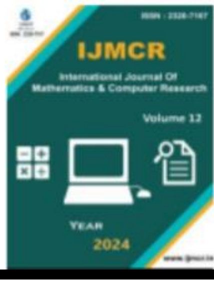
International Journal of Mathematics and Computer Research

ISSN: 2320-7167

Volume 12 Issue 07 July 2024, Page no. – 4339-4345

Index Copernicus ICV: 57.55, Impact Factor: 8.316

DOI: 10.47191/ijmcr/v12i7.02



Generalization of Banach Fixed Point Theorem for Ƭ-Orbitally Complete Dislocated Metric Space

Uttam Dolhare[1], Godavari Jojar[2*], Sachin Basude[3]

[1]Professor, Department of Mathematics, DSM College, Jintur-431509(India)

[2]Research Scholar, Department of Mathematics, NES Science College, Nanded-431603(India)

[3]Assistant Professor, Department of Mathematics, Maharashtra Mahavidyalaya, Nilanga-413521(India)

ARTICLE INFO	ABSTRACT
Published Online: **09 July 2024** Corresponding Author: **Godavari Jojar**	Generalization of Banach Fixed Point Theorem for different metric spaces like dislocated metric space, dislocated quasi metric space, b-metric space and S-metric space etc was studied by several authors using different conditions on self-mapping. Dolhare and Nalawade studied the generalization in $\mathcal{T}$-orbitally complete metric space. In this research article, we have proved two new results of the existence and uniqueness of fixed point for selfmapping in $\mathcal{T}$ -orbitally complete dislocated metric space by the rationalization condition on $\mathcal{T}$. Also, we have proved consequences of our main results.

KEYWORDS: Metric space, Dislocated metric space, fixed point, $\mathcal{T}$-orbitally complete metric space.

Mathematics Subject Classification: 47H10, 54H25.

INTRODUCTION AND PRELIMINARIES

The Banach fixed point theorem [1], which was published in 1922, is helpful in determining whether there are solutions to the various nonlinear issues that have arisen in the biological, physical, and social sciences, among other fields of study. Abramski and Jung [2] provided some information about dislocated metric space in 1994. In 2000, Hitzler and Seda [3] introduced the idea of dislocated metric space, which provided some modifications to the Banach contraction principle in complete dislocated metric space. For similar results and some generalizations of Fixed Point Theorems, one can refer [5, 8, 10].

Definition 1.1 [6]. For the set $\mathcal{X}(\neq \emptyset)$ consider the map $d: \mathcal{X} \times \mathcal{X} \to [0, \infty)$ which satisfies:

(d_1) $d(x, y) = d(y, x) = 0 \Longrightarrow x = y$

(d_2) $d(x, y) = d(y, x), \forall\, x, y \in X$

(d_3) $d(x, y) \leq d(x, z) + d(z, y), \ \forall\, x, y, z \in \mathcal{X}$

Here d is called dislocated metric (DM) on $\mathcal{X}$ and $(\mathcal{X}, d)$ is called as a dislocated metric space (DMS).

Definition 1.2 [7]. In a DMS $(\mathcal{X}, d)$, sequence $\{x_n\}$ converges to z, if $\lim_{n\to\infty} d(x_n, z) = \lim_{n\to\infty} d(z, x_n) = 0$.

Definition 1.3 [7]. In a DMS $(\mathcal{X}, d)$, $\{x_n\}$ is said to be Cauchy if for a given $\epsilon > 0$, $\exists\, n_0 \in \mathbb{N}$ such that, $\forall\, m, n \geq n_0$, $d(x_m, x_n) < \epsilon$.

Definition 1.4 [6]. A DMS $(\mathcal{X}, d)$ is complete, if each Cauchy sequence in $\mathcal{X}$ is convergent to a point in $\mathcal{X}$.

Definition 1.5 [4]. A function $\mathcal{T}: \mathcal{X} \to \mathcal{X}$ is a λ- generalized contraction if and only if for all $x, y \in \mathcal{X}$ there are nonnegative numbers $p(x, y), q(x, y), r(x, y), \ s(x, y)$, such that

$$\sup_{x,y\in X} \{p(x, y) + q(x, y) + r(x, y) + 2s(x, y)\} = \lambda < 1$$

and

$d(\mathcal{T}x, \mathcal{T}y) \leq p(x, y)d(x, y) + q(x, y)d(x, \mathcal{T}x) + r(x, y)d(y, \mathcal{T}y) + s(x, y)[d(x, \mathcal{T}y) +$

$d(y, \mathcal{T}x)]$ holds for all $x, y \in \mathcal{X}$.

