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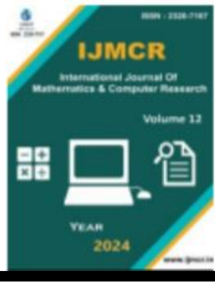

The Algebraic Solution of Dual Fuzzy Complex Linear Systems Formed $2n \times 2n$

Sri Wahyuni[1], Yuslenita Muda[2], Fitri Aryani[3], Rahmawati[4]

[1,2,3,4]Department of Mathematics, Faculty of Science and Technology, Universitas Islam Negeri Sultan Syarif Kasim Riau

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Corresponding Author:
Sri Wahyuni

ABSTRACT

The main objective of this research is to solve the system of equations of dual complex fuzzy matrices $A\tilde{X} + \tilde{Y} = B\tilde{X} + \tilde{Z}$ where A, B are crisp coefficient matrices of size n× n, and $\tilde{Y}, \tilde{Z}$ are complex fuzzy number matrices. By transforming it into the form of a system of equations of complex fuzzy matrices $S\tilde{X} + \tilde{Y} = R\tilde{X} + \tilde{Z}$ where $S \geq 0$ denotes the positive entries of matrix A, $S \leq 0$ denotes the absolute values of the negative entries of matrix A, $R \geq 0$ denotes the positive entries of matrix B, $R \leq 0$ denotes the absolute values of the negative entries of matrix B, and $\tilde{Y}, \tilde{Z}$ are matrices of complex fuzzy numbers. Based on the research, it is obtained that the solution of the system of equations of dual complex fuzzy matrices can be achieved by transforming the system of equations of $n \times n$ matrices into $2n \times 2n$ matrices, where $n \geq 2$.

KEYWORDS: Fuzzy Linear Systems, Dual Fuzzy Linear Systems, Complex Numbers

I. INTRODUCTION

One of the issues encountered in the field of mathematics is linear equation systems. Linear equation systems are widely used in various fields of science such as physics [1], optimization [2], economics [3], management and financial business [4], transportation planning [5], flow theory, and flow control [6]. In certain fields such as decision theory, control theory, and some areas of management science, the problems of linear equations involve not only real numbers but also fuzzy-based linear equations as their mathematical models [7].

The solution of fuzzy linear equation systems can be expressed as the matrix equation $A\tilde{X} = \tilde{Y}$, where A is a crisp-valued matrix and $\tilde{Y}$ is a vector-valued in fuzzy numbers, as first proposed by Fridman et al. [8], [9]. Furthermore, several methods for solving such fuzzy linear equation systems can be found in [10], [11], [12], [13], [14]. In paper [9], $n \times n$ complex fuzzy matrix is transformed into a $2n \times 2n$ form using the method provided in [15]. Subsequently, the complex form of $2n \times 2n$ fuzzy linear equation systems is developed into a dual form.

In recent years, the dual form of fuzzy linear equation systems has seen rapid development and has a broad application scope across various fields such as economics, finance, engineering, and physics [16]. In [17], the complex dual fuzzy linear equation systems are transformed into the form of complex dual fuzzy matrix equation systems written as $A\tilde{X} + \tilde{Y} = B\tilde{X} + \tilde{Z}$, where A and B are standard complex matrices, $\tilde{Y}$ and $\tilde{Z}$ are complex vectors on the left-hand side and right-hand side respectively, and $\tilde{X}$ is the unknown complex vector. Furthermore, in [18] the dual fuzzy linear systems are solved algebraically, with research findings indicating that the coefficient matrices are crisp-valued matrices and the left-hand and right-hand vectors are valued in fuzzy numbers.

Recent research on solving dual fuzzy matrix equation systems can be found in [19]. In [20], fuzzy linear equation systems are solved using numerical methods. Furthermore, [21] discusses systems of linear equations with coefficients as complex numbers and constants as complex fuzzy numbers using the Doolittle decomposition method. Based on the previous research, there has been no discussion on complex dual fuzzy linear equation systems. Hence, we propose a study on solving complex dual fuzzy linear equation systems.

II. PRELIMINARIES

Solving the system of equations of dual complex fuzzy matrices in $2n \times 2n$ form. The research method employed is a literature review. The following are some supporting theories used in the research.

Definition 2.1 [11]: A fuzzy set $\tilde{x}$ with membership function $\mu_{\tilde{x}}: \mathbb{R} \to [0,1]$, is called a fuzzy number if

i. There exists $t_0 \in \mathbb{R}$ such that $\mu_{\tilde{x}}(t_0) = 1$, i.e., $\tilde{x}$ normal.

ii. For any $\lambda \in [0,1]$ and $s, t \in \mathbb{R}$, we have $\mu_{\tilde{x}}(\lambda s + (1-\lambda)t) \geq min\,\{\mu_{\tilde{x}}(s), \mu_{\tilde{x}}(t)\}$, i.e., $\tilde{x}$ is a convex fuzzy set.

iii. For any $s \in \mathbb{R}$, the set $\{t \in \mathbb{R} : \mu_{\tilde{x}}(t) > s\}$ is an open set in $\mathbb{R}$, i.e., $\mu_{\tilde{x}}$ is upper semi-continuous on $\mathbb{R}$.

iv. The set $\overline{t \in \mathbb{R} : \mu_{\tilde{x}}(t) > 0\}}$ is a compact set in $\mathbb{R}$, where $\overline{A}$ denotes the closure of A.

We refer to the set of all fuzzy numbers as $\mathbb{F}$ in this study. In particular, we can argue that $\mathbb{R} \subset \mathbb{F}$ if we consider the real line $\mathbb{R}$ as $\mathbb{R} = \{X_t : t\}$, which represents real numbers [22]. additionally, the fuzzy number $\tilde{x}$ is α-level for $0 < \alpha \leq 1$ is defined as $[\tilde{x}]_\alpha = \{t \in R : \mu_{\tilde{x}}(t) \geq \alpha\}$ and for $\alpha = 0$ it is defined as $[\tilde{x}]_0 = \overline{\{t \in R : \mu_{\tilde{x}}(t) > 0\}}$. With the support of the fuzzy set defined as

$supp(\tilde{x}) = [\tilde{x}]_0 = \overline{\{t \in R : \mu_{\tilde{x}}(t) > 0\}}$.

Conditions (i) through (iv) suggest that for each $\alpha \in [0,1], [\tilde{x}]_\alpha$ is a bounded closed interval in $\mathbb{R}$ [15], by Definition 2.1. Here, the α-level of the fuzzy number $\tilde{x}$ is denoted by $[\tilde{x}]_\alpha = [\underline{x}(\alpha), \overline{x}(\alpha)]$,, for ever $\alpha \in [0,1]$. It is clear that if for every $\alpha \in [0,1]$, $\underline{x}(\alpha) = \overline{x}(\alpha)$, then $\tilde{x}$ is a crisp real number.

A necessary and sufficient condition for an interval member $[\underline{x}(\alpha), \overline{x}(\alpha)], 0 \leq \alpha \leq 1$, to be the α-level of a fuzzy number in $\mathbb{F}$ is stated by the following lemma.

Lemma 2.1 [23]: Let $\{[\underline{x}(\alpha), \overline{x}(\alpha)] : 0 \leq \alpha \leq 1\}$ be a certain non-empty set member in $\mathbb{R}$. If

i. $[\underline{x}(\alpha), \overline{x}(\alpha)]$ is a bounded closed interval, for every $\alpha \in [0,1]$,

ii. $[\underline{x}(\alpha_1), \overline{x}(\alpha_1)] \supseteq [\underline{x}(\alpha_2), \overline{x}(\alpha_2)]$ *for all* $0 \leq \alpha_1 \leq \alpha_2 \leq 1$,

iii. $\left[\lim_{k\to\infty} \underline{x}(\alpha_k), \lim_{k\to\infty} \overline{x}(\alpha_k)\right] = [\underline{x}(\alpha), \overline{x}(\alpha)]$ *whenever* $\{\alpha_k\}$ is a non-decreasing sequence in [0,1] converging to α.

