheses that there is a function $\Im \in \Psi(\wp, \Phi, \Lambda)$ such that,

(B1). $\Im : [0, \eta] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is measurable in $\wp$ for any $\Phi \in \mathbb{R}$ and satisfies the Lipschitz condition

$$|\Im(\wp, \Phi, \Lambda)-\Im(\wp, M, N)| \leq \Upsilon|\Phi-M|+\Upsilon^*|\Lambda-N|.$$

(B2). $\xi, \vartheta : [0, \eta] \times \mathbb{R} \rightarrow \mathbb{R}$ be measurable in $\wp$, $\forall\ \Phi \in \mathbb{R}$ and satisfies the Lipschitz conditions

$$|\xi(\wp, \Phi)-\xi(\wp, q)| \leq \Upsilon_1|\Phi-q|,\ |\vartheta(\wp, \Phi)-\vartheta(\wp, q)| \leq \Upsilon_2|\Phi-q|$$

(B3).

$$\left(\frac{1+2\sum_{\gamma=1}^{\Theta}\Delta_\gamma}{1+\sum_{\gamma=1}^{\Theta}\Delta_\gamma}\left((\Upsilon\Upsilon_1+\Upsilon^*\Upsilon_2)L\eta^2+(\Upsilon+\Upsilon^*)\eta\right)\right)<1.$$

Theorem 3.3. *If **(B1)-(B3)** hold, then the solution of system* (1.1)-(1.2) *is unique.*

Proof. Let ϖ, Φ be two the solutions of system (1.1)-(1.2). Then

$$\begin{aligned}
&|\varpi(\wp)-\Phi(\wp)|\\
\leq&\frac{1}{1+\sum_{\gamma=1}^{\Theta}\Delta_\gamma}\left(\sum_{\gamma=1}^{\Theta}\Delta_\gamma\int_0^{\Omega(\rho_\gamma)}\left|\Im\left(\wp,\int_0^{\Omega(\hbar)}\varpi\ \xi(\theta,\varpi(\theta))\mathrm{d}\theta,\int_0^{\eta}\varpi\ \vartheta(\theta,\varpi(\theta))\mathrm{d}\theta\right)\right.\right.\\
&\left.\left.-\Im\left(\wp,\int_0^{\Omega(\hbar)}\Phi\ \xi(\theta,\Phi(\theta))\mathrm{d}\theta,\int_0^{\eta}\Phi\ \vartheta(\theta,\Phi(\theta))\mathrm{d}\theta\right)\right|\mathrm{d}\hbar\right)\\
&+\int_0^{\wp}\left|\Im\left(\hbar,\int_0^{\Omega(\hbar)}\varpi\ \xi(\theta,\varpi(\theta))\mathrm{d}\theta,\int_0^{\eta}\varpi\ \vartheta(\theta,\varpi(\theta))\mathrm{d}\theta\right)\right.\\
&\left.-\Im\left(\hbar,\int_0^{\Omega(\hbar)}\Phi\ \xi(\theta,\Phi(\theta))\mathrm{d}\theta,\int_0^{\eta}\Phi\ \vartheta(\theta,\Phi(\theta))\mathrm{d}\theta\right)\right|\mathrm{d}\hbar
\end{aligned}$$

hence

$$\begin{aligned}
&|\varpi(\wp)-\Phi(\wp)|\\
\leq&\frac{\Upsilon}{1+\sum_{\gamma=1}^{\Theta}\Delta_\gamma}\sum_{\gamma=1}^{\Theta}\Delta_\gamma\int_0^{\Omega(\rho_\gamma)}\left|\int_0^{\Omega(\hbar)}\varpi\ \xi(\theta,\varpi(\theta))\mathrm{d}\theta-\int_0^{\Omega(\hbar)}\Phi\ \xi(\theta,\Phi(\theta))\mathrm{d}\theta\right|\mathrm{d}\hbar\\
&+\frac{\Upsilon^*}{1+\sum_{\gamma=1}^{\Theta}\Delta_\gamma}\sum_{\gamma=1}^{\Theta}\Delta_\gamma\int_0^{\Omega(\rho_\gamma)}\left|\int_0^{\eta}\varpi\ \vartheta(\theta,\varpi(\theta))\mathrm{d}\theta-\int_0^{\eta}\Phi\ \vartheta(\theta,\Phi(\theta))\mathrm{d}\theta\right|\mathrm{d}\hbar\\
&+\Upsilon\int_0^{\wp}\left|\int_0^{\Omega(\hbar)}\varpi\ \xi(\theta,\varpi(\theta))\mathrm{d}\theta-\int_0^{\Omega(\hbar)}\Phi\ \xi(\theta,\Phi(\theta))\mathrm{d}\theta\right|\mathrm{d}\hbar\\
&+\Upsilon^*\int_0^{\wp}\left|\int_0^{\eta}\varpi\ \vartheta(\theta,\varpi(\theta))\mathrm{d}\theta-\int_0^{\eta}\Phi\ \vartheta(\theta,\Phi(\theta))\mathrm{d}\theta\right|\mathrm{d}\hbar.
\end{aligned}$$