Definition 1.6 [4]. A space $\mathcal{X}$ with a function $\mathcal{T}: \mathcal{X} \to \mathcal{X}$ is called $\mathcal{T}$-orbitally complete metric space if every Cauchy sequence $\{\mathcal{T}^n ix: i \in \mathbb{N}, x \in \mathcal{X}\}$ has a limit point in $\mathcal{X}$.

In next section, we generalize the results given by Nalavade in [9].

1. MAIN RESULTS

Theorem 2.1. For a T-Orbitally complete dislocated metric space (X, d) with self mapping $T: X \to X$, define an operation $*$ on X as $\alpha * x$, for $\alpha \in \mathbb{R}$ and $x \in X$ such that $\alpha * x \in \mathbb{R}$ and for any $\alpha \in \mathbb{R}$, defined $d(\alpha x, \alpha y) = |\alpha| d(x, y)$. Also, for nonnegative numbers $p(x, y)$, $q(x, y)$, $r(x, y)$, $s(x, y)$, which depends on x, y let there is a numbers $k \in \mathbb{R}^+$ with

$$\sup_{x,y\in X} (p(x,y) + q(x,y) + r(x,y) + 2s(x,y)) = \lambda < \frac{1}{k} \ldots (1)$$

Define the function $\psi: X \to X$ as $\psi(x) = kx$, $k \in \mathbb{R}$, and T satisfies the condition

$$d(Tx, Ty) \le pd(\psi(x), \psi(y)) + q[d(\psi(x), \psi(Tx)) + d(\psi(y), \psi(Ty)]$$

$$+r[d(\psi(x), \psi(Ty)) + d(\psi(y), \psi(Tx))] + s\left[\frac{d(\psi(x),\psi(Tx)).d(\psi(y),\psi(Ty))}{d(\psi(x),\psi(y))}\right] \ldots (2)$$

for all $x, y \in X$. Then there exist unique $u \in T$ such that $T(u) = u$.

Proof. We will denote $\alpha * x$ by αx. Let $x_0 \in X$ be an arbitrary element and consider a sequence, $x_1 = Tx_0$, $x_2 = Tx_1$, … . $x_{n+1} = Tx_n$, …. Now by using the equation (2) and triangle inequality, gives

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n)$$

$$\le pd(\psi(x_{n-1}), \psi(x_n)) + q[d(\psi(x_{n-1}), \psi(Tx_{n-1})) + d(\psi(x_n), \psi(Tx_n)] + r[d(\psi(x_{n-1}), \psi(Tx_n)) + d(\psi(x_n), \psi(Tx_{n-1}))] + s\left[\frac{d(\psi(x_{n-1}),\psi(Tx_{n-1})).d(\psi(x_n),\psi(Tx_n))}{d(\psi(x_{n-1}),\psi(x_n))}\right]$$

$$\le pd(\psi(x_{n-1}), \psi(x_n)) + q[d(\psi(x_{n-1}), \psi(x_n)) + d(\psi(x_n), \psi(x_{n+1})] + r[d(\psi(x_{n-1}), \psi(x_{n+1})) + d(\psi(x_n), \psi(x_n))] + s\left[\frac{d(\psi(x_{n-1}),\psi(x_n)).d(\psi(x_n),\psi(x_{n+1}))}{d(\psi(x_{n-1}),\psi(x_n))}\right]$$

$$\le pkd(x_{n-1}, x_n) + qk[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] + rk[d(x_{n-1}, x_{n+1}) + d(x_n, x_n)] + ks[d(x_n, x_{n+1})]$$

$$\le pkd(x_{n-1}, x_n) + qk[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] + rk[d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + d(x_{n-1}, x_n) + d(x_n, x_{n+1})] + ks[d(x_n, x_{n+1})]$$

which gives,

$$(1 - qk - 2rk - ks)d(x_n, x_{n+1}) \le (pk + qk + 2rk)d(x_{n-1}, x_n)$$

$$\text{i.e., } \Rightarrow d(x_n, x_{n+1}) \le \left[\frac{p+q+2r}{\frac{1}{k}-q-2r-s}\right] d(x_{n-1}, x_n)$$

$\Rightarrow d(x_n, x_{n+1}) \le Rd(x_{n-1}, x_n)$, where $R = \frac{p+q+2r}{\frac{1}{k}-q-2r-s} < 1$ (by condition (1) in the theorem).

In general, $d(x_n, x_{n+1}) \le R^n d(x_0, x_1)$ ⋯ (3)

Taking limit as $n \to \infty$, as $R < 1$, so $R^n \to 0$ and therefore from equation (3), we have $d(x_n, x_{n+1}) \to 0$.

This shows that, $\{x_n\}$ is a Cauchy sequence in a complete dislocated metric space (X, d).

Therefore there exists $u \in X$ such that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} T^n x_0 = u$. ⋯ (4) Now, we will show that this u is a fixed point of T.

