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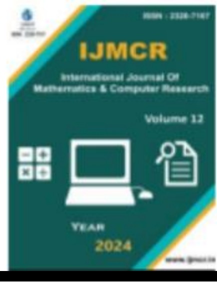

Certain Bilinear Generating Relations for the Extended Gauss Hypergeometric Function, Using Fractional Calculus

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ARTICLE INFO	ABSTRACT
Published Online: **20 June 2024** Corresponding Author: **Pankaj Kumar Shukla**	The main aim of the present paper is to apply the extended Riemann Liouville fractional derivative operator for finding some bilinear generating relations for extended Gauss hypergeometric function. Two main results are obtained, which are presented in the form of two theorems.

KEYWORDS: Gamma function, Beta function, Riemann-Liouville fractional derivative, hypergeometric function, Fox H-function, generating functions, extended Gauss hypergeometric function.

I. INTRODUCTION

The subject of fractional calculus, now a days is one of the most rapidly growing subjects of mathematical analysis. The fractional integral operators, involving various Special functions have found significant importance and applications in various sub fields of applicable mathematical analysis. The applications of fractional calculus are also seen in various fields, including turbulence and fluid dynamics, stochastic dynamical system, plasma physics and controlled thermal nuclear fusion, non-linear control theory, image processing, nonlinear biological system, astrophysics etc. (e.g.one can see [1,9,11,13,14,25])

In the last three decades, a number of workers like Love [16], Mc Bride [17], Kalla [18,19], Kalla and Saxena [20], Saigo[21,22], Kilbas [23],have studied the properties ,applications & different extensions of various operators of fractional calculus on a number of classical & non classical Special functions & polynomials. A sufficient account of fractional calculus operators along with their properties and applications can be found in the research monographs by Miller and Ross [25], & Kiryakova [24]. The first application of fractional calculus was due to Abel [27] in the solution to the fractional problem. In fractional calculus, the fractional derivatives are defined via fractional integrals.

In the recent years, certain extended fractional derivative operators, associated with Special functions have been actively investigated and applied on various Special functions. Authors Agarwal & Choi [12,20], have introduced certain extended fractional derivative operators, and applied them on various Special functions.

Motivated by these recent developments in the field of applications of extended fractional derivatives to various Special functions, in the present paper an attempt has been made to obtain some bilinear generating relations including extended Gauss hypergeometric functions, using extended Riemann Liouville fractional derivative operator, defined by Choi & Pairs in their very recent paper [1], published in the year 2015.

(1). The **extended Gauss hypergeometric function** $\mathbf{F_p^{(\alpha,\beta,\kappa,\mu)}(a,\ b,\ c;\ z)}$ is defined by Agrawal &Choi [1] as follows:

$$F_p^{(\alpha,\beta,\kappa,\mu)}(a, b, c; z) =\sum_{n=0}^{\infty}(a)_n \frac{B_p^{(\alpha,\beta,\kappa,\mu)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}, \quad (1.1)$$

($|z|<1$; min {Re(α), Re(β), Re(κ), Re(μ)}>0; Re(c)>Re(b)>0; Re(p)≥0),

where B(u,v) is the familiar Beta function defined as:

$$B(u,v)=\int_0^1 t^{u-1}(1-t)^{v-1}\,dt, \quad (\text{ Re(u)>; Re(v)>0 })\}.$$

$$=\frac{\Gamma u\Gamma v}{\Gamma v+v} \quad (u,v\in C), \quad (1.2)$$

where Γ denotes the Eulers Gamma function [4].

It is to note here that, for p=0, (1.1) reduces to the ordinary Gauss hypergeometric function ${}_2F_1(a, b, c; z)$.

(2). The **extended beta function** $\mathbf{B_p^{(\alpha,\beta,\kappa,\mu)}(x,\ y)}$ is defined by Srivastava [8] as:

$$B_p^{(\alpha,\beta,\kappa,\mu)}(x,y)= \left\{\int_0^1 t^{x-1}(1-t)^{y-1}\ {}_1F_1\left(\alpha;\beta;\frac{-p}{t^k(1-t)^k}\right)\right\}dt,$$

(1.3)

κ≥0, μ≥0, min {Re(α), Re(β),}>0; Re(x)>-Re(κa)>0; Re(y)≥-Re(μa) & Re(ρ).

