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A Note on Strongly Large Ideals

Deepali B. Banswal

Department of Mathematics, B.P. Arts, S.M.A. Science & K.K.C. Commerce College, Chalisgaon.

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Corresponding Author: **Deepali B. Banswal**

ABSTRACT

In this paper, I introduced Strongly Large Ideals and studied some properties. In this paper, I also studied Strongly Large Closed ideals and Strongly Large Complement Ideal. An Ideal is called an SL-Closed ideal if it has no proper Strongly Large Extension in L. I prove that direct summands of a lattice L are SL-Closed ideals. I give an example for, the intersection of SL-Closed ideals of a lattice L need not be SL-closed. I also show that , direct summands of SL-Complements are SL-Complements.

KEYWORDS: Ideal, Essential Ideal, Closed Ideal, Strongly Large Ideals, Strongly Large Closed ideal and Strongly Large Complement Ideal

I. INTRODUCTION

Inaam Hadi and Thaar Younis Ghawi[] introduced the concepts of a Strongly Large submodule. B. Ungor, S. Halicioglu, M.A. Kamal and A. Harmanci[] introduced the concept a Strongly Large Closed submodule and a Strongly Large Complement submodule. A submodule K of an R-Module M is called a strongly large submodule in M, if for any m ∈ M, s ∈ R with ms ≠ 0 there exists an r ∈ R such that mr ∈ K and mrs ≠ 0. A submodule N of R-module M is called SL-closed if, N has no proper strongly large extensions in M. A Submodule K is called an SL-complement in M if there exists a submodule L such that K is an SL-Compleent of L in M.

In this paper, I introduced Strongly Large Ideals and studied Closed ideals and Strongly Large Complement Ideal. An Ideal is called an SL-Closed ideal if it has no proper Strongly Large Extension in L. I prove that direct summands of a lattice L are SL-Closed ideals. I give an example for, the intersection of SL-Closed ideals of a lattice L need not be SL-closed. I also show that , direct summands of SL-Complements are SL-Complements.

Throughout in this paper L denotes a lattice with 0 and 1. Id(L) denotes the ideal lattice.

II. STRONGLY LARGE IDEALS

Definition 1 : Strongly large ideals : Let I ∈ Id(L) is called a Strongly Large Ideal in L, in case for any a,b∈ L with a^b ≠ 0 there exists c ∈ L such that a^c ∈ I and a^b^c ≠ 0. It is denoted by SL-ideals

Definition 2: Strongly large closed ideal: An ideal I ∈ Id(L) is called strongly large closed ideal if I has no proper strongly large extension in L. It is denoted by SL-closed ideal.

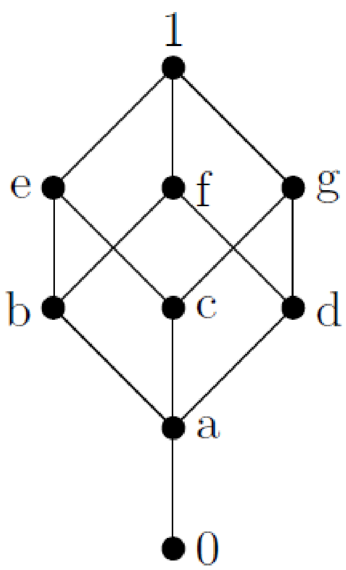

Figure 1

In the lattice shown in figure 1, Let I = {0,a,b,c,e}, for e, g ∈ L such that $e \wedge g = c \neq 0$ there exists f ∈ L such that $e \wedge f = b \in I$ and $e \wedge g \wedge f = a \neq 0$. Thus I is strongly large ideal in L. Similarly J = {0,a,d,g,c} is also strongly large ideal in L.

In the lattice shown in figure 2, there does not exists any strongly large ideal in L.

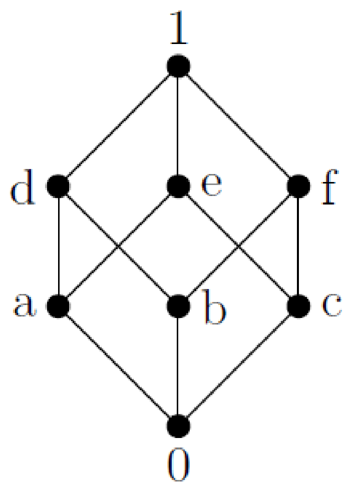

Figure 2

Lemma 1: Let I, J ∈ Id(L) with I ⊆ J. If I is strongly large in J then I ∩ J = (0] implies J ∩ K = (0] for any ideal K of L.

Proof: Suppose I ∈ Id(L) is strongly large in J. Let K ∈ Id(L) be such that I ∩ K = (0]. Clearly I ∩ K ∈ Id(J) and I ∩ (J ∩ K) = (0] and so (J ∩ K) = (0].

Lemma 2: Let I, J ∈ Id(L) with I ⊆ J. Then I is strongly large in J and J is strongly large in L then I is strongly large in L.