Using equations (1) and (2) in the theorem 2.1, we have

$$d(Tu, Tx_n) \le pd(\psi(u), \psi(x_n)) + q[d(\psi(u), \psi(Tu)) + d(\psi(x_n), \psi(Tx_n)]$$

$$+r[d(\psi(u), \psi(Tx_n)) + d(\psi(x_n), \psi(Tu))] + s\left[\frac{d(\psi(u),\psi(Tu)).d(\psi(x_n),\psi(Tx_n))}{d(\psi(u),\psi(x_n))}\right]$$

$$\le pkd(u, x_n) + qk[d(u, Tu) + d(x_n, Tx_n)] + rk[d(u, Tx_n) + d(x_n, Tu)] + sk\left[\frac{d(u,Tu)\cdot d(x_n,Tx_n)}{d(u,x_n)}\right]$$

which gives,

$$d(Tu, Tx_n) \le pkd(u, x_n) + qk[d(u, Tu) + d(x_n, x_{n+1})] + rk[d(u, x_{n+1}) + d(x_n, Tu)] + sk\left[\frac{d(u,Tu)\cdot d(x_n,x_{n+1})}{d(u,x_n)}\right]$$

$$\leq pkd(u,x_n) + qk[d(u,x_{n+1}) + d(x_{n+1},Tu) + d(x_n,x_{n+1})]$$

$$+rk[d(u,x_{n+1}) + d(x_n,Tu)] + sk\left[\frac{d(u,Tu)\cdot d(x_n,x_{n+1})}{d(u,x_n)}\right]$$

$$\leq pkd(u, x_n) + qkd(u, x_{n+1}) + qkd(Tx_n, Tu) + qkd(x_n, x_{n+1})$$

$$+rkd(u,x_{n+1}) + rkd(x_n,Tx_n) + rkd(Tx_n,Tu) + skd(x_n,x_{n+1})\left[\frac{d(u,Tu)}{d(u,x_n)}\right]$$

$$\leq pkd(u, x_n) + qkd(u, x_{n+1}) + qkd(Tx_n, Tu) + qkd(x_n, x_{n+1})$$

$$+rkd(u,x_{n+1}) + rkd(x_n,Tx_n) + rkd(Tx_n,Tu) + skd(x_n,x_{n+1})\left[\frac{d(u,Tx_n)}{d(u,x_n)}\right]$$

$$\leq pkd(u, x_n) + (qk + rk)d(u, x_{n+1}) + (qk + rk + sk)d(x_n, x_{n+1})$$

$$+(qk + rk)d(Tx_n, Tu)$$

$$\leq \lambda d(u, x_n) + \lambda d(u, x_{n+1}) + \lambda d(x_n, x_{n+1}) + \lambda d(Tx_n, Tu)$$

$$(1 - \lambda)d(Tu, Tx_n) \leq \lambda[d(u, x_n) + d(u, x_{n+1}) + d(x_n, x_{n+1})]$$

$$\therefore\ d(Tu,Tx_n) \leq \frac{\lambda}{1-\lambda}[d(u,x_n) + d(u,x_{n+1}) + d(x_n,x_{n+1})]$$

Therefore, $\lim_{n\to\infty} d(Tu,Tx_n) \leq \frac{\lambda}{1-\lambda}\lim_{n\to\infty}[d(u,x_n) + d(u,x_{n+1}) + d(x_n,x_{n+1})]$

By using equation (4) and that $\{x_n\}$ is a Cauchy sequence, we get

$\lim_{n\to\infty}[d(u,x_n) + d(u,x_{n+1}) + d(x_n,x_{n+1})] = 0$. Hence

$$Tu = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = u$$

Thus, u is a fixed point of T.

Uniqueness:

Suppose that u and v are two distinct fixed points T i.e., $u \neq v$ and $T(u) = u$, $T(v) = v$.

By equations (1) and (2), we have

$d(u, v) = d(Tu, Tv) \leq pd(\psi(u), \psi(v)) + q[d(\psi(u), \psi(Tu)) + d(\psi(v), \psi(Tv))]$

$$+r\big[d\big(\psi(u),\psi(Tv)\big) + d\big(\psi(v),\psi(Tu)\big)\big] + s\left[\frac{d(\psi(u),\psi(Tu)).d(\psi(v),\psi(Tv))}{d(\psi(u),\psi(v))}\right]$$

$$\leq pkd(u,v) + qk[d(u,Tu) + d(v,Tv)] + rk[d(u,Tv) + d(v,Tu)]$$

$$+sk\left[\frac{d(u,Tu)\cdot d(v,Tv)}{d(u,v)}\right]$$

$$\leq pkd(u, v) + qk[d(u, u) + d(v, v)] + rk[d(u, v) + d(v, u)]$$

$$+sk\left[\frac{d(u,u)\cdot d(v,v)}{d(u,v)}\right]$$

Therefore, $d(u, v) \leq pkd(u, v) + 2rkd(u, v)$

$\leq (pk + 2rk)d(u, v) \leq \lambda d(u, v)$

Thus, $(1 - \lambda)d(u, v) \leq 0$ i.e., $d(u, v) = 0 \Rightarrow u = v$, which is a contradiction to $u \neq v$.