(3). A further extension of the **extended Gauss hypergeometric function** $\mathbf{F_{p;\kappa,\mu}(a,b,c;z;m)}$ is defined by Srivastava as [1]:

$$F_{p;\kappa,\mu}(a,b,c;z;m)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{B_p^{(\alpha,\beta,\kappa,\mu)}(b+n,c-b+m)}{B(b+n,c-b+m)}\frac{z^n}{n!} \quad (1.4)$$

where, (p≥0; Re(κ)>0, Re(μ)}>0; Re(c)>Re(b)>m; Re(p)≥0).

(4). The **extended Riemann -Liouville fractional derivative of F(z)** of order υ is defined by Agarwal, Choi & Pairs [1] by the following relations:

$$D_z^{\upsilon,p,k,\mu}f(z)=\frac{1}{\Gamma-\upsilon}\int_0^z(z-t)^{-\upsilon-1}f(t)dt\ {}_1F_1\left(\alpha;\beta;\frac{pz^{k+\mu}}{t^k(z-t)^{\mu}}\right)dt, \quad (1.5)$$

and

$$D_z^{\upsilon,p,k,\mu}f(z)=\frac{d^m}{dz^m}D_z^{\upsilon-m;p;k,\mu}f(z)$$
$$=\frac{d^m}{dz^m}\left\{\frac{1}{\Gamma m-\upsilon}\int_0^z(z-t)^{m-\upsilon-1}f(t)dt\ {}_1F_1\left(\alpha;\beta\frac{pz^{k+\mu}}{t^k(z-t)^{\mu}}\right)dt\right\}, \quad (1.6)$$

where, $m-1\le Re(\upsilon)<m),(Re(\upsilon)<0;Re(p)>0;Re(k)>0;\ Re(\mu)>0)$.

From (1.5) & (1.6), it may easily be seen that for p=0, we obtain the classical-Riemann Liouville fractional derivative.

In the present paper, an attempt has been made to obtain certain bilinear generating relations involving extended Gauss hypergeometric function (1.1), using operators (1.5) & (1.6).

While proving the main results of the present paper, we will use the following well-known results seen in [1].

2. PRELIMINARIES

While proving the main results, the following well-known identities & results will be used.

2.1. The elementary identity ([24, p.291]):

$$[(1-x)-t]^{-\alpha}=(1-t)^{-\alpha}\left(1-\frac{x}{1-t}\right)^{-\alpha} \quad (2.1)$$

2.2. The identity ([7, p.595]):

$$[(1-x)-t]^{-\alpha}=(1-t)^{-\alpha}\left(1+\frac{xt}{1-t}\right)^{-\alpha} \quad (2.2)$$

2.3. The result [1, p. 458]:

$$\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}F_{p,k,\mu}(\alpha+n,\lambda,\upsilon;z;m)\,t^n=(1-t)^{-\alpha}F_{p,k,\mu}\left(\alpha,\lambda;\upsilon;\frac{x}{(1-t)};m\right) \quad (2.3)$$

2.4. The generalized binomial theorem, [1, p.456]

$$(1-z)^{-\alpha}=\sum_{l=0}^{\infty}\frac{(\alpha)_l}{l!},(|l|<1;\ \alpha\epsilon C) \quad (2.4)$$

2.5. $(z)_n=\frac{\Gamma z+n}{\Gamma z}$,[4] (2.5)

2.6. The result, [1, p. 456]:

$$D_z^{\lambda-\upsilon,p,k,\mu}\{z^{\lambda-1}(1-z)^{-\alpha}\}$$
$$=\frac{\Gamma(\lambda)z^{(\upsilon-1)}}{\Gamma\upsilon}\sum_{n=0}^{\infty}\frac{(\alpha)_n(\lambda)_n}{(\upsilon)_n}\frac{B_p^{\alpha,\beta,k,\mu}(\lambda+n,\upsilon-\lambda+m)}{B(\lambda+n,\upsilon-\lambda+m)}\frac{z^n}{n!},$$
$$=\frac{\Gamma(\lambda)z^{(\upsilon-1)}}{\Gamma\upsilon}F_{p,k,\mu}(\alpha,\lambda;\upsilon;z;m), \quad (2.6)$$

where, m-1≤Re(υ)<m, for some mϵN & Re(v) < Re(λ).

