International Journal of Mathematics and Computer Research

ISSN: 2320-7167

Volume 12 Issue 06 June 2024, Page no. – 4288-4292

Index Copernicus ICV: 57.55, Impact Factor: 8.316

DOI: 10.47191/ijmcr/v12i6.05

Common Fixed-Point Theorems for Expansive Mappings in Parametric Metric Spaces

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ARTICLE INFO

Published Online: **07 June 2024**

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ABSTRACT

In this paper, we establish some common fixed-point results in parametric metric spaces for pairs of expansive mappings. As a result, several findings from metric spaces are expanded to include this additional context. We also provide an example that shows the usefulness of these contractions.

2010 Mathematics Subject Classification: 47H10, 54H25

KEYWORDS: Fixed point; expansive mapping; parametric metric space.

1. INTRODUCTION

There are numerous ways to generalize the idea of metric spaces. Czerwik developed the idea of a b-metric space in [3-4], and many researchers have established fixed-point solutions for single-valued and multi-valued mappings in (ordered) b-metric spaces (see, e.g., [2, 10]). In 1965, Zadeh [25] developed the idea of a fuzzy set. The concept of fuzzy metric space was first proposed by Kramosil and Michalek [19] in 1975. It is a generalization of statistical (probabilistic) metric space. An essential foundation for the development of fixed-point theory in fuzzy metric spaces has been established by this study.

Park proposed the idea of an intuitionistic fuzzy metric space in 2004 [21]. In fixed point theory, the study of expansive mappings is an extremely fascinating field of study. Expanding mappings were first introduced and several fixed-point theorems in complete metric spaces were demonstrated by Wang et al. in 1984 [24]. Daffer and Kaneko [5] established several common fixed-point theorems for two mappings in complete metric spaces and provided an expanding condition for a pair of mappings in 1992.

In this study, we will extend results from [12, 24] and other papers by proving several additional fixed-point and common fixed-point theorems for expansion mappings in the context of parametric metric space.

2. DEFINITIONS AND PRELIMINARIES

The notion of parametric metric space was established and examined as follows by Hussain et al. [22] in 2014.

Definition 2.1 Let $\mathfrak{D}$ be a nonempty set and $\mathcal{P} : \mathfrak{D} \times \mathfrak{D} \times (0, +\infty) \to [0, +\infty)$ be a function. We say $\mathcal{P}$ is a parametric metric on $\mathfrak{D}$ if,

(1) $\mathcal{P}(\theta, \vartheta, t) = 0$ for all $t > 0$ if and only if $\theta = \vartheta$;

(2) $\mathcal{P}(\theta, \vartheta, t) = \mathcal{P}(\vartheta, \theta, t)$ for all $t > 0$;

(3) $\mathcal{P}(\theta, \vartheta, t) \leq \mathcal{P}(\theta, \omega, t) + \mathcal{P}(\omega, \vartheta, t), \forall\, \theta, \vartheta, \omega \in \mathfrak{D}$ and all $t > 0$.

and one says the pair $(\mathfrak{D}, \mathcal{P})$ is a parametric metric space.

The following definitions are required in the sequel which can be found in [22].

Definition 2.2 Let $\{\theta_n\}_{n=1}^{\infty}$ be a sequence in a parametric metric space $(\mathfrak{D}, \mathcal{P})$.

1. $\{\theta_n\}_{n=1}^{\infty}$ is said to be convergent to $\theta \in \mathfrak{D}$, written as $\lim_{n\to\infty} \theta_n = \theta$, for all $t > 0$, if $\lim_{n\to\infty} \mathcal{P}(\theta_n, \theta, t) = 0$.
2. $\{\theta_n\}_{n=1}^{\infty}$ is said to be a Cauchy sequence in $\mathfrak{D}$ if for all $t > 0$, if $\lim_{n,m\to\infty} \mathcal{P}(\theta_n, \theta_m, t) = 0$.

3. $(\mathfrak{D},\mathcal{P})$ is said to be complete if every Cauchy sequence is a convergent sequence.