Hence, T has a unique fixed point.

Example 1. For the metric space (X, d) where $X = [0,5]$, define $d(x, y) = \max\{x, y\}$ for all $x, y \in X$. Here $d(\alpha x, \alpha y) = \max\{\alpha x, \alpha y\} = |\alpha| \max\{x, y\} = |\alpha|d(x, y)$ for all $x, y \in X$.

Define $T: X \to X$ as $Tx = \frac{3}{5}x,\ \forall x \in X$. Then (X, d) is a T-orbitally complete dislocated metric space. For $k = \frac{1}{2}$, consider the numbers $p(x,y) = \frac{3}{5}, q(x,y) = \frac{1}{5}, r(x,y) = \frac{1}{5}, s(x,y) = \frac{1}{5}$.

Now, $\sup_{x,y\in X=[0,5]}\{p(x,y) + q(x,y) + r(x,y) + 2s(x,y)\} = \sup_{x,y\in X=[0,5]}\left\{\frac{3}{5} + \frac{1}{5} + \frac{1}{5} + 2.\frac{1}{5}\right\}$

$$= \frac{7}{5} = 1.4 < 2 = \frac{1}{\frac{1}{2}} = \frac{1}{k}$$

Define the function $\psi: X \to X$ by $\psi(x) = kx, \quad \forall x \in X$.

We have, $d(Tx,Ty) = d\left(\frac{3}{5}x,\frac{3}{5}y\right) = \max\left\{\frac{3}{5}x,\frac{3}{5}y\right\} = \frac{3}{5}\max\{x,y\} = \frac{3}{5}x$ or $\frac{3}{5}y$ and

$pd(\psi(x), \psi(y)) + q[d(\psi(x), \psi(Tx)) + d(\psi(y), \psi(Ty))]$

$$+r\left[d\big(\psi(x),\psi(Ty)\big)+d\big(\psi(y),\psi(Tx)\big)\right]+s\left[\frac{d\big(\psi(x),\psi(Tx)\big).d\big(\psi(y),\psi(Ty)\big)}{d\big(\psi(x),\psi(y)\big)}\right]$$

$$=\frac{3}{5}d\left(\frac{1}{2}x,\frac{1}{2}y\right)+\frac{1}{5}\left[d\left(\frac{1}{2}x,\frac{3}{10}x\right)+d\left(\frac{1}{2}y,\frac{3}{10}y\right)\right]+\frac{1}{5}\left[d\left(\frac{1}{2}x,\frac{3}{10}y\right)+d\left(\frac{1}{2}y,\frac{3}{10}y\right)\right]+$$

$$\frac{1}{5}\left[\frac{d\left(\frac{1}{2}x,\frac{3}{10}x\right)\cdot d\left(\frac{1}{2}y,\frac{3}{10}y\right)}{d\left(\frac{1}{2}x,\frac{1}{2}y\right)}\right]$$

$$=\frac{3}{5}\max\left\{\frac{1}{2}x,\frac{1}{2}y\right\}+\frac{1}{5}\left[\max\left\{\frac{1}{2}x,\frac{3}{10}x\right\}+\max\left\{\frac{1}{2}y,\frac{3}{10}y\right\}\right]+\frac{1}{5}\left[\max\left\{\frac{1}{2}x,\frac{3}{10}y\right\}+\right.$$

$$\left.\max\left\{\frac{1}{2}y,\frac{3}{10}y\right\}\right]+\frac{1}{5}\left[\frac{\max\left\{\frac{1}{2}x,\frac{3}{10}x\right\}\cdot\max\left\{\frac{1}{2}y,\frac{3}{10}y\right\}}{\max\left\{\frac{1}{2}x,\frac{1}{2}y\right\}}\right]$$

$$=\frac{3}{5}\frac{1}{2}\max\{x,y\}+\frac{1}{10}\left[\max\left\{x,\frac{3}{5}x\right\}+\max\left\{y,\frac{3}{5}y\right\}\right]+\frac{1}{10}\left[\max\left\{x,\frac{3}{5}y\right\}+\right.$$

$$\left.\max\left\{y,\frac{3}{5}y\right\}\right]+\frac{1}{10}\left[\frac{\max\left\{x,\ \frac{3}{5}x\right\}\cdot\max\left\{y,\ \frac{3}{5}y\right\}}{\max\{x,y\}}\right]$$