Proof: Let a, b ∈ L be such that $a \wedge b \neq 0$. Since J is strongly large in L, there exists $c_2 \in J$ such that $a \wedge c_1 \wedge c_2 \in I$ with $a \wedge b \wedge c_1 \wedge c_2 \neq 0$. Hence I is strongly large in L

Lemma 3: Let L be a lattice. If I_1 is strongly large in J_1 and I_2 is Strongly large in J_2 then $I_1 \cap I_2$ is strongly large in $J_1 \cap J_2$.

Proof: Let a,b ∈ $J_1 \cap J_2$ with $a \wedge b \neq 0$ then there exists $c_1 \in J_1 \cap J_2$ such that $a \wedge c_1 \in I_1$ and $a \wedge b \wedge c_1 \neq 0$. If $a \wedge c_1 \in I_2$ then prove is over. If $a \wedge c_1$ does not belong to I_2, Observe that $a \wedge c_1 \in J_2$ with $a \wedge b \wedge c_1 \neq 0$. So there exists $c_2 \in J_2$ such that $a \wedge c_1 \wedge c_2 \in I_2$ and $a \wedge b \wedge c_1 \wedge c_2 \neq 0$. Hence $a \wedge c_1 \wedge c_2 \in I_1 \cap I_2$ with $a \wedge b \wedge c_1 \wedge c_2 \neq 0$. Therefore $I_1 \cap I_2$ is strongly large in $J_1 \cap J_2$.

Lemma 4: Let I, J ∈ Id(L) with I ⊆ J. Then I is strongly large in L if and only if I is strongly large in J and J is strongly large in L.

Proof: Let I, J ∈ Id(L) with I ⊆ J. Since I is strongly large in J and J is strongly large in L then by Lemma 2, I is Strongly large in L.

Conversely, Suppose that I is Strongly large in L. To show that I is strongly large in J and J is strongly large in L. By Lemma 3, I is strongly large in L implies I ∩ J is strongly large in L ∩ J implies I is strongly large in J. Now let a,b ∈ L be such that $a \wedge b \neq 0$. Since I is strongly large in L there exists c ∈ L such that $a \wedge c \in I$ and $a \wedge b \wedge c \neq 0$. Since I ⊆ J we have $a \wedge c \in J$ with $a \wedge b \wedge c \neq 0$. Hence J is strongly large in L.

Lemma 5: Direct summands of a lattice L are SL-closed ideals.

Proof: Let I, J ∈ Id(L) be such that L = I ⊕ J. Assume that I is strongly large in K ∈ Id(L). Hence K ∩ J = (0] by lemma 1 and so K = I. Therefore I has no proper strongly large extension in L.

Remark: The intersection of SL-closed ideals of a lattice L need not be SL-closed.

Theorem 1: Let L be a lattice and I ∈ Id(L) be an SL-closed ideal in L. If K is strongly large in L then I ∩ K is SL-closed.

Proof: Let J ∈ Id(L) be a strongly large extension of I ∩ K in K that is I ∩ K is strongly large J in K. To show: I ∩ K is SL-closed in K i.e. I ∩ K = J. To show I is strongly large I ∩ J. Let a,b ∈ I ∩ J be such that $a \wedge b \neq 0$. Since K is strongly large in L, there exists c ∈ L such that $a \wedge c \in K$ and $a \wedge b \wedge c \neq 0$. So $a \wedge c \in I \cap K$ and hence $a \wedge b \wedge c \in J$ with $a \wedge b \wedge c \neq 0$. Since I ∩ K is strongly large in J, there exists $c_1 \in J$ such that $a \wedge b \wedge c \wedge c_1 \in I \cap K \subseteq I$ and $a \wedge b \wedge c \wedge c_1 \neq 0$. Therefore I is strongly large in I ∩ J. Since I is an SL-closed in L, it follows that I = I ∩ J. Hence I ∩ K = J and so I ∩ K is an SL-closed in K.

Definition 3: SL-Complement: Let I, J ∈ Id(L) with I ∩ J = (0]. Then J is called SL-complement of I in L if J is an SL-closed in L and I ⊕ J is strongly large in L.

An Ideal I ∈ Id(L) is called SL-complement in L if there exists an ideal J ∈ Id(L) such that I is an SL-complement of J in L.

Theorem 2: Let I, J ∈ Id(L) with I ∩ J = (0]. Then J is an SL-complement of I in L If and only if J is maximal with respect to being J ⊕ I strongly large in M.

Proof: Let J be an SL-complement of I in L and J ⊆ K ∈ L with K ⊕ I strongly large in L. Let P, Q ∈ Id(K) with P ∩ Q ≠(0]. There exists R ∈ Id(L) such that P ∩ R ∈ J ⊕ I and P ∩ Q ∩ R ≠(0]. Then P ∩ R ∈ Id(J). Hence J is strongly large in K. Since J is SL-closed in L, we have J = K. This shows the maximality of J with respect to being J ⊕ I strongly large in M.