Definition 2.3 Let $(\mathfrak{D},\mathcal{P})$ be a parametric metric space and $f:\mathfrak{D}\to\mathfrak{D}$ be a mapping. We say f is a continuous mapping at θ in $\mathfrak{D}$, if for any sequence $\{\theta_n\}_{n=1}^{\infty}$ in $\mathfrak{D}$ such that $\lim_{n\to\infty}\theta_n=\theta$, then $\lim_{n\to\infty}f\theta_n=f\theta$.

Example 2.4 Let $\mathfrak{D}$ denote the set of all functions $f:(0,+\infty)\to\mathbb{R}$. Define $\mathcal{P}:\mathfrak{D}\times\mathfrak{D}\times(0,+\infty)\to[0,+\infty)$ by $\mathcal{P}(f,g,t)=|f(t)-g(t)|\ \forall\ f,g\in\mathfrak{D}$ and all $t>0$. Then $\mathcal{P}$ is a parametric metric on $\mathfrak{D}$ and the pair $(\mathfrak{D},\mathcal{P})$ is a parametric metric space.

Let $\mathfrak{D}$ be a set. A point $\vartheta\in\mathfrak{D}$ is a point of coincidence of a pair of self-maps $f,g:\mathfrak{D}\to\mathfrak{D}$ and $\theta\in\mathfrak{D}$ is its coincidence point if $f\theta=g\theta=\vartheta$. Mappings f and g are weakly compatible if $fg\theta=gf\theta$ for each of their coincidence points θ [12,15,17] and occasionally weakly compatible if the same holds for some coincidence point [14].

The set of fixed points of a self-map $f:\mathfrak{D}\to\mathfrak{D}$ will be denoted as $\mathfrak{F}(f)$. The mapping f is said to possess property (P) if $\mathfrak{F}(f^n)=\mathfrak{F}(f)$ for each $n\in\mathbb{N}$ (see [12, 18]). A pair of self-maps $f,g:\mathfrak{D}\to\mathfrak{D}$ is said to have property (Q) if $\mathfrak{F}(f^n)\cap\mathfrak{F}(g^n)=\mathfrak{F}(f)\cap\mathfrak{F}(g)$ holds for each $n\in\mathbb{N}$ (see [12]).

3. MAIN RESULTS

We start with a straightforward yet useful lemma.

Lemma 3.1 Let $\{\theta_n\}_{n=1}^{\infty}$ be a sequence in a parametric metric space $(\mathfrak{D},\mathcal{P})$ such that

$$\mathcal{P}(\theta_n,\theta_{n+1},t)\le\ \mu^n\mathcal{P}(\theta_0,\theta_1,t)\tag{1}$$

where $\mu\in[0,1)$ and $n=1,2,\ldots$. Then $\{\theta_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $(\mathfrak{D},\mathcal{P})$.

Proof Let $m>n\ge1$. It follows that

$$\begin{aligned}\mathcal{P}(\theta_n,\theta_m,t)&\le\mathcal{P}(\theta_n,\theta_{n+1},t)+\mathcal{P}(\theta_{n+1},\theta_{n+2},t)\\&\quad+\cdots+\mathcal{P}(\theta_{m-1},\theta_m,t)\\&\le(\mu^n+\mu^{n+1}+\cdots+\mu^{m-1})\,\mathcal{P}(\theta_0,\theta_1,t)\\&\le\frac{\mu^n}{1-\mu}\,\mathcal{P}(\theta_0,\theta_1,t)\end{aligned}$$

for all $t>0$. Since $\mu<1$. Assume that $\mathcal{P}(\theta_0,\theta_1,t)>0$. By taking limit as $m,n\to+\infty$ in above inequality we get $\lim_{n,m\to\infty}\mathcal{P}(\theta_n,\theta_m,t)=0$. Therefore, $\{\theta_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathfrak{D}$. Also, if $\mathcal{P}(\theta_0,\theta_1,t)=0$, then $\mathcal{P}(\theta_n,\theta_m,t)=0$ for all $m>n$ and hence $\{\theta_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathfrak{D}$.