Then, we get

$$
\begin{aligned}
&|\varpi(\wp)-\Phi(\wp)|\\
\leq&\frac{\Upsilon}{1+\sum_{\gamma=1}^{\Theta}\Delta_\gamma}\sum_{\gamma=1}^{\Theta}\Delta_\gamma\int_0^{\Omega(\rho_\gamma)}\left|\int_0^{\Omega(\hbar)}\varpi\ \xi(\theta,\varpi(\theta))\mathrm{d}\theta-\int_0^{\Omega(\hbar)}\varpi\ \xi(\theta,\Phi(\theta))\mathrm{d}\theta\right|\mathrm{d}\hbar\\
&+\frac{\Upsilon}{1+\sum_{\gamma=1}^{\Theta}\Delta_\gamma}\sum_{\gamma=1}^{\Theta}\Delta_\gamma\int_0^{\Omega(\rho_\gamma)}\left|\int_0^{\Omega(\hbar)}\varpi\ \xi(\theta,\Phi(\theta))\mathrm{d}\theta-\int_0^{\Omega(\hbar)}\Phi\ \xi(\theta,\Phi(\theta))\mathrm{d}\theta\right|\mathrm{d}\hbar\\
&+\frac{\Upsilon^*}{1+\sum_{\gamma=1}^{\Theta}\Delta_\gamma}\sum_{\gamma=1}^{\Theta}\Delta_\gamma\int_0^{\Omega(\rho_\gamma)}\left|\int_0^{\eta}\varpi\ \vartheta(\theta,\varpi(\theta))\mathrm{d}\theta-\int_0^{\eta}\varpi\ \vartheta(\theta,\Phi(\theta))\mathrm{d}\theta\right|\mathrm{d}\hbar\\
&+\frac{\Upsilon^*}{1+\sum_{\gamma=1}^{\Theta}\Delta_\gamma}\sum_{\gamma=1}^{\Theta}\Delta_\gamma\int_0^{\Omega(\rho_\gamma)}\left|\int_0^{\eta}\varpi\ \vartheta(\theta,\Phi(\theta))\mathrm{d}\theta-\int_0^{\eta}\Phi\ \vartheta(\theta,\Phi(\theta))\mathrm{d}\theta\right|\mathrm{d}\hbar\\
&+\Upsilon\int_0^{\wp}\left|\int_0^{\Omega(\hbar)}\varpi\ \xi(\theta,\varpi(\theta))\mathrm{d}\theta-\int_0^{\Omega(\hbar)}\varpi\ \xi(\theta,\Phi(\theta))\mathrm{d}\theta\right|\mathrm{d}\hbar\\
&+\Upsilon\int_0^{\wp}\left|\int_0^{\Omega(\hbar)}\varpi\ \xi(\theta,\Phi(\theta))\mathrm{d}\theta-\int_0^{\Omega(\hbar)}\Phi\ \xi(\theta,\Phi(\theta))\mathrm{d}\theta\right|\mathrm{d}\hbar\\
&+\Upsilon^*\int_0^{\wp}\left|\int_0^{\eta}\varpi\ \vartheta(\theta,\varpi(\theta))\mathrm{d}\theta-\int_0^{\eta}\varpi\ \vartheta(\theta,\Phi(\theta))\mathrm{d}\theta\right|\mathrm{d}\hbar\\
&+\Upsilon^*\int_0^{\wp}\left|\int_0^{\eta}\varpi\ \vartheta(\theta,\Phi(\theta))\mathrm{d}\theta-\int_0^{\eta}\Phi\ \vartheta(\theta,\Phi(\theta))\mathrm{d}\theta\right|\mathrm{d}\hbar
\end{aligned}
$$