Then the set $[\underline{x}(\alpha), \overline{x}(\alpha)]$ represents the α-level of the fuzzy number $\tilde{x}$ in $\mathbb{F}$. Conversely, if $[\underline{x}(\alpha), \overline{x}(\alpha)], 0 \leq \alpha \leq 1$, is the α-level of a fuzzy number $\tilde{x} \in \mathbb{F}$, then conditions (i) - (iii) are satisfied.

Remark 2.1 [14] By Lemma 2.2 it is concluded that if the family

$$\{[\underline{x}(\alpha), \overline{x}(\alpha)] : 0 \leq \alpha \leq 1\},$$

presents the α-levels of a fuzzy number, then:

1) The condition (i) implies the functions $\underline{x}$ and $\overline{x}$ are bounded over [0,1] and $\underline{x}(\alpha) \leq \overline{x}(\alpha)$ for each $\alpha \in [0,1]$.
2) The condition (ii) implies the functions $\underline{x}$ and $\overline{x}$ are non-decreasing over [0,1], respectively.
3) The condition (iii) implies the functions $\underline{x}$ and $\overline{x}$ are left-continuous over [0,1].

For $x, y \in \mathbb{F}$, and $\lambda \in \mathbb{R}$, based on the extension principle, arithmetic operations on the fuzzy number are presented using the concept of α-levels of fuzzy numbers and interval arithmetic. Then the α-levels of the sum $\tilde{x} + \tilde{y}$ and the product $\lambda \cdot \tilde{x}$ are obtained as follows

$$[\tilde{x} + \tilde{y}]_\alpha = [\tilde{x}]_\alpha + [\tilde{y}]_\alpha = \{s + t : s \in [\tilde{x}]_\alpha, t \in [\tilde{y}]_\alpha\}$$
$$= [\underline{x}(\alpha) + \underline{y}(\alpha), \overline{x}(\alpha) + \overline{y}(\alpha)],$$
$$[\lambda \cdot \tilde{x}]_\alpha = \lambda \cdot [\tilde{x}]_\alpha = \{\lambda t : t \in [\tilde{x}]_\alpha\}$$
$$= \begin{cases} [\lambda\underline{x}(\alpha), \lambda\overline{x}(\alpha)], & \lambda \geq 0, \\ [\lambda\overline{x}(\alpha), \lambda\underline{x}(\alpha)], & \lambda < 0. \end{cases}$$

Definition 2.2 [24]: For any complex fuzzy number represented by $\tilde{z} = \tilde{x} + i\tilde{y}$, with $\tilde{x} = [\underline{x}(r), \overline{x}(r)]$ and $\tilde{y} = [\underline{y}(r), \overline{y}(r)], 0 \leq r \leq 1$, it can be written as $\tilde{z} = [\underline{x}(r), \overline{x}(r)] + i\,[\underline{y}(r), \overline{y}(r)] = [(\underline{x}(r) + i\underline{y}(r)), (\overline{x}(r) + i\overline{y}(r))]$.

Further, arithmetic operations of complex fuzzy numbers are discussed in [25], [26], and [27] as stated in Definition 2.3.

Definition 2.3 [28]: For any two rectangular complex fuzzy numbers $\tilde{z}_1 = \tilde{x}_1 + i\tilde{y}_1$ and $\tilde{z}_2 = \tilde{x}_2 + i\tilde{y}_2$ and a complex number $c = a + ib$, the α-cut of the sum $\tilde{z}_1 + \tilde{z}_2$ and the product $c \cdot \tilde{z}_1$ are determined based on interval arithmetic as follows:

$$[\tilde{z}_1 + \tilde{z}_2]_\alpha = ([\tilde{x}_1]_\alpha + [\tilde{x}_2]_\alpha) + i([\tilde{y}_1]_\alpha + [\tilde{y}_2]_\alpha)$$
$$= [\underline{x_1}(\alpha) + \underline{x_2}(\alpha), \overline{x_1}(\alpha) + \overline{x_2}(\alpha)]$$
$$+ i\,[\underline{y_1}(\alpha) + \underline{y_2}(\alpha), \overline{y_1}(\alpha) + \overline{y_2}(\alpha)]$$

and

$$[c \cdot \tilde{z}_1]_\alpha = [(a + ib) \cdot \tilde{z}_1]_\alpha = (a + ib) \cdot ([\tilde{x}_1]_\alpha + i[\tilde{y}_1]_\alpha)$$
$$= (a[\tilde{x}_1]_\alpha - b[\tilde{y}_1]_\alpha) + i(a[\tilde{x}_1]_\alpha + b[\tilde{y}_1]_\alpha).$$

Also, two rectangular complex fuzzy numbers $\tilde{z}_1 = \tilde{x}_1 + i\tilde{y}_1$ and $\tilde{z}_2 = \tilde{x}_2 + i\tilde{y}_2$ are said to be equal if and only if $\tilde{x}_1 = \tilde{x}_2 and\ \tilde{y}_1 = \tilde{y}_2$, i.e., $[\tilde{x}_1]_\alpha = [\tilde{x}_2]_\alpha$ and $[\tilde{y}_1]_\alpha = [y_2]_\alpha$, for every $\alpha \in [0,1]$.

Definition 2.4 [18]: We say that two fuzzy numbers $\tilde{x}$ and $\tilde{y}$ are equal if and only if for any $t \in \mathbb{R}, \mu\tilde{x}(t) = \mu\tilde{y}(t), [\tilde{x}]_\alpha = [\tilde{y}]_\alpha$ for any $\alpha \in [0,1]$. Also

$$\tilde{x} \subseteq \tilde{y} \Leftrightarrow [\tilde{x}]_\alpha \subseteq [\tilde{y}]_\alpha, \qquad \forall\alpha \in [0,1].$$

Furthermore, we define two concepts: "α-radius" and "α-center" of any rectangular complex fuzzy number.

Definition 2.5 [25]: We define the α-center of a rectangular complex fuzzy number $\tilde{z} = \tilde{x} + i\tilde{y}$ as follows:

$$[\tilde{z}]_\alpha^c = \left(\frac{\overline{x}(\alpha) + \underline{x}(\alpha)}{2}\right) + i\left(\frac{\overline{y}(\alpha) + \underline{y}(\alpha)}{2}\right), \alpha \in [0,1].$$

Where $[\tilde{x}]_\alpha = [\underline{x}(\alpha), \overline{x}(\alpha)]$ and $[\tilde{y}]_\alpha = [\underline{y}(\alpha), \overline{y}(\alpha)]$

Definition 2.6 [25]: We define the α-radius of a rectangular complex fuzzy number $\tilde{z} = \tilde{x} + i\tilde{y}$ as follows:

$$[\tilde{z}]_\alpha^r = \left(\frac{\overline{x}(\alpha) - \underline{x}(\alpha)}{2}\right) + i\left(\frac{\overline{y}(\alpha) - \underline{y}(\alpha)}{2}\right), \alpha \in [0,1].$$

Where $[\tilde{x}]_\alpha = [\underline{x}(\alpha), \overline{x}(\alpha)]$ and $[\tilde{y}]_\alpha = [\underline{y}(\alpha), \overline{y}(\alpha)]$

For any $\alpha \in [0,1]$, it is evident that the α-radius and α-center of a rectangular complex fuzzy number are complex numbers In the section that follows, we define a system of complex fuzzy linear equations so that the α square matrix with complex values represents its coefficient matrix.

Remark 2.2 [18]: Clearly, the α-radius and α-center of any fuzzy number are definite real functions of α.

Remark 2.3 [18]: For a fuzzy number $\tilde{x}$, if for every $\alpha \in [0,1], x^R(\alpha) = 0$ then it can be easily concluded that $\tilde{x}$ is a definite real number.

Remark 2.4 [18]: For a fuzzy number $\tilde{x}$, if for every $\alpha \in [0,1], x^C(\alpha) = \underline{x}(\alpha)$ or $x^C(\alpha) = \overline{x}(\alpha)$, then according to Statement 2.3, it can be shown that $\tilde{x}$ is a definite real (crisp) number.