Now, our first main result as follows.

Theorem 3.2 Let $(\mathfrak{D},\mathcal{P})$ be a complete parametric metric space and $f,g:\mathfrak{D}\to\mathfrak{D}$ be two maps such that $f\mathfrak{D}\supset g\mathfrak{D}$ and one of the subsets $f\mathfrak{D}$ and $g\mathfrak{D}$ is complete. Suppose that

$$\mathcal{P}(f\theta,f\vartheta,t)\ge\mu\mathcal{P}(g\theta,g\vartheta,t)\tag{2}$$

for some $\mu>1$ and all $\theta,\vartheta\in\mathfrak{D}$ and for all $t>0$. Then f and g have a unique point of coincidence. If, moreover, the pair (f,g) is (occasionally) weakly compatible, then f and g have a unique common fixed point.

Proof Take arbitrary $\theta_0\in\mathfrak{D}$. Construct sequences $\{\theta_n\}$ and $\{\vartheta_n\}$ such that $\vartheta_n=g\theta_n=f\theta_{n+1}$ for $n=0,1,2,\ldots$. For all $t>0$, condition (2) implies that

$$\begin{aligned}\mathcal{P}(\vartheta_n,\vartheta_{n-1},t)&=\mathcal{P}(f\theta_{n+1},f\theta_n,t)\\&\ge\mu\mathcal{P}(g\theta_{n+1},g\theta_n,t)\\&=\mu\mathcal{P}(\vartheta_{n+1},\vartheta_n,t)\end{aligned}$$

Hence

$$\begin{aligned}\mathcal{P}(\vartheta_{n+1},\vartheta_n,t)&\le\mu^{-1}\mathcal{P}(\vartheta_n,\vartheta_{n-1},t)\\&\le\cdots\\&\le(\mu^{-1})^n\mathcal{P}(\vartheta_1,\vartheta_0,t)\end{aligned}$$

Since $\mu^{-1}\in(0,1)$, Lemma 3.1 implies that $\{\vartheta_n\}$ is a Cauchy sequence. Let, e.g., $f\mathfrak{D}$ be complete. Then there exists $\omega\in\mathfrak{D}$ such that $\vartheta_n\to f\omega$, when $n\to\infty$. Let us prove that $f\omega=g\omega$. Putting $\theta=\theta_n,\vartheta=\ \omega$ in (2) we obtain for all $t>0$,

$$\mathcal{P}(f\theta_n,f\omega,t)\ge\mu\mathcal{P}(g\theta_n,g\omega,t)$$

and $\vartheta_{n-1}=f\theta_n\to f\omega$ implies that $g\theta_n\to g\omega$. Since also $g\theta_n\to f\omega$ it follows that $f\omega=g\omega$. Thus, $f\omega=g\omega=\varpi$ is a point of coincidence for (f,g). Suppose that there is another point of coincidence $\varpi_1=f\omega_1=g\omega_1$. Then for all $t>0$,

$$\begin{aligned}\mathcal{P}(\varpi,\varpi_1,t)&=\mathcal{P}(f\omega,f\omega_1,t)\\&\ge\mu\mathcal{P}(g\omega,g\omega_1t)\\&=\mu\mathcal{P}(\varpi,\varpi_1,t),\end{aligned}$$

implying (since $\mu>1$) that $\mathcal{P}(\varpi,\varpi_1,t)=0$. Thus, the point of coincidence is unique. If the pair (f,g) is weakly compatible, applying [12, Proposition 1.12] we conclude that f and g have a unique common fixed point. If (f,g) is occasionally weakly compatible, the same conclusion follows from [13, Lemma 1.6].

Now we give an example illustrating Theorem 3.2.