which diminishes to

$$
\begin{aligned}
&|\varpi(\wp)-\Phi(\wp)|\\
\leq&\frac{\Upsilon L}{1+\sum_{\gamma=1}^{\Theta}\Delta_\gamma}\sum_{\gamma=1}^{\Theta}\Delta_\gamma\left(\int_0^{\Omega(\rho_\gamma)}\int_0^{\Omega(\hbar)}|\xi(\theta,\varpi(\theta))-\xi(\theta,\Phi(\theta))|\mathrm{d}\theta\mathrm{d}\hbar+\eta\|\varpi-\Phi\|\right)\\
&+\frac{\Upsilon^* L}{1+\sum_{\gamma=1}^{\Theta}\Delta_\gamma}\sum_{\gamma=1}^{\Theta}\Delta_\gamma\left(\int_0^{\Omega(\rho_\gamma)}\int_0^{\eta}|\vartheta(\theta,\varpi(\theta))-\vartheta(\theta,\Phi(\theta))|\mathrm{d}\theta\mathrm{d}\hbar+\eta\|\varpi-\Phi\|\right)\\
&+\Upsilon L\int_0^{\wp}\int_0^{\Omega(\hbar)}|\xi(\theta,\varpi(\theta))-\xi(\theta,\Phi(\theta))|\mathrm{d}\theta\mathrm{d}\hbar+\Upsilon\eta\|\varpi-\Phi\|\\
&+\Upsilon^* L\int_0^{\wp}\int_0^{\eta}|\vartheta(\theta,\varpi(\theta))-\vartheta(\theta,\Phi(\theta))|\mathrm{d}\theta\mathrm{d}\hbar+\Upsilon^*\eta\|\varpi-\Phi\|.
\end{aligned}
$$

Then, we get

$$
\begin{aligned}
&|\varpi(\wp)-\Phi(\wp)| \\
\leq&\frac{\Upsilon L}{1+\sum_{\gamma=1}^{\Theta}\Delta_\gamma}\sum_{\gamma=1}^{\Theta}\Delta_\gamma\left(\Upsilon_1\int_0^{\Omega(\rho_\gamma)}\int_0^{\Omega(\hbar)}|\varpi(\theta)-\Phi(\theta)|\mathrm{d}\theta\mathrm{d}\hbar+\eta\|\varpi-\Phi\|\right) \\
&+\frac{\Upsilon^* L}{1+\sum_{\gamma=1}^{\Theta}\Delta_\gamma}\sum_{\gamma=1}^{\Theta}\Delta_\gamma\left(\Upsilon_2\int_0^{\Omega(\rho_\gamma)}\int_0^{\eta}|\varpi(\theta)-\Phi(\theta)|\mathrm{d}\theta\mathrm{d}\hbar+\eta\|\varpi-\Phi\|\right) \\
&+\Upsilon\Upsilon_1 L\int_0^{\wp}\int_0^{\Omega(\hbar)}|\varpi(\theta)-\Phi(\theta)|\mathrm{d}\theta\mathrm{d}\hbar+\Upsilon\eta\|\varpi-\Phi\| \\
&+\Upsilon^*\Upsilon_2 L\int_0^{\wp}\int_0^{\eta}|\varpi(\theta)-\Phi(\theta)|\mathrm{d}\theta\mathrm{d}\hbar+\Upsilon^*\eta\|\varpi-\Phi\| \\
\leq&\frac{\sum_{\gamma=1}^{\Theta}\Delta_\gamma}{1+\sum_{\gamma=1}^{\Theta}\Delta_\gamma}\left(\Upsilon\Upsilon_1 L\eta^2+\Upsilon\eta\right)\|\varpi-\Phi\|+\left(\Upsilon\Upsilon_1 L\eta^2+\Upsilon\eta\right)\|\varpi-\Phi\| \\
&+\frac{\sum_{\gamma=1}^{\Theta}\Delta_\gamma}{1+\sum_{\gamma=1}^{\Theta}\Delta_\gamma}\left(\Upsilon^*\Upsilon_2 L\eta^2+\Upsilon^*\eta\right)\|\varpi-\Phi\|+\left(\Upsilon^*\Upsilon_2 L\eta^2+\Upsilon^*\eta\right)\|\varpi-\Phi\| \\
=&\frac{1+2\sum_{\gamma=1}^{\Theta}\Delta_\gamma}{1+\sum_{\gamma=1}^{\Theta}\Delta_\gamma}\left((\Upsilon\Upsilon_1+\Upsilon^*\Upsilon_2)L\eta^2+(\Upsilon+\Upsilon^*)\eta\right)\|\varpi-\Phi\|.
\end{aligned}
$$