Remark 2.5 [18]: Two fuzzy numbers $\tilde{x}$ and $\tilde{y}$ are equal if and only if $x^C(\alpha) = y^C(\alpha)$ and $x^R(\alpha) = y^R(\alpha)$, for every $\alpha \in [0,1]$.

The following statements can be deduced from [29].

Remark 2.6 [18]: Let $\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n \in \mathbb{F}, \lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}$ *and* $\tilde{u} = \sum_{i=1}^n \lambda_i \tilde{x}_i$ then

$$u^C(\alpha) = \sum_{i=1}^n \lambda_i {x_i}^C(\alpha), u^R(\alpha) = \sum_{i=1}^n |\lambda_i| {x_i}^R(\alpha).$$

Definition 2.7 [18]: We define a vector-valued fuzzy number as $\tilde{X} = (\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)^T$, where, $i = 1,2,\dots,n$, is a fuzzy number. Also, we denote $[\tilde{X}]_\alpha$ with $[\tilde{X}]_\alpha = ([\tilde{x}_1]_\alpha, [\tilde{x}_2]_\alpha, \dots, [\tilde{x}_n]_\alpha)^T$ and consequently $x^C(\alpha) = \left(x_1^C(\alpha), x_2^C(\alpha), \dots, x_N^C(\alpha)\right)^T$ and $x^R(\alpha) = \left(x_1^R(\alpha), x_2^R(\alpha), \dots, x_n^R(\alpha)\right)^T$.

Furthermore, for two vector-valued fuzzy numbers $\tilde{X}$ and $\tilde{Y}$, we define

$$\tilde{X} \subseteq \tilde{Y} \Leftrightarrow [\tilde{X}]_\alpha \subseteq [\tilde{Y}]_{\alpha,} \forall \alpha \in [0,1]$$
$$\Leftrightarrow [\tilde{x}_i]_\alpha \subseteq [\tilde{y}_i]_{\alpha,} i = 1,2,\dots,n, \forall \alpha \in [0,1].$$

Theorem 2.1 [18]: Let $A = (a_{ij})_{n \times n}$ be an arbitrary crisp-valued matrix and also $\tilde{X} = (\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)^T$ be an arbitrary vector-valued fuzzy number. Then

$$(A \cdot \tilde{X})^C(\alpha) = A \cdot X^C(\alpha),$$

and

$$(A \cdot \tilde{X})^R(\alpha) = |A| \cdot X^R(\alpha),$$

where $|A| = (|a_{ij}|)_{n \times n}$

Proof. The proof follows from Statement 2.6 and Definition 2.7. ■

One intriguing topic in fuzzy mathematics with numerous applications across various branches of science is the dual fuzzy linear equation system. These systems are defined as follows.

Definition 2.8 [9]: A fuzzy equation system

$$\begin{aligned} a_{11}\tilde{x}_1 + a_{12}\tilde{x}_2 + \cdots + a_{1n}\tilde{x}_n &= \tilde{y}_1, \\ a_{21}\tilde{x}_1 + a_{22}\tilde{x}_2 + \cdots + a_{2n}\tilde{x}_n &= \tilde{y}_2, \\ &\vdots \\ a_{n1}\tilde{x}_1 + a_{n2}\tilde{x}_2 + \cdots + a_{nn}\tilde{x}_n &= \tilde{y}_n. \end{aligned} \tag{1}$$

Is called a fuzzy linear equation system with $A = [a_{ij}]_{i,j=1}^n$ being a crisp coefficient matrix and $\tilde{y}_i$ being a fuzzy number.

Definition 2.9 [9]: A vector of fuzzy numbers $(x_1, x_2, \dots, x_n)^T$ is given by

$$x_i = \left(\underline{x_i}(r), \overline{x}_i(r)\right), \quad 1 \le i \le n, \qquad 0 \le r \le 1,$$

it is called a solution to the fuzzy linear system if

$$\begin{aligned} \underline{\sum_{j=1}^n a_{ij}x_j} &= \sum_{j=1}^n \underline{a_{ij}x_j} = \underline{y_i}, \\ \overline{\sum_{j=1}^n a_{ij}x_j} &= \sum_{j=1}^n \overline{a_{ij}x_j} = \overline{y_i}, \end{aligned} \tag{2}$$

If, for a specific i, $a_{ij} > 0, 1 \le j \le n$, we only obtain

$$\sum_{j=1}^n a_{ij}\underline{x_j} = \underline{y_i}, \qquad \sum_{j=1}^n a_{ij}\overline{x_j} = \overline{y_i}. \tag{3}$$

Considering the fuzzy linear equation system in Equation (1). Transform the $n \times n$ coefficient matrix A into a $2n \times 2n$ matrix as follows:

$$\begin{aligned} &s_{11}\underline{x_1} + \cdots + s_{1n}\underline{x_n} + s_{1,n+1}(-\overline{x_1}) \\ &\quad + s_{1,n+2}(-\overline{x_2}) + \cdots \\ &\quad + s_{1,2n}(-\overline{x_n}) = \underline{y_1}, \\ &\vdots \\ &s_{n1}\underline{x_1} + \cdots + s_{nn}\underline{x_n} + s_{n,n+1}(-\overline{x_1}) \\ &\quad + s_{n,n+2}(-\overline{x_2}) + \cdots \\ &\quad + s_{n,2n}(-\overline{x_n}) = \underline{y_n}, \\ &s_{n+1,1}\underline{x_1} + \cdots + s_{n+1,n}\underline{x_n} + s_{n+1,n+1}(-\overline{x_1}) \\ &\quad + \cdots + s_{n+1,2n}(-\overline{x_n}) = -\overline{y_1}, \\ &\vdots \\ &s_{2n,1}\underline{x_1} + \cdots + s_{2n,n}\underline{x_n} + s_{2n,n+1}(-\overline{x_1}) + \cdots \\ &\quad + s_{2n,2n}(-\overline{x_n}) = -\overline{y}_n, \end{aligned} \tag{4}$$

where

$$\begin{aligned} s_{ij} &= s_{i+n,j+n} = a_{ij}, \text{ if } a_{ij} \ge 0, \\ s_{i,j+n} &= s_{i+n,j} = a_{ij}, \text{ if } a_{ij} < 0. \end{aligned} \tag{5}$$

And any s_{ij} not determined by Equation (5) is zero. Using matrix notation.

$$S\tilde{X} = \tilde{Y}, \tag{6}$$

where

$$\tilde{X} = \left(\underline{x_1}, \dots, \underline{x_n}, -\overline{x_1}, \dots, -\overline{x_n}\right)^T,$$

$$\tilde{Y} = \left(\underline{y_1}, \dots, \underline{y_n}, -\overline{y}, \dots, -\overline{y_n}\right)^T.$$

The structure of S expresses $s_{ij}, 1 \le i, j \le n$, and

$$S = \begin{bmatrix} B & C \\ C & B \end{bmatrix}, \tag{7}$$

with B being the positive entries in matrix A and C the absolute values of the negative entries in matrix A, and $A = B - C$. Now we need to compute S^{-1} (if it exists) and then obtain

$$X = S^{-1}Y. \quad (8)$$

Theorem 2.2 [9]: If S^{-1} exists, it must have the same structure as S, namely

$$S^{-1} = \begin{bmatrix} D & E \\ E & D \end{bmatrix}. \quad (9)$$

Theorem 2.3 [9]: The unique solution X of Equation (6) is a fuzzy vector for any arbitrary Y if and only if is S^{-1} non-negative, that is,

$$(S^{-1})_{ij} \geq 0, \qquad 1 \leq i, j \geq 2n. \quad (10)$$

Further explanations for the proofs of Theorem 2.2 and Theorem 2.3 are discussed in [[9] pages 204-205].