Example 3.3 Let $\mathfrak{D}=[0,+\infty)$ be endowed with parametric metric,

$$\mathcal{P}(\theta,\vartheta,t)=\begin{cases}t\max\{\theta,\vartheta\}, & \theta\ne\vartheta\\0, & \theta=\vartheta\end{cases}$$

for all $\theta,\vartheta\in\mathfrak{D}$ and $t>0$. Consider functions $f,g:\mathfrak{D}\to\mathfrak{D}$ defined by

$$f\theta=\frac{\theta}{3},\ g\theta=\frac{\theta}{5}$$

and take arbitrary $1<\mu\le\frac{5}{3}$. Then all the conditions of Theorem 3.2 are fulfilled. Obviously, f and g have a unique common fixed point.

Taking $g=i_D$ in Theorem 3.2 we obtain the following parametric metric version of [11].

Corollary 3.4 Let $(\mathfrak{D},\mathcal{P})$ be a complete parametric metric space and let $f:\mathfrak{D}\to\mathfrak{D}$ be a surjection. If there is a constant $\mu>1$ such that

$$\mathcal{P}(f\theta,f\vartheta,t)\ge\mu\mathcal{P}(\theta,\vartheta,t)\tag{3}$$

$\forall\ \theta,\vartheta\in\mathfrak{D}$ and all $t>0$. Then f has a unique fixed point.

Lemma 3.5 Let $(\mathfrak{D},\mathcal{P})$ be a complete parametric metric space and let $f:\mathfrak{D}\to\mathfrak{D}$ such that $\mathfrak{F}(f)\neq\emptyset$ and that

$$\mathcal{P}(f\theta,f^2\theta,t)\geq\mu\mathcal{P}(\theta,f\theta,t) \quad (4)$$

holds for some $\mu>1$ and all $t>0$ and either (i) for all $\theta\in\mathfrak{D}$, or (ii) for all $\theta\in\mathfrak{D},\theta=f\theta$. Then f has the property (P).

Proof. (i) Suppose that (4) holds for some $\mu>1$ and each $\theta\in\mathfrak{D}$ and let $\xi\in\mathfrak{F}(f^n)$ for some $n>1$. Then for all $t>0$,

$$\begin{aligned}\mathcal{P}(\xi,f\xi,t)&=\mathcal{P}(f^n\xi,f^{n+1}\xi,t)\\&=\mathcal{P}(ff^{n-1}\xi,f^2f^{n-1}\xi,t)\\&\geq\mu\mathcal{P}(f^{n-1}\xi,f^n\xi,t)\\&=\mu\mathcal{P}(ff^{n-2}\xi,f^2f^{n-2}\xi,t)\\&\geq\mu^2\mathcal{P}(f^{n-2}\xi,f^{n-1}\xi,t)\\&\geq\cdots\\&\geq\mu^n\mathcal{P}(\xi,f\xi,t)\end{aligned}$$

If $\mathcal{P}(\xi,f\xi,t)>0$ then $1\geq\mu^n$ which is a contradiction. It follows that $\xi\in\mathfrak{F}(f)$ and $\mathfrak{F}(f^n)=\mathfrak{F}(f)$.

(ii) Let (4) holds whenever $\theta\neq f\theta$, and let $\xi\in\mathfrak{F}(f^n)$ for some $n>1$. If $\xi=f\xi$, the proof is complete. Suppose $\xi\neq f\xi$. Then, similarly as in the case (i) we get that for all $t>0$,

$$\mathcal{P}(\xi,f\xi,t)=\mathcal{P}(ff^{n-1}\xi,f^2f^{n-1}\xi,t).$$

In order to use (4) we need that $f^{n-1}\xi\neq ff^{n-1}\xi=f^n\xi$. But, if this is not the case, then $f^n\xi=\xi$ and so $\xi=f^n\xi=f\xi$, a contradiction. Hence, applying (4) we obtain that

$$\begin{aligned}\mathcal{P}(\xi,f\xi,t)&=\mathcal{P}(ff^{n-1}\xi,f^2f^{n-1}\xi,t)\\&\geq\mu\mathcal{P}(f^{n-1}\xi,f^n\xi,t)\\&=\mu\mathcal{P}(ff^{n-2}\xi,f^2f^{n-2}\xi,t)\end{aligned}$$

Repeating the same argument several times we finally obtain, similarly as in case (i), that $\mathcal{P}(\xi,f\xi,t)\geq\mu^n\mathcal{P}(\xi,f\xi,t)$, which again implies $\xi=f\xi$ since $\mu>1$. Contradiction.