Hence,

$$\left(1-\left(\frac{1+2\sum_{\gamma=1}^{\Theta}\Delta_\gamma}{1+\sum_{\gamma=1}^{\Theta}\Delta_\gamma}\left((\Upsilon\Upsilon_1+\Upsilon^*\Upsilon_2)L\eta^2+(\Upsilon+\Upsilon^*)\eta\right)\right)\right)\|\varpi-\Phi\|\leq 0.$$

Since

$$\frac{1+2\sum_{\gamma=1}^{K}\Delta_\gamma}{1+\sum_{\gamma=1}^{K}\Delta_\gamma}\left((\Upsilon\Upsilon_1+\Upsilon^*\Upsilon_2)L\eta^2+(\Upsilon+\Upsilon^*)\eta\right)<1,$$

which implies $\Phi(\wp)=\varpi(\wp)$ and the solution of the system (1.1)-(1.2) is unique.

Theorem 3.4. *Let **(A4)**, **(B1)-(B3)** hold, then the solution of the system* (1.1)-(1.2) *depends continuously on* Φ_0.

Proof. Let Φ^* be a solution of the integral equation

$$\begin{aligned}
&\Phi(\wp)\\
=&\frac{1}{1+\sum_{\gamma=1}^{\Theta}\Delta_\gamma}\\
&\times\left[\Phi_0^* - \sum_{\gamma=1}^{\Theta}\Delta_\gamma \int_0^{\Omega(\rho_\gamma)} \Im\left(\hbar, \int_0^{\Omega(\hbar)} \Phi\ \xi(\theta,\Phi(\theta))\mathrm{d}\theta, \int_0^{\eta} \Phi\ \vartheta(\theta,\Phi(\theta))\mathrm{d}\theta\right)\mathrm{d}\hbar\right]\\
&+\int_0^{\wp} \Im\left(\hbar, \int_0^{\Omega(\hbar)} \Phi\ \xi(\theta,\Phi(\theta))\mathrm{d}\theta, \int_0^{\eta} \Phi\ \vartheta(\theta,\Phi(\theta))\mathrm{d}\theta\right)\mathrm{d}\hbar,
\end{aligned}$$

such that $|\Phi_0 - \Phi_0^*| < \delta$. Then

$$\begin{aligned}
&|\Phi(\wp)-\Phi^*(\wp)|\\
\le&\frac{1}{1+\sum_{\gamma=1}^{\Theta}\Delta_\gamma}|\Phi_0-\Phi_0^*| + \frac{1}{1+\sum_{\gamma=1}^{\Theta}\Delta_\gamma}\\
&\times\left(\sum_{\gamma=1}^{\Theta}\Delta_\gamma \int_0^{\Omega(\rho_\gamma)} \left|\Im\left(\wp, \int_0^{\Omega(\hbar)} \Phi\ \xi(\theta,\Phi(\theta))\mathrm{d}\theta, \int_0^{\eta} \Phi\ \vartheta(\theta,\Phi(\theta))\mathrm{d}\theta\right)\right.\right.\\
&\left.\left.-\Im\left(\wp, \int_0^{\Omega(\hbar)} \Phi^*\ \xi\left(\theta,\Phi^*(\theta)\right)\mathrm{d}\theta, \int_0^{\eta} \Phi^*\ \vartheta\left(\theta,\Phi^*(\theta)\right)\mathrm{d}\theta\right)\right|\mathrm{d}\hbar\right)\\
&+\int_0^{\wp}\left|\Im\left(\hbar, \int_0^{\Omega(\hbar)} \Phi\ \xi(\theta,\Phi(\theta))\mathrm{d}\theta, \int_0^{\eta} \Phi\ \vartheta(\theta,\Phi(\theta))\mathrm{d}\theta\right)\right.\\
&\left.-\Im\left(\hbar, \int_0^{\Omega(\hbar)} \Phi^*\ \xi\left(\theta,\Phi^*(\theta)\right)\mathrm{d}\theta, \int_0^{\eta} \Phi^*\ \vartheta\left(\theta,\Phi^*(\theta)\right)\mathrm{d}\theta\right)\right|\mathrm{d}\hbar
\end{aligned}$$