Definition 2.10 [18]: A linear system of $n \times n$

$$\begin{aligned} a_{11}\tilde{x}_1 + a_{12}\tilde{x}_2 + \cdots + a_{1n}\tilde{x}_n + \tilde{y}_1 &= b_{11}\tilde{x}_1 + b_{12}\tilde{x}_2 + \cdots + b_{1n}\tilde{x}_n + \tilde{z}_1 \\ a_{21}\tilde{x}_1 + a_{22}\tilde{x}_2 + \cdots + a_{2n}\tilde{x}_n + \tilde{y}_2 &= b_{21}\tilde{x}_1 + b_{22}\tilde{x}_2 + \cdots + b_{2n}\tilde{x}_n + \tilde{z}_1 \\ &\vdots \\ a_{n1}\tilde{x}_1 + a_{n2}\tilde{x}_2 + \cdots + a_{nn}\tilde{x}_n + \tilde{y}_n &= b_{n1}\tilde{x}_1 + b_{n2}\tilde{x}_2 + \cdots + b_{nn}\tilde{x}_n + \tilde{z}_n \end{aligned} \quad (11)$$

Where the coefficient matrices $A = (a_{ij})_{n\times n}, B = (b_{ij})_{n\times n}$ are two real crisp $n\,x\,n$ matrices, and vectors $\tilde{Y} = (\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_n)^T, \tilde{Z} = (\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_n)^T$ are complex fuzzy vectors with the form $\tilde{y} = \tilde{g} + i\tilde{h}$ and $\tilde{z} = \tilde{p} + i\tilde{q}$ which are unknown $\tilde{X} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)^T$ is a complex fuzzy number vector with the form $\tilde{x} = \tilde{e} + i\tilde{f}$. Also, it is referred to as a complex dual fuzzy linear system.

Based on Definition 2.10, the matrix form of the complex dual fuzzy linear system in Equation (11) is as follows:

$$A\tilde{X} + \tilde{Y} = B\tilde{X} + \tilde{Z}, \quad (12)$$

where matrices A, B and vectors $\tilde{X}, \tilde{Y}$ and $\tilde{Z}$ are defined in Definition 2.10. Additionally, it is important to note that for any complex fuzzy number $\tilde{x} \in \mathbb{F}_{\mathbb{C}}$ there is no element $\tilde{y} \in \mathbb{F}_{\mathbb{C}}$ such that $\tilde{x} + \tilde{y} = 0$. In other words, for any $\tilde{x} \in \mathbb{F}_{\mathbb{C}} - \mathbb{R}$, we have $\tilde{x} + (-\tilde{x}) \neq 0$ and consequently, we cannot equivalently replace the complex dual fuzzy linear system in Equation (12) with the complex fuzzy linear system $(A - B)\tilde{X} = \tilde{Z} - \tilde{Y}$.

From Equation (12), we transform it into the matrix form of the complex dual fuzzy linear system of $2n \times 2n$ as follows:

$$S\tilde{X} + \tilde{Y} = R\tilde{X} + \tilde{Z}, \quad (13)$$

Thus,

$$\begin{aligned} & s_{11}\underline{x_1} + \cdots + s_{1n}\underline{x_n} + s_{1,n+1}(-\overline{x_1}) + \cdots + s_{1,2n}(-\overline{x_n}) + \underline{y_1} \\ &= r_{11}\underline{x_1} + \cdots + r_{1n}\underline{x_n} + r_{1,n+1}(-\overline{x_1}) + \cdots + r_{1,2n}(-\overline{x_n}) + \underline{z_1}, \\ &\vdots \\ & s_{n1}\underline{x_1} + \cdots + s_{nn}\underline{x_n} + s_{n,n+1}(-\overline{x_1}) + \cdots + s_{n,2n}(-\overline{x_n}) + \underline{y_n} = r_{n1}\underline{x_1} + \cdots + r_{nn}\underline{x_n} + r_{n,n+1}(-\overline{x_1}) + \cdots + r_{n,2n}(-\overline{x_n}) + \underline{z_n}, \\ & s_{n+1,1}\underline{x_1} + \cdots + s_{n+1,n}\underline{x_n} + s_{n+1,n+1}(-\overline{x_1}) + \cdots + s_{n+1,2n}(-\overline{x_n}) + (-\overline{y_1}) \\ &= r_{n+1,1}\underline{x_1} + \cdots + r_{n+1,n}\underline{x_n} + r_{n+1,n+1}(-\overline{x_1}) + \cdots + r_{n+1,2n}(-\overline{x_n}) + (-\overline{z_1}), \\ &\vdots \\ & s_{2n,1}\underline{x_1} + \cdots + s_{2n,n}\underline{x_n} + s_{2n,n+1}(-\overline{x_1}) + \cdots + s_{2n,2n}(-\overline{x_n}) + (-\overline{y_n}) \\ &= r_{2n,1}\underline{x_1} + \cdots + r_{2n,n}\underline{x_n} + r_{2n,n+1}(-\overline{x_1}) + \cdots + r_{2n,2n}(-\overline{x_n}) + (-\overline{z_n}), \end{aligned} \quad (14)$$

where

$$\begin{aligned} & S = s_{ij} = s_{i+n,j+n} = a_{ij}, \text{ if } a_{ij} \geq 0, \\ & S = s_{i,j+n} = s_{i+n,j} = a_{ij}, \text{ if } a_{ij} < 0, \\ & R = r_{ij} = r_{i+n,j+n} = b_{ij}, \text{ if } b_{ij} \geq 0, \\ & R = r_{i,j+n} = r_{i+n,j} = b_{ij}, \text{ if } b_{ij} < 0, \\ & \tilde{X} = \left(\underline{x_1}, \ldots, \underline{x_n}, -\overline{x_1}, \ldots, -\overline{x_n}\right)^T = \left(\underline{e_1} + i\underline{f}, \ldots, \underline{e_n} + i\underline{f_n}, -\overline{e_1} + (-i\overline{f_1}), \ldots, -\overline{e_n} + (-i\overline{f_n})\right)^T, \\ & \tilde{Y} = \left(\underline{y_1}, \ldots, \underline{y_n}, -\overline{y}, \ldots, -\overline{y_n}\right)^T = \left(\underline{g_1} + i\underline{h}, \ldots, \underline{g_n} + i\underline{h_n}, -\overline{g_1} + (-i\overline{h_1}), \ldots, -\overline{g_n} + (-i\overline{h_n})\right)^T, \\ & \tilde{Z} = \left(\underline{z_1}, \ldots, \underline{z_n}, -\overline{z}, \ldots, -\overline{z_n}\right)^T = \left(\underline{p_1} + i\underline{q}, \ldots, \underline{p_n} + i\underline{q_n}, -\overline{p_1} + (-i\overline{q_1}), \ldots, -\overline{p_n} + (-i\overline{q_n})\right)^T. \end{aligned} \quad (15)$$

Definition 2.11 [18]: If for the system in Equation (12), $det\,(A - B) \neq 0$, then we define the extended solution of the system in Equation (12) as follows:

$$\tilde{X}_E = (A - B)^{-1}(\tilde{Z} - \tilde{Y}). \quad (16)$$

Also, if $det(A - B) = 0$, then we say that the extended solution does not exist.