Lemma 3.6 Let $(\mathfrak{D},\mathcal{P})$ be a complete parametric metric space and let $f:\mathfrak{D}\to\mathfrak{D}$ be a continuous surjective self-map. If

$$\mathcal{P}(f\theta,f^2\theta,t)\geq\mu\mathcal{P}(\theta,f\theta,t) \quad (5)$$

holds for some $\mu>1$, for all $\theta\in\mathfrak{D}$ and all $t>0$. Then $\mathfrak{F}(f)\neq\emptyset$.

Proof Let $\theta_0\in\mathfrak{D}$ be arbitrary and choose a sequence $\{\theta_n\}$ such that $\theta_n=f\theta_{n+1}$ $n=0,1,2,\ldots$ Then using (5), we get

$$\begin{aligned}\mathcal{P}(\theta_n,\theta_{n-1},t)&=\mathcal{P}(f\theta_{n+1},f^2\theta_{n+1},t)\\&\geq\mu\mathcal{P}(\theta_{n+1},f\theta_{n+1},t)\\&=\mu\mathcal{P}(\theta_{n+1},\theta_n,t),\end{aligned}$$

for each n $\in\mathbb{N}$. Hence, $\mathcal{P}(\theta_{n+1},\theta_n,t)\leq\mu^{-1}\mathcal{P}(\theta_n,\theta_{n-1},t)$ and, by Lemma 3.1, $\{\theta_n\}$ is a Cauchy sequence in $\mathfrak{D}$. If $\theta_n\to\theta$, when $n\to\infty$, then, using continuity of f, we easily get that $f\theta=\theta$. Hence $\mathfrak{F}(f)\neq\emptyset$.

Theorem 3.7 Let $(\mathfrak{D},\mathcal{P})$ be a complete parametric metric space and $f,g:\mathfrak{D}\to\mathfrak{D}$ be two maps such that $f\mathfrak{D}\supset g\mathfrak{D}$ and that at least one of these subspaces is complete. Suppose that there exist $a,b,c\geq0$ with $a+b+c>1$ such that

$$\begin{aligned}\mathcal{P}(f\theta,f\vartheta,t)&\geq a\,\mathcal{P}(g\theta,g\vartheta,t)\\&+b\,\mathcal{P}(g\theta,f\theta,t)\\&+c\,\mathcal{P}(g\vartheta,f\vartheta,t) \quad (6)\end{aligned}$$

$\forall\,\theta,\vartheta\in\mathfrak{D}$ with $\theta\neq\vartheta$ and all $t>0$. Then f and g have a unique point of coincidence. If, moreover, the pair (f,g) is (occasionally) weakly compatible, then f and g have a unique common fixed point.

Proof Let $\theta_0\in\mathfrak{D}$ be arbitrary. As in the proof of Theorem 3.2 choose sequences $\{\theta_n\}$ and $\{\vartheta_n\}$ such that $\vartheta_n=g\theta_n=f\theta_{n+1}$ for $n=0,1,2,\ldots$ For all $t>0$, applying (6), we obtain

$$\begin{aligned}\mathcal{P}(\vartheta_n,\vartheta_{n-1},t)&=\mathcal{P}(f\theta_{n+1},f\theta_n,t)\\&\geq a\,\mathcal{P}(g\theta_{n+1},g\theta_n,t)\\&+b\,\mathcal{P}(g\theta_{n+1},f\theta_{n+1},t)\\&+c\,\mathcal{P}(g\theta_n,f\theta_n,t)\\&\geq a\,\mathcal{P}(\vartheta_{n+1},\vartheta_n,t)\\&+b\,\mathcal{P}(\vartheta_{n+1},\vartheta_n,t)\\&+c\,\mathcal{P}(\vartheta_n,\vartheta_{n-1},t)\\&=(a+b)\,\mathcal{P}(\vartheta_{n+1},\vartheta_n,t)\\&+c\,\mathcal{P}(\vartheta_n,\vartheta_{n-1},t)\end{aligned}$$