then

$$\begin{aligned}
&|\Phi(\wp)-\Phi^*(\wp)|\\
\le&\frac{1}{1+\sum_{\gamma=1}^{\Theta}\Delta_\gamma}\delta\\
&+\Upsilon\frac{1}{1+\sum_{\gamma=1}^{\Theta}\Delta_\gamma}\sum_{\gamma=1}^{\Theta}\Delta_\gamma\int_0^{\Omega(\rho_\gamma)}\left|\int_0^{\Omega(\hbar)} \Phi\ \xi(\theta,\Phi(\theta))\mathrm{d}\theta - \int_0^{\Omega(\hbar)} \Phi^*\ \xi\left(\theta,\Phi^*(\theta)\right)\mathrm{d}\theta\right|\mathrm{d}\hbar\\
&+\Upsilon^*\frac{1}{1+\sum_{\gamma=1}^{\Theta}\Delta_\gamma}\sum_{\gamma=1}^{\Theta}\Delta_\gamma\int_0^{\Omega(\rho_\gamma)}\left|\int_0^{\eta} \Phi\ \vartheta(\theta,\Phi(\theta))\mathrm{d}\theta - \int_0^{\eta} \Phi^*\ \vartheta\left(\theta,\Phi^*(\theta)\right)\mathrm{d}\theta\right|\mathrm{d}\hbar
\end{aligned}$$

$$+\Upsilon\int_0^{\wp}\left|\int_0^{\Omega(\hbar)}\Phi\ \xi(\theta,\Phi(\theta))\mathrm{d}\theta-\int_0^{\Omega(\hbar)}\Phi^*\ \xi\left(\theta,\Phi^*(\theta)\right)\mathrm{d}\theta\right|\mathrm{d}\hbar$$
$$+\Upsilon^*\int_0^{\wp}\left|\int_0^{\eta}\Phi\ \vartheta(\theta,\Phi(\theta))\mathrm{d}\theta-\int_0^{\eta}\Phi^*\ \vartheta\left(\theta,\Phi^*(\theta)\right)\mathrm{d}\theta\right|\mathrm{d}\hbar.$$