Definition 2.12 [18]: A vector-valued fuzzy number $\tilde{X}_A = (\tilde{x}_{1_A}, \tilde{x}_{2_A}, \ldots, \tilde{x}_{n_A})^T$ is called an algebraic solution of the dual fuzzy linear system in Equation (12) if

$$A\tilde{X}_A + \tilde{Y} = B\tilde{X}_A + \tilde{Z},$$

or in other words,

$$\sum_{j=1}^{n} A_{ij}\tilde{X}_{jA} + \tilde{Y}_i = \sum_{j=1}^{n} B_{ij}\tilde{X}_{jA} + \tilde{Z}_i, \ \ \forall i = 1,2,\dots,n.$$

Remark 2.7 [18]: Based on Definition 2.4 and Definition 2.12, it is clear that $\tilde{X}_A = \left(\tilde{x}_{1_A}, \tilde{x}_{2_A}, \dots, \tilde{x}_{n_A}\right)^T$ is an algebraic solution of the system in Equation (12) if

$$\sum_{j=1}^{n} A_{ij}\left[\tilde{X}_{jA}\right]_\alpha + \left[\tilde{Y}_i\right]_\alpha = \sum_{j=1}^{n} B_{ij}\left[\tilde{X}_{jA}\right]_\alpha + [\tilde{Z}_i]_\alpha,$$

for every $\alpha \in [0,1]$ and $i = 1,2,\dots,n$. also, if $\det(A - B) \neq 0$, according to Definitions 2.4 and 2.11, we have

$$\left[\tilde{X}_E\right]_\alpha = (A-B)^{-1}\left([\tilde{Z}]_\alpha - \left[\tilde{Y}\right]_\alpha\right),$$

III. DISCUSSION RESULTS

Since the research objective is to solve the system of equation complex dual fuzzy matrices, here we present the algebraic steps involved:

1. Given the system of equations dual complex fuzzy matrices in the form $A\tilde{X} + \tilde{Y} = B\tilde{X} + \tilde{Z}$ where A, B are crisp coefficient matrices, and $\tilde{Y}, \tilde{Z}$ are complex fuzzy numbers.
2. Next, the first step is to transform it into a system of equations dual complex fuzzy matrices in the form $S\tilde{X} + \tilde{Y} = R\tilde{X} + \tilde{Z}$ where $S \geq 0$ denotes the positive entries of matrix A, $S \leq 0$ denotes the absolute values of the negative entries of matrix A, $R \geq 0$ denotes the positive entries of matrix B, $R \leq 0$ denotes the absolute values of the negative entries of matrix B, and $\tilde{Y}, \tilde{Z}$ are complex fuzzy numbers.
3. Then, determine $det(S-R) \neq 0$ so that the inverse of $(S-R)$ exists, and the system of equations of dual complex fuzzy matrices has a solution.
4. Determining the algebraic solution of the complex dual fuzzy linear equation based on Theorem 3.1 using $\tilde{X} = (S-R)^{-1}(\tilde{Z} - \tilde{Y})$.

Example 3.1 Given the system of equations of a complex dual fuzzy matrix as follows:

$$\begin{bmatrix} 6 & 2 \\ 1 & 7 \end{bmatrix}\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} (\alpha + i(-2+2\alpha), (2-\alpha) + i(2-2\alpha)) \\ (\alpha + i(-4+\alpha), (2.5 - 1.5\alpha) + i(-1-2\alpha)) \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} ((4+\alpha) + i(1-\alpha), (6-\alpha) + i(-1+\alpha)) \\ ((-2+\alpha) + i(-3+\alpha), (-\alpha) + i(-1-\alpha)) \end{bmatrix},$$

Determine the solution of the system of equations of a complex dual fuzzy matrix.

Solution:

Given

$$A = \begin{bmatrix} 6 & 2 \\ 1 & 7 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix},$$

if it is transformed into a $2n \times 2n$ matrix equation, then it is obtained as:

$$\begin{bmatrix} 6 & 2 \\ 1 & 7 \end{bmatrix} \to \begin{bmatrix} 6 & 2 & 0 & 0 \\ 1 & 7 & 0 & 0 \\ 0 & 0 & 6 & 2 \\ 0 & 0 & 1 & 7 \end{bmatrix},$$

and

$$\begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix} \to \begin{bmatrix} 2 & 4 & 0 & 0 \\ 4 & 2 & 0 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 4 & 2 \end{bmatrix},$$

so the system of equations is

$$\begin{bmatrix} 6 & 2 & 0 & 0 \\ 1 & 7 & 0 & 0 \\ 0 & 0 & 6 & 2 \\ 0 & 0 & 1 & 7 \end{bmatrix}\begin{bmatrix} \underline{x_1} \\ \underline{x_2} \\ -\overline{x_1} \\ -\overline{x_2} \end{bmatrix} + \begin{bmatrix} \alpha + i(-2+2\alpha) \\ \alpha + i(-4+\alpha) \\ -((2-\alpha) + i(2-2\alpha)) \\ -((2.5-1.5\alpha) + i(-1-2\alpha)) \end{bmatrix} = \begin{bmatrix} 2 & 4 & 0 & 0 \\ 4 & 2 & 0 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 4 & 2 \end{bmatrix}\begin{bmatrix} \underline{x_1} \\ \underline{x_2} \\ -\overline{x_1} \\ -\overline{x_2} \end{bmatrix} + \begin{bmatrix} (4+\alpha) + i(1-\alpha) \\ (-2+\alpha) + i(-3+\alpha) \\ -((6-\alpha) + i(-1+\alpha)) \\ -(-\alpha + i(-1-\alpha)) \end{bmatrix},$$

since $det(S-R) = 196$ then $(S-R)^{-1}$ exists and has the algebraic solution as follows.

$$\tilde{X} = (S-R)^{-1}(\tilde{Z} - \tilde{Y})$$

$$= \left(\begin{bmatrix} 4 & -2 & 0 & 0 \\ -3 & 5 & 0 & 0 \\ 0 & 0 & 4 & -2 \\ 0 & 0 & -3 & 5 \end{bmatrix}\right)^{-1}\left(\begin{bmatrix} (4+\alpha) + i(1-\alpha) \\ (-2+\alpha) + i(-3+\alpha) \\ -((6-\alpha) + i(-1+\alpha)) \\ -(-\alpha + i(-1-\alpha)) \end{bmatrix} - \begin{bmatrix} \alpha + i(-2+2\alpha) \\ \alpha + i(-4+\alpha) \\ -((2-\alpha) + i(2-2\alpha)) \\ -((2.5-1.5\alpha) + i(-1-2\alpha)) \end{bmatrix}\right)$$

$$= \begin{bmatrix} 5/14 & 1/7 & 0 & 0 \\ 3/14 & 2/7 & 0 & 0 \\ 0 & 0 & 5/14 & 1/7 \\ 0 & 0 & 3/14 & 2/7 \end{bmatrix}\begin{bmatrix} 4 + i(3-3\alpha) \\ -2 + i \\ -4 + i(3-3\alpha) \\ (2.5 + 0.5\alpha) + i(-\alpha) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{8}{7} + i\left(\frac{17}{14} - \frac{15}{14}\alpha\right) \\ \frac{2}{7} + i\left(\frac{13}{14} - \frac{9}{14}\alpha\right) \\ \left(-\frac{1}{14}\alpha - \frac{15}{14}\right) + i\left(\frac{15}{14} - \frac{17}{14}\alpha\right) \\ \left(-\frac{1}{7}\alpha - \frac{1}{7}\right) + i\left(\frac{9}{14} - \frac{13}{14}\alpha\right) \end{bmatrix},$$

so, the solution of the system of equations of a complex dual fuzzy matrix is:

$$x_1 = \left[\frac{8}{7}, \frac{15}{14} + \frac{1}{14}\alpha\right] + i\left(\left[\frac{17}{14} - \frac{15}{14}\alpha, -\frac{15}{14} + \frac{17}{14}\alpha\right]\right),$$

$$x_2 = \left[\frac{2}{7}, \frac{1}{7} + \frac{1}{7}\alpha\right] + i\left(\left[\frac{13}{14} - \frac{9}{14}\alpha, -\frac{9}{14} + \frac{13}{14}\alpha\right]\right).$$

Example 3.2 Given the system of equations of a complex dual fuzzy matrix as follows:

$$\begin{bmatrix}2 & 2 & 0\\ 3 & 1 & -1\\ 1 & -2 & 3\end{bmatrix}\begin{bmatrix}\tilde{x}_1\\ \tilde{x}_2\\ \tilde{x}_3\end{bmatrix} + \begin{bmatrix}(\alpha+i(-2+2\alpha),(2-\alpha)+i(2-2\alpha))\\ ((2+\alpha)+i(2\alpha+2),3+i(6-2\alpha))\\ (-2+i(2\alpha+4),(-1-\alpha)+i(8-2\alpha))\end{bmatrix}$$

$$= \begin{bmatrix}1 & 1 & 1\\ 2 & 3 & -2\\ 1 & -3 & 0\end{bmatrix}\begin{bmatrix}\tilde{x}_1\\ \tilde{x}_2\\ 3\end{bmatrix} + \begin{bmatrix}((4\alpha-1)+i(1+8\alpha),(4-\alpha)+i(6+5\alpha))\\ ((4\alpha-2)+i(4\alpha-2),(3-\alpha)+i(8+2\alpha))\\ ((4\alpha+1)+i(-5\alpha+3),(6-\alpha)+i(-4+3\alpha))\end{bmatrix},$$

Determine the solution of the system of equations of a complex dual fuzzy matrix.