Hence

$$(1-c)\,\mathcal{P}(\theta_{n-1},\theta_n,t)\geq(a+b)\,\mathcal{P}(\theta_{n+1},\theta_n,t)$$

If $a+b=0$, then $c>0$. The above inequality implies that a negative number is greater then or equal to zero. This is impossible. So, $a+b\neq0$ and $(1-c)>0$. Therefore,

$$\mathcal{P}(\theta_{n+1},\theta_n,t)\leq\mu\mathcal{P}(\theta_{n-1},\theta_n,t) \quad (7)$$

where $\mu=\frac{1-c}{a+b}<1$ for all $n\in\mathbb{N}\cup\{0\}$ and $t>0$. Repeating (7) n-times, we get

$$\mathcal{P}(\theta_{n+1},\theta_n,t)\leq\mu^n\,\mathcal{P}(\theta_0,\theta,t)$$

for all $t>0$. By Lemma 3.1, $\{\theta_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Suppose that, e.g., $f\mathfrak{D}$ is complete. Then there exists $\theta^\star\in\mathfrak{D}$ such that $f\theta_n\to f\theta^\star$ when n $\to\infty$. Let us prove that $f\theta^\star=g\theta^\star$. Then, using (6), we get

$$\begin{aligned}\mathcal{P}(\theta_n,f\theta^\star,t)&=\mathcal{P}(f\theta_{n+1},f\theta^\star,t)\\&\geq a\,\mathcal{P}(g\theta_{n+1},g\theta^\star,t)\\&+b\,\mathcal{P}(g\theta_{n+1},f\theta_{n+1},t)\\&+c\,\mathcal{P}(g\theta^\star,f\theta^\star,t)\\&=a\,\mathcal{P}(f\theta_{n+2},g\theta^\star,t)\\&+b\,\mathcal{P}(f\theta_{n+2},f\theta_{n+1},t)\\&+c\,\mathcal{P}(g\theta^\star,f\theta^\star,t)\end{aligned}$$

which implies that as $n\to+\infty$,

$$0\geq(a+c)\,\mathcal{P}(g\theta^\star,f\theta^\star,t)$$

Hence $g\theta^\star=f\theta^\star$. Thus, $g\theta^\star=f\theta^\star=\varpi$ is a point of coincidence for (f,g). Suppose that there is another point of coincidence $\varpi_1=f\omega_1=g\omega_1$. Then for all $t>0$,

$$\begin{aligned}\mathcal{P}(\varpi,\varpi_1,t)&=\mathcal{P}(f\theta^\star,f\omega_1,t)\\&\geq a\,\mathcal{P}(g\theta^\star,g\omega_1,t)\\&+b\,\mathcal{P}(g\theta^\star,f\theta^\star,t)\\&+c\,\mathcal{P}(g\omega_1,f\omega_1,t)\\&=a\,\mathcal{P}(\varpi,\varpi_1,t)\\&+b\,\mathcal{P}(\varpi,\varpi,t)\\&+c\,\mathcal{P}(\varpi_1,\varpi_1,t)\\&=a\,\mathcal{P}(\varpi,\varpi_1,t),\end{aligned}$$

implying (since $a>1$) that $\mathcal{P}(\varpi,\varpi_1,t)=0$. Thus, the point of coincidence is unique. If the pair (f,g) is weakly compatible, applying [13, Proposition 1.12] we conclude that

f and g have a unique common fixed point. If (f, g) is occasionally weakly compatible, the same conclusion follows from [14, Lemma 1.6]. This completes the proof.

Setting $b = c = 0$ and $a = \mu$ in Theorem 7.3.2, we can obtain the following result.