Then, we get

$$\begin{aligned}
&|\Phi(\wp)-\Phi^*(\wp)|\\
\leq&\frac{1}{1+\sum_{\gamma=1}^{\Theta}\Delta_\gamma}\delta\\
&+\Upsilon\frac{1}{1+\sum_{\gamma=1}^{\Theta}\Delta_\gamma}\sum_{\gamma=1}^{\Theta}\Delta_\gamma\int_0^{\Omega(\rho_\gamma)}\left|\int_0^{\Omega(\hbar)}\Phi\ \xi(\theta,\Phi(\theta))\mathrm{d}\theta-\int_0^{\Omega(\hbar)}\Phi\ \xi\left(\theta,\Phi^*(\theta)\right)\mathrm{d}\theta\right|\mathrm{d}\hbar\\
&+\Upsilon\frac{1}{1+\sum_{\gamma=1}^{\Theta}\Delta_\gamma}\sum_{\gamma=1}^{\Theta}\Delta_\gamma\int_0^{\Omega(\rho_\gamma)}\left|\int_0^{\Omega(\hbar)}\Phi\ \xi\left(\theta,\Phi^*(\theta)\right)\mathrm{d}\theta-\int_0^{\Omega(\hbar)}\Phi^*\ \xi\left(\theta,\Phi^*(\theta)\right)\mathrm{d}\theta\right|\mathrm{d}\hbar\\
&+\Upsilon^*\frac{1}{1+\sum_{\gamma=1}^{\Theta}\Delta_\gamma}\sum_{\gamma=1}^{\Theta}\Delta_\gamma\int_0^{\Omega(\rho_\gamma)}\left|\int_0^{\eta}\Phi\ \vartheta(\theta,\Phi(\theta))\mathrm{d}\theta-\int_0^{\eta}\Phi\ \vartheta\left(\theta,\Phi^*(\theta)\right)\mathrm{d}\theta\right|\mathrm{d}\hbar\\
&+\Upsilon^*\frac{1}{1+\sum_{\gamma=1}^{\Theta}\Delta_\gamma}\sum_{\gamma=1}^{\Theta}\Delta_\gamma\int_0^{\Omega(\rho_\gamma)}\left|\int_0^{\eta}\Phi\ \vartheta\left(\theta,\Phi^*(\theta)\right)\mathrm{d}\theta-\int_0^{\eta}\Phi^*\ \vartheta\left(\theta,\Phi^*(\theta)\right)\mathrm{d}\theta\right|\mathrm{d}\hbar\\
&+\Upsilon\int_0^{\wp}\left|\int_0^{\Omega(\hbar)}\Phi\ \xi(\theta,\Phi(\theta))\mathrm{d}\theta-\int_0^{\Omega(\hbar)}\Phi\ \xi\left(\theta,\Phi^*(\theta)\right)\mathrm{d}\theta\right|\mathrm{d}\hbar\\
&+\Upsilon\int_0^{\wp}\left|\int_0^{\Omega(\hbar)}\Phi\ \xi\left(\theta,\Phi^*(\theta)\right)\mathrm{d}\theta-\int_0^{\Omega(\hbar)}\Phi^*\ \xi\left(\theta,\Phi^*(\theta)\right)\mathrm{d}\theta\right|\mathrm{d}\hbar\\
&+\Upsilon^*\int_0^{\wp}\left|\int_0^{\eta}\Phi\ \vartheta(\theta,\Phi(\theta))\mathrm{d}\theta-\int_0^{\eta}\Phi\ \vartheta\left(\theta,\Phi^*(\theta)\right)\mathrm{d}\theta\right|\mathrm{d}\hbar\\
&+\Upsilon^*\int_0^{\wp}\left|\int_0^{\eta}\Phi\ \vartheta\left(\theta,\Phi^*(\theta)\right)\mathrm{d}\theta-\int_0^{\eta}\Phi^*\ \vartheta\left(\theta,\Phi^*(\theta)\right)\mathrm{d}\theta\right|\mathrm{d}\hbar
\end{aligned}$$

which diminishes to

$$\begin{aligned}
&|\Phi(\wp)-\Phi^*(\wp)|\\
\leq&\frac{1}{1+\sum_{\gamma=1}^{\Theta}\Delta_\gamma}\delta\\
&+\Upsilon L\frac{1}{1+\sum_{\gamma=1}^{\Theta}\Delta_\gamma}\sum_{\gamma=1}^{\Theta}\Delta_\gamma\left(\int_0^{\Omega(\rho_\gamma)}\int_0^{\Omega(\hbar)}\left|\xi(\theta,\Phi(\theta))-\xi\left(\theta,\Phi^*(\theta)\right)\right|\mathrm{d}\theta\mathrm{d}\hbar+\eta\left\|\Phi-\Phi^*\right\|\right)
\end{aligned}$$

$$+\Upsilon^* L \frac{1}{1+\sum_{\gamma=1}^{\Theta}\Delta_\gamma}\sum_{\gamma=1}^{\Theta}\Delta_\gamma\left(\int_0^{\Omega(\rho_\gamma)}\int_0^{\eta}|\vartheta(\theta,\Phi(\theta))-\vartheta(\theta,\Phi^*(\theta))|\,\mathrm{d}\theta\mathrm{d}\hbar+\eta\|\Phi-\Phi^*\|\right)$$
$$+\Upsilon L\int_0^{\wp}\int_0^{\Omega(\hbar)}|\xi(\theta,\Phi(\theta))-\xi(\theta,\Phi^*(\theta))|\,\mathrm{d}\theta\mathrm{d}\hbar+\Upsilon\eta\|\Phi-\Phi^*\|$$
$$+\Upsilon^* L\int_0^{\wp}\int_0^{\eta}|\vartheta(\theta,\Phi(\theta))-\vartheta(\theta,\Phi^*(\theta))|\,\mathrm{d}\theta\mathrm{d}\hbar+\Upsilon^*\eta\|\Phi-\Phi^*\|.$$