$$=\frac{3}{10}x+\frac{1}{10}[x+y]+\frac{1}{10}[x+y]+\frac{1}{10}\left[\frac{x\cdot y}{x}\right]$$

$$=\frac{3}{10}x+\frac{2}{10}x+\frac{2}{10}y+\frac{1}{10}y$$

$$=\frac{1}{2}x+\frac{3}{10}y$$

OR

$$=\frac{3}{10}y+\frac{1}{10}[x+y]+\frac{1}{10}[x+y]+\frac{1}{10}\left[\frac{x\cdot y}{y}\right]$$

$$=\frac{3}{10}y+\frac{2}{10}x+\frac{2}{10}y+\frac{1}{10}x$$

$$=\frac{1}{2}y+\frac{3}{10}x$$

All the conditions of theorem 2.1 are satisfied here. Also $x = 0 \in X$ is the only fixed point of T.

Theorem 2.2. For a T-Orbitally complete dislocated metric space (X, d) with self mapping $T: X \to X$, define an operation $*$ on X as $\alpha * x$, for $\alpha \in \mathbb{R}$ and $x \in X$ such that $\alpha * x \in \mathbb{R}$ and for any $\alpha \in \mathbb{R}$, define $d(\alpha x, \alpha y) = |\alpha| d(x, y)$. Also, for nonnegative numbers $p(x, y)$, $q(x, y)$, $r(x, y)$, $s(x, y)$, which depends on x, y let there is a numbers $k \in \mathbb{R}^{+}$with

$$\sup_{x,y\in X}\big(p(x,y)+q(x,y)+r(x,y)+2s(x,y)\big)=\lambda<\frac{1}{k}. \qquad \ldots(5)$$

Define the function $\psi: X \to X$ as $\psi(x) = kx$, $k \in \mathbb{R}$, and T satisfies the condition

$$d(Tx, Ty) \le pd(\psi(x), \psi(y)) + q[d(\psi(x), \psi(Tx)) + d(\psi(y), \psi(Ty)]$$
$$+r[d(\psi(x), \psi(Ty)) + d(\psi(y), \psi(Tx))] +$$
$$s\left[\frac{d\big(\psi(y),\psi(Ty)\big)\big[1+d\big(\psi(x),\psi(Tx)\big)\big]}{1+d\big(\psi(x),\psi(y)\big)}\right] \qquad \ldots(6)$$

for all $x, y \in X$. Then there exist unique $u \in T$ such that $T(u) = u$.

Proof. We will denote $\alpha * x$ by αx. Let $x_0 \in X$ be an arbitrary element and consider a sequence, $x_1 = Tx_0$, $x_2 = Tx_1$, … . $x_{n+1} = Tx_n$, …. Now by using the equation (2) and triangle inequality, gives

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n)$$
$$\le pd(\psi(x_{n-1}), \psi(x_n)) + q[d(\psi(x_{n-1}), \psi(Tx_{n-1})) + d(\psi(x_n), \psi(Tx_n)] \quad +r[d(\psi(x_{n-1}), \psi(Tx_n)) + d(\psi(x_n), \psi(Tx_n))] +$$
$$s\left[\frac{d\big(\psi(x_n),\psi(Tx_n)\big)\big[1+d\big(\psi(x_{n-1}),\psi(Tx_{n-1})\big)\big]}{1+d\big(\psi(x_{n-1}),\psi(x_n)\big)}\right]$$
$$\le pd(\psi(x_{n-1}), \psi(x_n)) + q[d(\psi(x_{n-1}), \psi(x_n)) + d(\psi(x_n), \psi(x_{n+1})] \quad +r[d(\psi(x_{n-1}), \psi(x_{n+1})) + d(\psi(x_n), \psi(x_{n+1}))] +$$
$$s\left[\frac{d\big(\psi(x_n),\psi(x_{n+1})\big)\big[1+d\big(\psi(x_{n-1}),\psi(x_n)\big)\big]}{1+d\big(\psi(x_{n-1}),\psi(x_n)\big)}\right]$$