Solution:

Given

$$A = \begin{bmatrix}2 & 2 & 0\\ 3 & 1 & -1\\ 1 & -2 & 3\end{bmatrix} \text{ and } B = \begin{bmatrix}1 & 1 & 1\\ 2 & 3 & -2\\ 1 & -3 & 0\end{bmatrix},$$

if it is transformed into a $2n \times 2n$ matrix equation, then it is obtained as:

$$\begin{bmatrix}2 & 2 & 0\\ 3 & 1 & -1\\ 1 & -2 & 3\end{bmatrix} \to \begin{bmatrix}2 & 2 & 0 & 0 & 0 & 0\\ 3 & 1 & 0 & 0 & 0 & 1\\ 1 & 0 & 3 & 0 & 2 & 0\\ 0 & 0 & 0 & 2 & 2 & 0\\ 0 & 0 & 1 & 3 & 1 & 0\\ 0 & 2 & 0 & 1 & 0 & 3\end{bmatrix}$$

and

$$\begin{bmatrix}1 & 1 & 1\\ 2 & 3 & -2\\ 1 & -3 & 0\end{bmatrix} \to \begin{bmatrix}1 & 1 & 1 & 0 & 0 & 0\\ 2 & 3 & 0 & 0 & 0 & 2\\ 1 & 0 & 0 & 0 & 3 & 0\\ 0 & 0 & 0 & 1 & 1 & 1\\ 0 & 0 & 2 & 2 & 3 & 0\\ 0 & 3 & 0 & 1 & 0 & 0\end{bmatrix},$$

so the system of equation is

$$\begin{bmatrix}2 & 2 & 0 & 0 & 0 & 0\\ 3 & 1 & 0 & 0 & 0 & 1\\ 1 & 0 & 3 & 0 & 2 & 0\\ 0 & 0 & 0 & 2 & 2 & 0\\ 0 & 0 & 1 & 3 & 1 & 0\\ 0 & 2 & 0 & 1 & 0 & 3\end{bmatrix}\begin{bmatrix}\underline{x}_1\\ \underline{x}_2\\ \underline{x}_3\\ -\overline{x}_1\\ -\overline{x}_2\\ -\overline{x}_3\end{bmatrix} + \begin{bmatrix}\alpha+i(-2+2\alpha)\\ (2+\alpha)+i(2\alpha+2)\\ -2+i(2\alpha+4)\\ -((2-\alpha)+i(2-2\alpha))\\ -(3+i(6-2\alpha))\\ -((-1-\alpha)+i(8-2\alpha))\end{bmatrix}$$

$$= \begin{bmatrix}1 & 1 & 1 & 0 & 0 & 0\\ 2 & 3 & 0 & 0 & 0 & 2\\ 1 & 0 & 0 & 0 & 3 & 0\\ 0 & 0 & 0 & 1 & 1 & 1\\ 0 & 0 & 2 & 2 & 3 & 0\\ 0 & 3 & 0 & 1 & 0 & 0\end{bmatrix}\begin{bmatrix}\underline{x}_1\\ \underline{x}_2\\ \underline{x}_3\\ -\overline{x}_1\\ -\overline{x}_2\\ -\overline{x}_3\end{bmatrix} + \begin{bmatrix}(4\alpha-1)+i(1+8\alpha)\\ (4\alpha-2)+i(4\alpha-2)\\ (4\alpha+1)+i(-5\alpha+3)\\ -((4-\alpha)+i(6+5\alpha))\\ -((3-\alpha)+i(8+2\alpha))\\ -((6-\alpha)+i(-4+3\alpha))\end{bmatrix},$$

since $det(S-R) = 70$ then $(S-R)^{-1}$ exists and has the algebraic solution as follows.

$$\tilde{X} = (S-R)^{-1}(\tilde{Z}-\tilde{Y})$$

$$= \begin{bmatrix}1 & 1 & -1 & 0 & 0 & 0\\ 1 & -2 & 0 & 0 & 0 & -1\\ 0 & 0 & 3 & 0 & -1 & 0\\ 0 & 0 & 0 & 1 & 1 & -1\\ 0 & 0 & -1 & 1 & -2 & 0\\ 0 & -1 & 0 & 0 & 0 & 3\end{bmatrix}^{-1}\left(\begin{bmatrix}(4\alpha-1)+i(1+8\alpha)\\ (4\alpha-2)+i(4\alpha-2)\\ (4\alpha+1)+i(-5\alpha+3)\\ -((4-\alpha)+i(6+5\alpha))\\ -((3-\alpha)+i(8+2\alpha))\\ -((6-\alpha)+i(-4+3\alpha))\end{bmatrix} - \begin{bmatrix}\alpha+i(-2+2\alpha)\\ (2+\alpha)+i(2\alpha+2)\\ -2+i(2\alpha+4)\\ -((2-\alpha)+i(2-2\alpha))\\ -(3+i(6-2\alpha))\\ -((-1-\alpha)+i(8-2\alpha))\end{bmatrix}\right)$$

$$= \begin{bmatrix}\frac{70}{99} & \frac{29}{99} & \frac{7}{33} & \frac{7}{99} & -\frac{7}{99} & \frac{4}{33}\\ \frac{10}{33} & -\frac{10}{33} & \frac{1}{11} & \frac{1}{33} & -\frac{1}{33} & -\frac{1}{11}\\ \frac{1}{99} & -\frac{1}{99} & \frac{10}{33} & \frac{10}{99} & -\frac{10}{99} & \frac{1}{33}\\ \frac{7}{99} & -\frac{7}{99} & \frac{4}{33} & \frac{70}{99} & \frac{29}{99} & \frac{7}{33}\\ \frac{1}{33} & -\frac{1}{33} & -\frac{1}{11} & \frac{10}{33} & -\frac{10}{33} & \frac{1}{11}\\ \frac{10}{99} & -\frac{10}{99} & \frac{1}{33} & \frac{1}{99} & -\frac{1}{99} & \frac{10}{33}\end{bmatrix}\begin{bmatrix}3\alpha-1+i(3+6\alpha)\\ 3\alpha+i(2\alpha)\\ 4\alpha+3+i(-7\alpha-1)\\ -4+i(-4-7\alpha)\\ \alpha+i(-2-4\alpha)\\ -7+i(12-5\alpha)\end{bmatrix}$$

$$= \begin{bmatrix}\frac{34}{9}\alpha-\frac{221}{99}+i\left(\frac{250}{99}\alpha+\frac{203}{99}\right)\\ \frac{1}{3}\alpha+\frac{58}{33}+i\left(\frac{31}{33}\alpha+\frac{29}{33}\right)\\ \frac{10}{9}\alpha+\frac{52}{99}+i\left(-\frac{251}{99}\alpha-\frac{7}{99}\right)\\ \frac{7}{9}\alpha-\frac{230}{99}+i\left(-\frac{767}{99}\alpha-\frac{49}{99}\right)\\ -\frac{2}{3}\alpha-\frac{47}{33}+i\left(-\frac{20}{33}\alpha+\frac{26}{33}\right)\\ \frac{1}{9}\alpha-\frac{173}{99}+i\left(-\frac{134}{99}\alpha+\frac{425}{99}\right)\end{bmatrix},$$

so, the solution of the system of equations of a complex dual fuzzy matrix is:

$$x_1 = \left[\frac{34}{9}\alpha-\frac{221}{99},-\frac{7}{9}\alpha+\frac{230}{99}\right] + i\left(\left[\frac{250}{99}\alpha+\frac{203}{99},\frac{767}{99}\alpha+\frac{49}{99}\right]\right),$$

$$x_2 = \left[\frac{1}{3}\alpha+\frac{58}{33},\frac{2}{3}\alpha+\frac{47}{33}\right] + i\left(\left[\frac{31}{33}\alpha+\frac{29}{33},\frac{20}{33}\alpha-\frac{26}{33}\right]\right),$$