Corollary 3.8 $(\mathfrak{D}, \mathcal{P})$ be a complete parametric metric space and $f, g: \mathfrak{D} \to \mathfrak{D}$ be two maps such that $f\mathfrak{D} \supset g\mathfrak{D}$ and that at least one of these subspaces is complete. Suppose that there exists a real $\mu > 1$ such that

$$\mathcal{P}(f\theta, f\vartheta, t) \geq \mu\mathcal{P}(g\theta, g\vartheta, t) \tag{8}$$

$\forall\, \theta, \vartheta \in \mathfrak{D}$ with $\theta \neq \vartheta$ and all $t > 0$. Then f and g have a unique point of coincidence. If, moreover, the pair (f, g) is (occasionally) weakly compatible, then f and g have a unique common fixed point.

Corollary 3.9 Let $(\mathfrak{D}, \mathcal{P})$ be a complete parametric metric space and $f: \mathfrak{D} \to \mathfrak{D}$ be a surjection. Suppose that there exists a constant $\mu > 1$ such that

$$\mathcal{P}(f\theta, f\vartheta, t) \geq \mu\mathcal{P}(\theta, \vartheta, t) \tag{9}$$

$\forall\, \theta, \vartheta \in \mathfrak{D}$ and all $t > 0$. Then f has a unique fixed point in $\mathfrak{D}$.

Proof From Corollary 3.8, it follows that f has a fixed point $\theta^\star$ in $\mathfrak{D}$ by setting $g = i_D$.

Uniqueness. Suppose that $\theta^\star \neq \vartheta^\star$ is also another fixed point of f, then from condition (9), we obtain

$$\begin{aligned}\mathcal{P}(\theta^\star, \vartheta^\star, t) &= \mathcal{P}(f\theta^\star, f\vartheta^\star, t)\\ &\geq \mu\mathcal{P}(\theta^\star, \vartheta^\star, t)\end{aligned}$$

which implies $\mathcal{P}(\theta^\star, \vartheta^\star, t) = 0$, that is $\theta^\star = \vartheta^\star$. This completes the proof.

Corollary 3.10 Let $(\mathfrak{D}, \mathcal{P})$ be a complete parametric metric space and $f: \mathfrak{D} \to \mathfrak{D}$ be a surjection. Suppose that there exists a positive integer n and a real number $\mu > 1$ such that

$$\mathcal{P}(f^n\theta, f^n\vartheta, t) \geq \mu\mathcal{P}(\theta, \vartheta, t) \tag{10}$$

$\forall\, \theta, \vartheta \in \mathfrak{D}$ and all $t > 0$. Then T has a unique fixed point in $\mathfrak{D}$.

Proof From Corollary 3.9, f^n has a fixed point $\theta^\star$. But $f^n(f\theta^\star) = f(f^n\theta^\star) = f\theta^\star$, So $f\theta^\star$ is also a fixed point of f^n. Hence $f\theta^\star = \theta^\star, \theta^\star$ is a fixed point of f. Since the fixed point of f is also fixed point of f^n , the fixed point of f is unique.

Example 3.11 Let $\mathfrak{D} = [0,1]$ be endowed with parametric metric $\mathcal{P}(\theta, \vartheta, t) = t|\theta - \vartheta|$ for all $\theta, \vartheta \in \mathfrak{D}$ and all $t > 0$. Then $(\mathfrak{D}, \mathcal{P})$ is a complete parametric metric space. Define $f: \mathfrak{D} \to \mathfrak{D}$ by $f(\theta) = 3\theta$ for all $\theta \in \mathfrak{D}$. Then f is surjection on $\mathfrak{D}$. Further

$$\begin{aligned}\mathcal{P}(f\theta, f\vartheta, t) &= t|3\theta - 3\vartheta|\\ &= 3t|\theta - \vartheta|\\ &\geq 2\mathcal{P}(\theta, \vartheta, t)\\ &= \mu\mathcal{P}(\theta, \theta, t)\end{aligned}$$

for all $\theta, \vartheta \in D$ and all $t > 0$,where $\mu = 2 > 1$. Then (9) is satisfied. Thus, all conditions of Corollary 3.9 are satisfied and $\theta^\star = 0$ is a fixed point of f.

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