Then, we have

$$\begin{aligned}
&|\Phi(\wp)-\Phi^*(\wp)|\\
\leq&\frac{1}{1+\sum_{\gamma=1}^{\Theta}\Delta_\gamma}\delta\\
&+\Upsilon L\frac{1}{1+\sum_{\gamma=1}^{\Theta}\Delta_\gamma}\sum_{\gamma=1}^{\Theta}\Delta_\gamma\left(\Upsilon_1\int_0^{\Omega(\rho_\gamma)}\int_0^{\Omega(\hbar)}|\Phi(\theta)-\Phi^*(\theta)|\,\mathrm{d}\theta\mathrm{d}\hbar+\eta\|\Phi-\Phi^*\|\right)\\
&+\Upsilon^* L\frac{1}{1+\sum_{\gamma=1}^{\Theta}\Delta_\gamma}\sum_{\gamma=1}^{\Theta}\Delta_\gamma\left(\Upsilon_2\int_0^{\Omega(\rho_\gamma)}\int_0^{\eta}|\Phi(\theta)-\Phi^*(\theta)|\,\mathrm{d}\theta\mathrm{d}\hbar+\eta\|\Phi-\Phi^*\|\right)\\
&+\Upsilon\Upsilon_1 L\int_0^{\wp}\int_0^{\Omega(\hbar)}|\Phi(\theta)-\Phi^*(\theta)|\,\mathrm{d}\theta\mathrm{d}\hbar+\Upsilon\eta\|\Phi-\Phi^*\|\\
&+\Upsilon^*\Upsilon_2 L\int_0^{\wp}\int_0^{\eta}|\Phi(\theta)-\Phi^*(\theta)|\,\mathrm{d}\theta\mathrm{d}\hbar+\Upsilon^*\eta\|\Phi-\Phi^*\|\\
\leq&\frac{\delta}{1+\sum_{\gamma=1}^{K}\Delta_\gamma}+\frac{\sum_{\gamma=1}^{\Theta}\Delta_\gamma}{1+\sum_{\gamma=1}^{K}\Delta_\gamma}\left((\Upsilon\Upsilon_1+\Upsilon^*\Upsilon_2)L\eta^2+(\Upsilon+\Upsilon^*)\eta\right)\|\Phi-\Phi^*\|\\
&+\left((\Upsilon\Upsilon_1+\Upsilon^*\Upsilon_2)L\eta^2+(\Upsilon+\Upsilon^*)\eta\right)\|\Phi-\Phi^*\|\\
=&\frac{\delta}{1+\sum_{\gamma=1}^{K}\Delta_\gamma}+\frac{1+2\sum_{\gamma=1}^{\Theta}\Delta_\gamma}{1+\sum_{\gamma=1}^{\Theta}\Delta_\gamma}\left((\Upsilon\Upsilon_1+\Upsilon^*\Upsilon_2)L\eta^2+(\Upsilon+\Upsilon^*)\eta\right)\|\Phi-\Phi^*\|.
\end{aligned}$$

Hence

$$\|\Phi-\Phi^*\|\leq\frac{\frac{\delta}{1+\sum_{\gamma=1}^{k}\Delta_\gamma}}{1-\left(\frac{1+2\sum_{\gamma=1}^{K}\Delta_\gamma}{1+\sum_{\gamma=1}^{\Theta}\Delta_\gamma}\left((\Upsilon\Upsilon_1+\Upsilon^*\Upsilon_2)L\eta^2+(\Upsilon+\Upsilon^*)\eta\right)\right)}=\epsilon.$$

Then the solution of the system (1.1)-(1.2) depends continuously on Φ_0.