2.7. The result, [1, p. 457]:

$$D_z^{\lambda-\upsilon,p,k,\mu}\{(1-az)^{-\alpha}(1-az)^{-\beta}\}$$
$$=\frac{\Gamma(\lambda)z^{(\upsilon-1)}}{\Gamma\upsilon}F_{1,p,k,\mu}(\alpha,\beta,\lambda;\upsilon;az;bz;m), \quad (2.7)$$

where, m-1≤Re(υ)<m, for some mϵN & Re(v) < Re(λ).

2.8. The result, [1, p. 457]:

$$D_z^{\lambda-\upsilon,p,k,\mu}\left\{(1-z)^{-\alpha}z^{\lambda-1}F_{p,k,\mu}\left(\alpha,\lambda;\upsilon;\frac{x}{(1-t)};m\right)\right\},$$
$$=\frac{\Gamma(\lambda)z^{(\upsilon-1)}}{\Gamma\upsilon}F_{2,p,k,\mu}(\alpha,\beta,\lambda;y;\upsilon;x;z;m), \quad (2.8)$$

where, m-1≤Re(υ)<m, for some mϵN & Re(v) < Re(λ).

3. MAINS RESULT

3.1. Bilinear generating relations for the extended Gauss hypergeometric function $\mathbf{F_{p;\kappa,\mu}(a,b,c;z;m)}$:

We use extended fractional derivatives, defined in (1.6), for establishing some bilinear generating relations for extended Gauss hypergeometric function $F_{p;\kappa,\mu}(a,b,c;z;m)$:

:

Theorem. I. The following bilinear generating relation holds:

$$\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}F_{p,k,\mu}(\alpha+n,\lambda;\upsilon;z;m)\,F_{p,k,\mu}(\alpha n;\lambda;\upsilon;z;m)\frac{(t)^n}{(1-y)^n}$$
$$=F_{2,p,\kappa,\mu}\left(\delta,\alpha,\lambda,y;\upsilon;z;\frac{z}{\left(1-\frac{t}{1-y}\right)};m\right), \quad (3.1)$$

where, {|x|<min(1,|1-t)|}, (αϵC,|z|<1; $\left|\frac{t}{1-y}\right|$<1;), & m-1≤Re(β-y) <m<Re(β), for some mϵN & Re(λ) < Re(υ).

Proof of (3.1): Replacing t by $\frac{t}{(1-y)}$ in (2.3) & multiplying both sides of the resulting equation by y^{y-1}, we obtain:

$$\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}F_{p,k,\mu}(\alpha+n,\lambda;\upsilon;z;m)\{y^{y-1}(1-y)^{-n}\}t^n$$
$$=y^{y-1}\left[1-\frac{t}{1-y}\right]^{-\alpha}F_{p,k,\mu}\left(\alpha;\lambda;\upsilon;\frac{z}{\left(1-\frac{t}{1-y}\right)};m\right),$$

On operating both sides of the above equation by the fractional derivative $D_y^{y-\delta,p,\kappa,\mu}$, we obtain:

$$D_y^{y-\delta,p,\kappa,\mu}\left[\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}F_{p,k,\mu}(\alpha+n,\lambda;\upsilon;z;m)\{y^{y-1}(1-y)^{-n}\}t^n\right]$$
$$=\{D_y^{y-\delta,p,\kappa,\mu}\,y^{y-1}\left[1-\left(\frac{t}{1-y}\right)\right]^{-\alpha}F_{p,k,\mu}\left(\alpha;\lambda;\upsilon;\frac{z}{\left(1-\frac{t}{1-y}\right)};m\right)$$

Changing the order of the summation &the fractional derivatives in the last equation, we obtain:

$$\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}F_{p,k,\mu}(\alpha+n,\lambda;\upsilon;z;m)\,D_y^{y-\delta,p,\kappa,\mu}\{y^{y-1}(1-y)^{-n}\}t^n$$
$$=\{D_y^{y-\delta,p,\kappa,\mu}\,y^{y-1}\left[1-\left(\frac{t}{1-y}\right)\right]^{-\alpha}F_{p,k,\mu}\left(\alpha;\lambda;\upsilon;\frac{z}{\left(1-\frac{t}{1-y}\right)};m\right)$$