$\leq pkd(x_{n-1}, x_n) + qk\,[d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)] + rk[d(x_{n-1}, Tx_n) +$
$d(x_n, Tx_{n-1})] + ks\left[\frac{d(x_n,Tx_n)[1+d(x_{n-1},Tx_{n-1})]}{1+d(x_{n-1},x_n)}\right]$
$\leq pkd(x_{n-1}, x_n) + qk[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] + rk[d(x_{n-1}, x_{n+1}) +$
$d(x_n, x_n)] + ks\left[\frac{d(x_n,x_{n+1})[1+d(x_{n-1},x_n)]}{1+d(x_{n-1},x_n)}\right]$
$\leq pkd(x_{n-1}, x_n) + qk[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] + rk[d(x_{n-1}, x_n) +$
$d(x_n, x_{n+1}) + d(x_{n-1}, x_n) + d(x_n, x_{n+1})] + ks[d(x_n, x_{n+1})]$

which gives,

$$(1 - qk - 2rk - ks)d(x_n, x_{n+1}) \leq (pk + qk + 2rk)d(x_{n-1}, x_n)$$

i.e., $\Rightarrow d(x_n, x_{n+1}) \leq \left[\frac{pk+qk+2rk}{1-qk-2rk-sk}\right] d(x_{n-1}, x_n)$

$$\Rightarrow d(x_n, x_{n+1}) \leq \left[\frac{p+q+2r}{\frac{1}{k}-q-2r-s}\right] d(x_{n-1}, x_n)$$

$\Rightarrow d(x_n, x_{n+1}) \leq Rd(x_{n-1}, x_n)$, where $R = \frac{p+q+2r}{\frac{1}{k}-q-2r-s} < 1$

(by condition (5) in the theorem 2.2).

In general, $d(x_n, x_{n+1}) \leq R^n d(x_0, x_1)$ ⋯ (7)

Taking limit as $n \to \infty$, as $R < 1$, so $R^n \to 0$ and therefore from equation (7), we have $d(x_n, x_{n+1}) \to 0$.
This shows that, $\{x_n\}$ is a Cauchy sequence in a complete dislocated metric space (X, d).
Therefore there exists $u \in X$ such that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} T^n x_0 = u.$⋯ (8) Now, we will show that this u is a fixed point of T.

Using equations (5) and (6) in the theorem 2.2, we have

$d(Tu, Tx_n) \leq pd(\psi(u), \psi(x_n)) + q[d(\psi(u), \psi(Tu)) + d(\psi(x_n), \psi(Tx_n)]$
$+r[d(\psi(u),\psi(Tx_n)) + d(\psi(x_n),\psi(Tu))]$
$+ s\left[\frac{d(\psi(x_n),\psi(Tx_n))[1+d(\psi(u),\psi(Tu))]}{1+d(\psi(u),\psi(x_n))}\right]$
$\leq pkd(u, x_n) + qk[d(u, Tu) + d(x_n, Tx_n)] + rk[d(u, Tx_n) + d(x_n, Tu)]$
$+sk\left[\frac{d(x_n,Tx_n)[1+d(u,Tu)]}{1+d(u,x_n)}\right]$

which gives,

$d(Tu, Tx_n) \leq pkd(u, x_n) + qk[d(u, Tu) + d(x_n, x_{n+1})] + rk[d(u, x_{n+1}) + d(x_n, Tu)]$
$+sk\left[\frac{d(x_n,x_{n+1})[1+d(u,Tu)]}{1+d(u,x_n)}\right]$
$\leq pkd(u, x_n) + qkd(u, x_{n+1}) + qkd(x_{n+1}, Tu) + qkd(x_n, x_{n+1})$
$+rkd(u,x_{n+1}) + rkd(x_n,x_{n+1}) + rkd(x_{n+1},Tu) +$
$+sk\left[\frac{d(x_n,x_{n+1})[1+d(u,x_n)]}{1+d(u,x_n)}\right]$
$\leq pkd(u, x_n) + (qk + rk)d(u, x_{n+1}) + (qk + rk + sk)d(x_n, x_{n+1})$
$+(qk + rk)d(Tx_n, Tu)$
$\leq \lambda d(u, x_n) + \lambda d(u, x_{n+1}) + \lambda d(x_n, x_{n+1}) + \lambda d(Tx_n, Tu)$
$(1 - \lambda)d(Tu, Tx_n) \leq \lambda[d(u, x_n) + d(u, x_{n+1}) + d(x_n, x_{n+1})]$

$$\therefore\ d(Tu,Tx_n) \leq \frac{\lambda}{1-\lambda}[d(u,x_n) + d(u,x_{n+1}) + d(x_n,x_{n+1})]$$

Therefore, $\lim_{n\to\infty} d(Tu,Tx_n) \leq \frac{\lambda}{1-\lambda}\lim_{n\to\infty}[d(u,x_n) + d(u,x_{n+1}) + d(x_n,x_{n+1})]$

By using equation (8) and that $\{x_n\}$ is a Cauchy sequence, we get

$\lim_{n\to\infty}[d(u,x_n) + d(u,x_{n+1}) + d(x_n,x_{n+1})] = 0$. Hence

$$Tu = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = u$$

Thus, u is a fixed point of T.