Here we present the theorem regarding the solution of complex dual fuzzy linear equations.

$$x_3 = \left[\frac{10}{9}\alpha+\frac{52}{99},-\frac{1}{9}\alpha+\frac{173}{99}\right] + i\left(\left[-\frac{251}{99}\alpha-\frac{7}{99},\frac{134}{99}\alpha-\frac{425}{99}\right]\right),$$

Example 3.3 Given the system of equations of a complex dual fuzzy matrix as follows:

$$\begin{bmatrix}2 & -3\\ 4 & 2\end{bmatrix}\begin{bmatrix}\tilde{x}_1\\ \tilde{x}_2\end{bmatrix}+\begin{bmatrix}((\alpha+i(-2+3\alpha),(2-\alpha)+i(2-2\alpha))\\ ((\alpha+i(\alpha+4),(2-\alpha)+i(-1-2\alpha))\end{bmatrix}$$
$$=\begin{bmatrix}1 & -2\\ 3 & 1\end{bmatrix}\begin{bmatrix}\tilde{x}_1\\ \tilde{x}_2\end{bmatrix}$$
$$+\begin{bmatrix}((2\alpha+i(1+\alpha),(2-\alpha)+i(3-\alpha))\\ ((4+2\alpha)+i(2\alpha-4),(4-2\alpha)+i(-1-2\alpha))\end{bmatrix},$$

Determine the solution of the system of equations of a complex dual fuzzy matrix.

Solution:

Given

$$A=\begin{bmatrix}2 & -3\\ 4 & 2\end{bmatrix} \quad \text{and} \quad B=\begin{bmatrix}1 & -2\\ 3 & 1\end{bmatrix},$$

if it is transformed into a $2n\times 2n$ matrix equation, then it is obtained as:

$$\begin{bmatrix}2 & -3\\ 4 & 2\end{bmatrix}\to\begin{bmatrix}2 & 0 & 0 & 3\\ 4 & 2 & 0 & 0\\ 0 & 3 & 2 & 0\\ 0 & 0 & 4 & 2\end{bmatrix},$$

and

$$\begin{bmatrix}1 & -2\\ 3 & 1\end{bmatrix}\to\begin{bmatrix}1 & 0 & 0 & 2\\ 3 & 1 & 0 & 0\\ 0 & 2 & 1 & 0\\ 0 & 0 & 3 & 1\end{bmatrix},$$

so the system of equations is

$$\begin{bmatrix}2 & 0 & 0 & 3\\ 4 & 2 & 0 & 0\\ 0 & 3 & 2 & 0\\ 0 & 0 & 4 & 2\end{bmatrix}\begin{bmatrix}\underline{x_1}\\ \underline{x_2}\\ -\overline{x_1}\\ -\overline{x_2}\end{bmatrix}+\begin{bmatrix}(\alpha+i(-2+3\alpha))\\ (\alpha+i(\alpha+4))\\ -((2-\alpha)+i(2-2\alpha))\\ -((2-\alpha)+i(-1-2\alpha))\end{bmatrix}$$
$$=\begin{bmatrix}1 & 0 & 0 & 2\\ 3 & 1 & 0 & 0\\ 0 & 2 & 1 & 0\\ 0 & 0 & 3 & 1\end{bmatrix}\begin{bmatrix}\underline{x_1}\\ \underline{x_2}\\ -\overline{x_1}\\ -\overline{x_2}\end{bmatrix}$$
$$+\begin{bmatrix}((2\alpha+i(1+\alpha))\\ ((4+2\alpha)+i(2\alpha-4))\\ -((2-\alpha)+i(3-\alpha))\\ -((4-2\alpha)+i(-1-2\alpha))\end{bmatrix},$$

since $det(S-R)=0$ then $(S-R)^{-1}$ does not exist so the system of equations of a complex dual fuzzy matrix does not have a solution.

Based on the steps of solving the system of equations of a complex dual fuzzy matrix and the provided example, here we present the theorem regarding the solution of the system of equations of a complex dual fuzzy matrix.

Theorem 3.1: Let $A\tilde{X}+\tilde{Y}=B\tilde{X}+\tilde{Z}$ where A,B are coefficient matrices of size $n\times n$ and $\tilde{Y},\tilde{Z}$ are complex fuzzy numbers. If the equation is transformed into $S\tilde{X}+\tilde{Y}=R\tilde{X}+\tilde{Z}$, where $S=(s_{ij})$, $R=(r_{ij}), 1\le i,j\le n$, be nonnegative matrices. The solution of $S\tilde{X}+\tilde{Y}=R\tilde{X}+\tilde{Z}$ has a unique fuzzy solution if and only if $(S-R)^{-1}$ exists. Furthermore, that solution is also a solution for $A\tilde{X}+\tilde{Y}=B\tilde{X}+\tilde{Z}$.

Proof.

($\Rightarrow$)Given $S=(s_{ij})$, $R=(r_{ij}), 1\le i,j\le n$, are nonnegative matrices, and the linear system of dual fuzzy $S\tilde{X}+\tilde{Y}=R\tilde{X}+\tilde{Z}$ has a unique solution, it will be satisfied if $(S-R)$ exists.

With Equations (2) and (3), the dual fuzzy linear system

$$\sum_{j=1}^{n}s_{ij}\tilde{x_j}+\tilde{y_i}=\sum_{j=1}^{n}r_{ij}x_j+\tilde{z_i}, \tag{21}$$

is equivalent to (since $s_{ij}\ge 0$ and $r_{ij}\ge 0$ for all i,j)

$$\begin{aligned}\sum_{j=1}^{n}s_{ij}\underline{x_j}+\underline{y_i}&=\sum_{j=1}^{n}r_{ij}\underline{x_j}+\underline{z_i},\\ \sum_{j=1}^{n}s_{ij}\overline{x_j}+\overline{y_i}&=\sum_{j=1}^{n}r_{ij}\overline{x_j}+\overline{z_i},\end{aligned} \tag{22}$$

with $\tilde{X},\tilde{Y}$, and $\tilde{Z}$ given as in equation (15). consequently,

$$\sum_{j=1}^{n}(s_{ij}-r_{ij})\underline{x_j}=\underline{z_i}-\underline{y_i}, \tag{23}$$

$$\sum_{j=1}^{n}(s_{ij}-r_{ij})\overline{x_j}=\overline{z_i}-\overline{y_i}. \tag{24}$$

If $det(S-R)\neq 0$, then $(S-R)^{-1}$ exists, so equations (23) and (24) have a unique solution $\left(\underline{x_j}\right)_1^n,\left(\overline{x_j}\right)_1^n$.