Conversely, Let J be maximal with respect to being J ⊕ I strongly large in L and K be strongly large extension of J in L. Then J ∩ K = (0] and hence by Lemma 4, K ⊕ I is strongly large in M. By the maximality of J, J = K, Therefore J is SL-closed in L and J is an SL-complement of I in L

Theorem 3: Let I, J ∈ Id(L). If J is an SL-complement of I in L, then J is a complement of I in L.

Proof: Let K∈ Id(L) with J ⊆ K and I ∩ K = (0]. Since L is a strongly large extension of J ⊕ I, K ⊕ J is a strongly large ideal of L by Lemma 4. So we have J=K from Theorem 2. Hence J is maximal with respect to being I ∩ J =(0].

Remark: Complements need not be SL-Complements.

Theorem 4: Let K ∈ Id(L) be an SL-complement of I ∈ Id(L) and K = K_1 ⊕ K_2 then K_1 is an SL-complement of K_2 ⊕ I in L. In particular, direct summands of SL-complements are SL-complements.

Proof: Let P ∈ Id(L) with K_1 ⊆ P and P ⊕ K_2 ⊕ I is strongly large in L. Then K = K_1 ⊕ K_2 ⊆ P ⊕ K_2. Since K is maximal with respect to being K ⊕ I strongly large in L, then K = K_1

$\oplus$ K_2 = P $\oplus$ K_2. So K_1= P. Thus by Theorem 2, $K_1$is an SL-complement of $K_2 \oplus$ I in L.

Theorem 5: Let I, J ∈ Id(L). If I is an SL-complement in J and J is an SL-complement in L then J contains every strongly large extension of I in L.

Proof: Let I be an SL-complement of an ideal I' in J and J be an SL-complement of an ideal J' in L. By Theorem 2, I is maximal in J with respect to I' $\oplus$ I strongly large in J and J is maximal in L with respect to J' $\oplus$ J strongly closed in L. Let K ∈ Id(L) be a strongly large extension of I in L. Now since I ⊆(K $\oplus$ J') ∩ J ⊆ J and [(K $\oplus$ J') ∩ J] $\oplus$ I' is strongly large in J, we have (K $\oplus$ J') ∩ J = I. We know that (K ∨ J)∩ J' = 0. It is clear that L is a strongly large extension of (K ∨ J) $\oplus$ J'. By the maximality of J, we have K ∨ J and so K ⊆ J.

Theorem 6: Let I, J ∈ Id(L) with I ⊆ J, If K ∈ Id(L) is an SL-complement of I in L then K ∩ J is an SL-complement of I in J.

Proof: Since K $\oplus$ I is strongly large in L by Lemma 3 (K $\oplus$ I) ∩ J = (K ∩ J) $\oplus$ I is strongly large in J. We show that K ∩ J is an SL-closed in J. Let P ∈ Id(L) be strongly large extension of K ∩ J in J. Then (K ∨ P) ∩ I = (K ∨ P) ∩ J ∩ I = [(K ∩ J) ∨ P] ∩ I = P ∩ I = (0], It is clear that (K ∨ P) $\oplus$ I is strongly large in L and hence K = K ∨ P by the maximality of K. Thus P = K ∩ J and therefore K ∩ J is an SL-closed ideal of J.

Corollary 1: I, J ∈ Id(L) with I ⊆J. If I is an SL-complement of J and J is an SL-complement in L then I is an SL-closed in L.

Proof: Clear from Theorem 5.

Theorem 7: Let I ∈ Id(L) be a SL-complement in L and J ∈ Id(L) be a SL-complement in L for some ideal in I. Then I ∩ J is an SL-closed ideal of L.

Proof: Let an ideal K ⊆ I with J an SL-complement of K in L. Then by Theorem 6, I ∩ J is an SL-complement of K in I. Since I is an SL-complement in L, we have Corollary 1, that I ∩ J is an SL-closed ideal of L.

Theorem 8: Let L be a modular lattice. Let I, J, K ∈ Id(L) be such that L = I $\oplus$ J, K ⊆ I and P ∈ Id(L) an SL-complement of K $\oplus$ J in L. If P ⊆ I then P is an SL-complement of K in L.

Proof: Since P$\oplus$(K$\oplus$J) is strongly large in L, By Lemma 3, [P$\oplus$(K$\oplus$J)] ∩ I is strongly large in I. By modularity condition we have, [P$\oplus$(K$\oplus$J)] ∩ I = [I ∩ (K $\oplus$ J)] $\oplus$ P = [K $\oplus$ (I ∩ J)] $\oplus$ P = K $\oplus$ P. So I is a strongly large extension of K $\oplus$ P. Since P is SL-closed in L and P is SL-closed in I. Therefore P is an SL-complement of K in I.

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