Uniqueness:

Suppose that u and v are two distinct fixed points T i.e., $u \neq v$ and $T(u) = u$, $T(v) = v$.

By equations (1) and (2), we have

$d(u, v) = d(Tu, Tv) \leq pd(\psi(u), \psi(v)) + q[d(\psi(u), \psi(Tu)) + d(\psi(v), \psi(Tv)]$

$$+r\left[d\big(\psi(u),\psi(Tv)\big) + d\big(\psi(v),\psi(Tu)\big)\right] + s\left[\frac{d\big(\psi(v),\psi(Tv)\big)\left[1+d\big(\psi(u),\psi(Tu)\big)\right]}{1+d\big(\psi(u),\psi(v)\big)}\right]$$

$$\leq pkd(u,v) + qk[d(u,Tu) + d(v,Tv)] + rk[d(u,Tv) + d(v,Tu)]$$

$$+sk\left[\frac{d(v,Tv)[1+d(u,Tu)]}{1+d(u,v)}\right]$$

$$\leq pkd(u, v) + qk[d(u, u) + d(v, v)] + rk[d(u, v) + d(v, u)]$$

$$+sk\left[\frac{d(v,v)[1+d(u,u)]}{1+d(u,v)}\right]$$

Therefore, $d(u, v) \leq pkd(u, v) + 2rkd(u, v)$

$\leq (pk + 2rk)d(u, v) \leq \lambda d(u, v)$

Thus, $(1 - \lambda)d(u, v) \leq 0$ i.e., $d(u, v) = 0 \Rightarrow u = v$, which is a contradiction to $u \neq v$.

Hence, T has a unique fixed point.

Example 2. For the metric space (X, d), where $X = [0,6]$, define $d(x, y) = \max\{x, y\}$ for all $x, y \in X$. Here $d(\alpha x, \alpha y) = \max\{\alpha x, \alpha y\} = |\alpha| \max\{x, y\} = |\alpha| d(x, y)$ for all $x, y \in X$.

Define $T: X \to X$ as $Tx = \frac{4}{5}x, \ \forall x \in X$. Then (X, d) is a T-orbitally complete dislocated metric space. For $k = \frac{1}{2}$, consider the numbers $p(x,y) = \frac{3}{5}, q(x,y) = \frac{1}{5}, r(x,y) = \frac{2}{5}, s(x,y) = \frac{1}{5}$.

Now, $\sup_{x,y\in X=[0,6]} \{p(x,y) + q(x,y) + r(x,y) + 2s(x,y)\} = \sup_{x,y\in X=[0,5]} \left\{\frac{3}{5} + \frac{1}{5} + \frac{2}{5} + 2.\frac{1}{5}\right\}$

$$= \frac{8}{5} = 1.6 < 2 = \frac{1}{\frac{1}{2}} = \frac{1}{k}$$

Define the function $\psi: X \to X$ by $\psi(x) = kx, \quad \forall x \in X$.

We have, $d(Tx,Ty) = d\left(\frac{4}{5}x, \frac{4}{5}y\right) = \max\left\{\frac{4}{5}x, \frac{4}{5}y\right\} = \frac{4}{5}\max\{x,y\} = \frac{4}{5}x$ or $\frac{4}{5}y$ and

$pd(\psi(x), \psi(y)) + q[d(\psi(x), \psi(Tx)) + d(\psi(y), \psi(Ty)]$

$$+r\left[d\big(\psi(x),\psi(Ty)\big) + d\big(\psi(y),\psi(Tx)\big)\right] + s\left[\frac{d\big(\psi(y),\psi(Ty)\big)\left[1+d\big(\psi(x),\psi(Tx)\big)\right]}{1+d\big(\psi(x),\psi(y)\big)}\right]$$

$$= \frac{3}{5}d\left(\frac{1}{2}x,\frac{1}{2}y\right) + \frac{1}{5}\left[d\left(\frac{1}{2}x,\frac{4}{10}x\right) + d\left(\frac{1}{2}y,\frac{4}{10}y\right)\right] + \frac{2}{5}\left[d\left(\frac{1}{2}x,\frac{4}{10}y\right) + d\left(\frac{1}{2}y,\frac{4}{10}y\right)\right] + \frac{1}{5}\left[\frac{d\left(\frac{1}{2}y,\frac{4}{10}y\right)\left[1+d\left(\frac{1}{2}x,\frac{4}{10}y\right)\right]}{1+d\left(\frac{1}{2}x,\frac{1}{2}y\right)}\right]$$

$$= \frac{3}{5}\max\left\{\frac{1}{2}x,\frac{1}{2}y\right\} + \frac{1}{5}\left[\max\left\{\frac{1}{2}x,\frac{4}{10}x\right\} + \max\left\{\frac{1}{2}y,\frac{4}{10}y\right\}\right] + \frac{2}{5}\left[\max\left\{\frac{1}{2}x,\frac{4}{10}y\right\} + \max\left\{\frac{1}{2}y,\frac{3}{10}y\right\}\right] + \frac{1}{5}\left[\frac{\max\left\{\frac{1}{2}y,\frac{4}{10}y\right\}\left[1+\max\left\{\frac{1}{2}x,\frac{4}{10}x\right\}\right]}{1+\max\left\{\frac{1}{2}x,\frac{1}{2}y\right\}}\right]$$