4. Applications

Example 1. Consider the V-FIDE

$$\frac{\mathrm{d}\Phi}{\mathrm{d}\wp} = \frac{1}{3}\wp^3 + \frac{1}{\wp^2+5}\int_0^{\beta\wp} \frac{\Phi\cos^2\Phi}{1+e^{\Phi(\hbar)}}d\hbar + \frac{1}{\sqrt{\wp+16}}\int_0^1 \frac{\Phi\sin^2(\Phi)}{1+\hbar^2}\,\mathrm{d}\hbar, \quad \wp,\beta\in(0,1], \tag{4.1}$$

$$\Phi(0) + \sum_{\gamma=1}^{\infty}\frac{1}{\gamma^4}\Phi\left(\frac{\gamma-1}{\gamma}\right) = 1 \tag{4.2}$$

$$\begin{aligned}
&\Phi(\wp) \\
=&\frac{1}{1+\sum_{\gamma=1}^{\Theta}\gamma^4}\Big[1-\sum_{\gamma=1}^{\Theta}\gamma^6\int_0^{\frac{\gamma-1}{\gamma}} \\
&\times\left(\frac{1}{3}s^3+\frac{1}{s^2+5}\int_0^{\beta s}\frac{\Phi\ \cos^2\Phi}{1+e^{\Phi(\hbar)}}\mathrm{d}\hbar+\frac{1}{\sqrt{s+16}}\int_0^1\frac{\Phi\sin^2(\Phi)}{1+\hbar^2}\,\mathrm{d}\hbar\right)\mathrm{d}s\Big] \\
&+\int_0^{\wp}\left(\frac{1}{3}s^3+\frac{1}{s^2+5}\int_0^{\beta s}\frac{\Phi\ \cos^2\Phi}{1+e^{\Phi(\hbar)}}\mathrm{d}\hbar+\frac{1}{\sqrt{s+16}}\int_0^1\frac{\Phi\sin^2(\Phi)}{1+\hbar^2}\,\mathrm{d}\hbar\right)\mathrm{d}s.
\end{aligned} \tag{4.3}$$

Set

$$\begin{aligned}
&\Im\left(\wp,\int_0^{\Omega(\wp)}\Phi\ \xi(\hbar,\Phi(\hbar))\mathrm{d}\hbar,\int_0^{\eta}\Phi\ \vartheta(\hbar,\Phi(\hbar))\mathrm{d}\hbar\right) \\
=\ &\frac{1}{3}\wp^3+\frac{1}{\wp^2+5}\int_0^{\beta\wp}\frac{\Phi\ \cos^2\Phi}{1+e^{\Phi(\hbar)}}\mathrm{d}\hbar+\frac{1}{\sqrt{\wp+16}}\int_0^1\frac{\Phi\sin^2(\Phi)}{1+\hbar^2}\,\mathrm{d}\hbar.
\end{aligned}$$

Then

$$\left|\Im\left(\wp,\int_0^{\Omega(\wp)}\Phi\ \xi(\hbar,\Phi(\hbar))\mathrm{d}\hbar,\int_0^{\eta}\Phi\ \vartheta(\hbar,\Phi(\hbar))\mathrm{d}\hbar\right)\right| \le \frac{\wp^3}{3}+\frac{\Phi}{5}+\frac{\Phi}{4},$$

and also

$$|\xi(\hbar,\Phi(\hbar))|\le 1,\ \ |\vartheta(\hbar,\Phi(\hbar))|\le 1.$$

We can see that the hypotheses **(A1)-(A4)** of Theorem 3.1 are hold with $|c(\wp)| = \left|\frac{1}{3}\wp^3\right| \le \frac{1}{3}$ is measurable bounded, $\Upsilon = \frac{1}{5}, \Upsilon^* = \frac{1}{4}, L = \left(1+2\frac{\pi^4}{1035}\right)(\frac{1}{5}+\frac{1}{4}) \simeq 0.5347$ and the series $\sum_{\gamma=1}^{\infty}\frac{1}{\gamma^4}$ is convergent. Then, by applying to Theorem 3.1, the given system (4.1)-(4.2) has a solution given by (4.3).

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