Finally, using (2.6) & (2.8) on the right-hand side of the above equation, we obtain:

$$\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}\, F_{p,k,\mu}(\alpha+n,\lambda;\upsilon;z;m)\times F_{p,k,\mu}(\alpha n;\lambda;\upsilon;z;m)\,\frac{(t)^n}{(1-y)^n}$$
$$= F_{2,p,\kappa,\mu}\left(\delta,\alpha,\ \lambda;y;\ \upsilon;\ z;\ \frac{z}{\left(1-\frac{t}{1-y}\right)};m\right) \qquad (3.2)$$

which is the desired result (3.1), and thus theorem I is established.

Theorem. II. The following bilinear generating relation holds:

$$\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}\, F_{p,k,\mu}(\alpha+n,\lambda;\upsilon;z;m)\, F_{p,k,\mu}(\alpha n;\lambda;\upsilon;z;m)(yt)^n$$
$$= F_{2,p,\kappa,\mu}\left(\delta,\alpha,\lambda,y;\upsilon;z;\frac{z}{yt};m\right), \qquad (3.3)$$

where, {|x|<min (1, |1-t)|}, (αϵ C, |z|<1; |yt|<1;), & m-1≤Re(β-y) <m<Re(β), for some mϵN & Re(λ) < Re(υ).

Proof of (3.3): Replacing t by yt in (2.3) & multiplying both sides of the resulting equation by y^{y-1}, we obtain:

$$\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}F_{p,k,\mu}(\alpha+n,\ \lambda;\ \upsilon;\ z;\ m)\,\{y^{y-1}(1-yt)^n\}$$
$$= y^{y-1}[1-(yt)]^{-\alpha}F_{p,k,\mu}\left(\alpha;\ \lambda;\ \upsilon;\ \frac{z}{(1-yt)};\ m\right),$$

On operating both sides of the above equation by the fractional derivative $D_y^{y-\delta,p,\kappa,\mu}$, we obtain:

$$D_y^{y-\delta,p,\kappa,\mu}\left[\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}F_{p,k,\mu}(\alpha+n,\lambda;\upsilon;z;m)\ \times\{y^{y-1}(1-yt)^n\}t^n\right]$$
$$=\left\{D_y^{y-\delta,p,\kappa,\mu}\ y^{y-1}[1-(yt)]^{-\alpha}F_{p,k,\mu}\left(\alpha;\ \lambda;\ \upsilon;\ \frac{z}{(1-yt)};\ m\right)\right\},$$

Changing the order of the summation & the fractional derivatives in the last equation, we obtain:

$$\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}F_{p,k,\mu}(\alpha+n,\lambda;\upsilon;z;m)\times D_y^{y-\delta,p,\kappa,\mu}\{y^{y-1}(1-yt)^n\} =$$
$$\left\{D_y^{y-\delta,p,\kappa,\mu}\ y^{y-1}[1-(yt)]^{-\alpha}F_{p,k,\mu}\left(\alpha;\ \lambda;\ \upsilon;\ \frac{z}{(1-yt)};\ m\right)\right\},$$

Finally, using (2.6) & (2.8) on the right -hand side of above equation, we obtain:

$$\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}\, F_{p,k,\mu}(\alpha+n,\lambda;\upsilon;z;m)\times F_{p,k,\mu}(\alpha n;\lambda;\upsilon;z;m)(yt)^n$$
$$= F_{2,p,\kappa,\mu}\left(\delta,\alpha,\ \lambda;y;\ \upsilon;\ z;\ \frac{z}{(1-yt)};m\right) \qquad (3.4)$$

which is the desired result (3.3), and thus theorem II is established.

4. CONCLUDING REMARKS

Linear, bilinear and bilateral generating relations have been of much interest to various researchers in the recent past. Various mathematicians investigating and introducing certain extended fractional derivative and integral operators and applying them on various Special functions and obtaining linear, bilinear and bilateral generating relations involving some Special functions.

In the present paper, an attempt has been made to obtain some bilinear generating relations for extended Gauss hypergeometric function, applying the extended Riemann Liouville fractional derivative operator.

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