Case 1: If $x > y$

$$= \frac{3}{5}\frac{1}{2}x + \frac{1}{5}\left[\frac{1}{2}x + \frac{1}{2}y\right] + \frac{2}{5}\left[\frac{1}{2}x + \frac{1}{2}y\right] + \frac{1}{5}\left[\frac{\frac{1}{2}y\left[1+\frac{1}{2}x\right]}{1+\frac{1}{2}x}\right]$$

$$= \frac{3}{10}x + \frac{1}{10}[x+y] + \frac{1}{5}[x+y] + \frac{1}{10}y$$

$$= \left(\frac{3}{10} + \frac{1}{10} + \frac{1}{5}\right)x + \left(\frac{1}{10} + \frac{1}{5} + \frac{1}{10}\right)y$$

$$= \frac{3}{5}x + \frac{2}{10}y$$

Case 2: If $y > x$

$$= \frac{3}{5}\frac{1}{2}y + \frac{1}{5}\left[\frac{1}{2}x + \frac{1}{2}y\right] + \frac{2}{5}\left[\frac{1}{5}x + \frac{1}{2}y\right] + \frac{1}{5}\left[\frac{\frac{1}{2}y\left[1+\frac{1}{2}x\right]}{1+\frac{1}{2}y}\right]$$

$$= \frac{3}{10}y + \frac{1}{10}[x + y] + \frac{2}{10}x + \frac{2}{10}y + \frac{1}{10}\left[\frac{y+\frac{1}{2}xy}{1+\frac{1}{2}y}\right]$$

$$= \frac{3}{5}y + \frac{3}{10}x + \frac{1}{10}\left[\frac{y+\frac{1}{2}xy}{1+\frac{1}{2}y}\right]$$

All the conditions of theorem 2.1 are satisfied here. Also $x = 0 \in X$ is the only fixed point of T.

CONCLUSION

In this research article, we have proved two new results of the existence and uniqueness of fixed point with examples for surjective self-mapping in complete dislocated metric space. Also, we have proved consequences on our main results and generalizes some well-known similar results in the literature of fixed point theory.

ACKNOWLEDGEMENT

First author acknowledges Chhatrapati Shahu Maharaj Research Training and Human Development Institute (SARTHI), Pune for providing the CSMNRF-20 fellowship for carrying out this research work.

REFERENCES

1. S. Banach, Sur les operations dans les ensembles abstraits et leur application aux equations integrals, Fundamental Mathematicae. 3, 133-181 (1922).
2. S. Abramsky and A. Jung, Domain Theory in Hand Book of Logic in Computer Sciences, Oxford University Press,3, (1994).
3. P. Hitzler and A. K. Seda, Dislocated topologies, J. Electronic Engg., 51, 3-7 (2000).
4. Ciric LB, *Generalized Contractions and Fixed-Point Theorems*, Publ. Inst. Math. 1971, 12(26), 19-26.
5. C. T. Aage and J. N. Salunke, The Results on Fixed Points in Dislocated and Dislocated Quasi-Metric Space, Applied Math. Sci., 2(59), 2941-2948 (2008).
6. T. Senthil kumar, A. Muraliraj and R. Jahir Hussain, Some Fixed Point Theorems in Dislocated Metric Spaces, Pure Mathematical Sciences, 3(1), 35-43, (2014).
7. Mujeeb Ur Rahman and Muhammad Sarwar, Fixed Point Theorems for Expanding Mappings in Dislocated Metric Space, Mathematical Sciences Letters, 4(1), 69-73 (2015).
8. G. Jojar, U. Dolhare, S. Basude, N. Darkunde and A. Chavan, Some Results on Fixed Point Theorems in Complete Dislocated Quasi Metric Space, Advances and Applications in Mathematical Sciences, 21(7), 4111-4119 (2022).
9. Nalawade VV, Dolhare UP, *Some Fixed Points Theorems,* Int. Journal of Engineering Research and Application, 2017, 7(7), 51-56.
10. A. Tenguria and P. Chouhan, Some Fixed Point Theorems for Expansive type Mappings in Dislocated Metric Space, Global Journal of Mathematical Sciences: Theory and Practical, 4(2), 107-112 (2012).