($\Leftarrow$)Given $S=(s_{ij})$, $R=(r_{ij}), 1\le i,j\le n$, and $(S-R)^{-1}$ exists, it will be shown that the linear system of dual fuzzy $S\tilde{X}+\tilde{Y}=R\tilde{X}+\tilde{Z}$ has a unique solution. Because $(S-R)^{-1}$ exists, it has a unique solution $\left(\underline{x_j}\right)_1^n,\left(\overline{x_j}\right)_1^n$.

$$\sum_{j=1}^{n}\underline{x_i}=\sum_{j=1}^{n}\left(s_{ij}-r_{ij}\right)^{-1}\left(\underline{z_i}-\underline{y_i}\right),$$
$$\sum_{j=1}^{n}\overline{x_j}=\sum_{j=1}^{n}\left(s_{ij}-r_{ij}\right)^{-1}\left(\overline{z_i}-\overline{y_i}\right).$$

Next, both sides are multiplied by $\sum_{j=1}^{n}\left(s_{ij}-r_{ij}\right)$ resulting in

$$\sum_{j=1}^{n}\left(s_{ij}-r_{ij}\right)\underline{x_j}=\underline{z_i}-\underline{y_{i,}}$$
$$\sum_{j=1}^{n}\left(s_{ij}-r_{ij}\right)\overline{x_j}=\overline{z_i}-\overline{y_i}.$$

Then,

$$\sum_{j=1}^{n}s_{ij}\underline{x_j}+\underline{y_i}=\sum_{j=1}^{n}r_{ij}\underline{x_j}+\underline{z_{i,}}$$
$$\sum_{j=1}^{n}s_{ij}\overline{x_j}+\overline{y_i}=\sum_{j=1}^{n}r_{ij}\overline{x_j}+\overline{z_i}.$$

Are equivalent (since $s_{ij}\ge 0$ and $r_{ij}\ge 0$ for all i,j)

$$\sum_{j=1}^{n}s_{ij}\tilde{x_j}+\tilde{y_i}=\sum_{j=1}^{n}r_{ij}x_j+\tilde{z_i}. \blacksquare$$

IV. CONCLUSION

Based on the discussion above, the system of equations of complex dual fuzzy matrices $A\tilde{X}+\tilde{Y}=B\tilde{X}+\tilde{Z}$ where

A, B are coefficient matrices of size $n \times n$ and $\tilde{Y}, \tilde{Z}$ are complex fuzzy numbers that can be solved by transforming them into a $2n \times 2n$ complex fuzzy matrix. The system of equations referred to is $S\tilde{X} + \tilde{Y} = R\tilde{X} + \tilde{Z}$ where $S \geq 0$ are the nonnegative entries of matrix A, $S \leq 0$ are the absolute values of the negative entries of matrix A, $R \geq 0$ are the nonnegative entries of matrix B, $R \leq 0$ are the absolute values of the negative entries of matrix B, and $\tilde{Y}, \tilde{Z}$ are matrices of complex fuzzy numbers with the solution $\tilde{X} = (S - R)^{-1}(\tilde{Z} - \tilde{Y})$.

REFERENCES

1. Y. Zheng et al., "Solving Systems of Linear Equations with a Superconducting Quantum Processor," Phys. Rev. Lett., vol. 210504, no. May, pp. 1–6, 2017.
2. E. A, Youness and I. M, Mekawy, "A Study on Fuzzy Complex Linear Programming Problems," Int. J. Contemp. Math. Sci., vol. 7, no. 19, pp. 897–908, 2012.
3. A. C. Illeigo and H. N. Ojogbane, "Application of Matrices and Linear Programming in Economics," Am. J. IT Appl. Sci. Res., vol. 1, no. 1, pp. 1–15, 2022.
4. G. Bojadziev and M. Bojadziev, Advances in Fuzzy Systems Aplicationd and Theory, 2nd Editio. World Scientific, 2007.
5. A. Sarkar, G. Sahoo, and U. C, Sahoo, "Application of Fuzzy Logic in Transport Planning," Int. J. Soft Comput, vol. 3, no. 2, pp. 1–21, 2012.
6. D. Driankov, H. Hellendoorn, and M. Reinfrank, An Introduction to Fuzzy Control, 2nd Editio. Springer, 1996.
7. I. Irmawati, I. Sukarsih, and R. Respitawulan, "Solusi Sistem Persamaan Linear Fuzzy," Matematika, vol. 16, no. 2, pp. 1–8, 2017.
8. T. Allahviranloo, M. Ghanbari, A. A. Hosseinzadeh, E. Haghi, and R. Nuraei, "A note on 'fuzzy linear systems,'" Fuzzy Sets Syst., vol. 177, no. 1, pp. 87–92, 2011.
9. M. Friedman, M. Ming, and A. Kandel, "Fuzzy linear systems," Fuzzy Sets Syst., vol. 96, no. 2, pp. 201–209, 1998.
10. F. Babakordi and T. Allahviranloo, "A Cramer Method for Solving Fully Fuzzy Linear Systems Based on Transmission Average," Control Optim. Appl. Math., vol. 7, no. 2, pp. 115–130, 2022.
11. T. Allahviranloo, R. Nuraei, M. Ghanbari, E. Haghi, and A. A. Hosseinzadeh, "A new metric for L-R fuzzy numbers and its application in fuzzy linear systems," Soft Comput., vol. 16, no. 10, pp. 1743–1754, 2012.
12. M. Ghanbari, T. Allahviranloo, and W. Pedrycz, "On the rectangular fuzzy complex linear systems," Appl. Soft Comput. J., vol. 91, p. 106196, 2020.
13. M. Ghanbari and R. Nuraei, "Convergence of a semi-analytical method on the fuzzy linear systems," Iran. J. Fuzzy Syst., vol. 11, no. 4, pp. 45–60, 2014.
14. R. Nuraei, T. Allahviranloo, and M. Ghanbari, "Finding an inner estimation of the solution set of a fuzzy linear system," Appl. Math. Model., vol. 37, no. 7, pp. 5148–5161, 2013.
15. W. Cong-Xin and M. Ming, "Embedding problem of fuzzy number space: Part I," Fuzzy Sets Syst., vol. 44, no. 1, pp. 33–38, 1991.
16. M. Ma, M. Friedman, and A. Kandel, "Duality in fuzzy linear systems," Fuzzy Sets Syst., vol. 109, no. 1, pp. 55–58, 2000.
17. M. Chehlabi, "Solving fuzzy dual complex linear systems," J. Appl. Math. Comput., vol. 60, no. 1–2, pp. 87–112, 2019.
18. M. Ghanbari, T. Allahviranloo, and W. Pedrycz, "A straightforward approach for solving dual fuzzy linear systems," Fuzzy Sets Syst., vol. 435, pp. 89–106, 2022.
19. . Gong, J. Wu, and K. Liu, "The dual fuzzy matrix equations: Extended solution, algebraic solution, and solution," AIMS Math., vol. 8, no. 3, pp. 7310–7328, 2023.
20. M. . Jahantigh, S. Khezerloo, and M. Khezerloo, "Complex Fuzzy Linear Systems," Int. J. Ind. Math., vol. 2, no. 1, pp. 21–28, 2010.
21. C. C. Marzuki and N. Hasmita, "Penyelesaian Sistem Persamaan Linear Fuzzy Kompleks Menggunakan Metode Dekomposisi Doolittle," J. Sains, Teknol. dan Ind., vol. 11, no. 2, pp. 167–174, 2014.
22. B. Bede and S. G. Gal, "Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations," Fuzzy Sets Syst., vol. 151, no. 3, pp. 581–599, 2005.
23. K. Chakrabarty, R. Biswas, and S. Nanda, "On fuzzy metric spaces," Fuzzy Sets Syst., vol. 99, no. 1, pp. 111–114, 1998.
24. S. V. Rivika, "Penyelesaian Sistem Persamaan Linear Fuzzy Kompleks," Universitas Riau, 2020.
25. D. Behera and S. Chakraverty, "A new method for solving real and complex fuzzy systems of linear equations," Comput. Math. Model., vol. 23, no. 4, pp. 507–518, 2012.
26. K. Zhang and X. Guo, "Solving Complex Fuzzy Linear System of Equations by using QR-Decomposition Method," vol. 2, no. 9, pp. 54–63, 2016.

27. D. Behera and S. Chakraverty, "Solving fuzzy complex system of linear equations," Inf. Sci. (Ny)., vol. 277, pp. 154–162, 2014.
28. M. Ghanbari, "A discussion on 'solving fuzzy complex system of linear equations,'" Inf. Sci. (Ny)., vol. 402, pp. 165–169, 2017.
29. S. Abbasbandy, E. Babolian, and M. Alavi, "Numerical method for solving linear Fredholm fuzzy integral equations of the second kind," Chaos, Solitons and Fractals, vol. 31, no. 1, pp. 138–146, 2007