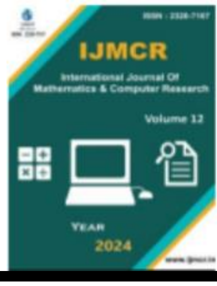
International Journal of Mathematics and Computer Research

ISSN: 2320-7167

Volume 12 Issue 07 July 2024, Page no. –4359-4362

Index Copernicus ICV: 57.55, Impact Factor: 8.316

DOI: 10.47191/ijmcr/v12i7.05



Subgroup Structure in semigroup with Source of Semiprimeness

Barış Albayrak

Department of Finance and Banking, Çanakkale Onsekiz Mart University, Çanakkale, Türkiye.

ARTICLE INFO

Published Online: 23 July 2024

Corresponding Author: Barış Albayrak

ABSTRACT

In this study, the $|S_S| -$ subgroup structure for $|S_S| -$ group obtained by using a given S semigroup is defined and its properties are examined. As in group theory, the relationships of the defined $|S_S| -$ subgroup structure with $|S_S| -$ group is investigated. Some well-known theorems in group theory have been adapted to the new subgroup structure and generalizations have been obtained. Additionally, their relationships with homomorphisms are examined and examples of the obtained results are given.

KEYWORDS: Semigroup, subgroup, semiprime semigroup, source of semiprimeness, $|S_S| -$ group

I. INTRODUCTION

In group theory, groups and their subgroups and the relationships between them have a very important place. Over time, many different researchers have studied groups and their subgroups and found new features. Then, new group and subgroup structures were defined and generalizations in group theory were reached. $|S_S| -$ group structure is one of these newly defined group structures. The studies started with the definition of the set of source of semiprimeness on the S, where S is a semigroup. For detailed information about this set, see [3].

After defining the source of semiprimeness, new structures were defined by using this set. One of these structures is the $|S_S| -$ group structure. In this study, the $|S_S| -$ subgroup structure for the $|S_S| -$ group obtained by using a given S semigroup is defined, and its properties are examined. The obtained theorems are supported with examples and a generalization of the subgroup structure is achieved.

Before going through the definitions and theorems used in the study, let's briefly take a look at the results obtained. The basis of the study is the $|S_S| -$ subgroup structure. First of all, the features of this structure are examined. Then, some well-known results and theorems in group theory are adapted to the new subgroup structure. Examples of the features found are given and the subject is explained in detail. Finally, the relations between the defined transformation f and the $|S_S| -$ subgroup are examined, and the study is completed with a general example.

II. PRELIMINARIES

First, let us give the definitions of special elements and semigroup types used in our study. Definitions are referenced by [4] and [5]. Let $(S,\cdot)$ be a semigroup. An identity element of a semigroup S is an element $1_S \in S$ such that $1_S x = x1_S = x$ for all $x \in S$. Semigroup that does have an identity element is called monoid. Similarly, a zero element of a semigroup S is an element $0_S \in S$ such that $0_S x = x0_S = 0_S$ for all $x \in S$. In this study, semigroups containing zero were studied. An element x of a semigroup S with identity element is called unit element if there exist $y \in S$ such that $xy = yx = 1_S$. We donate y by x^{-1}. If $xy = yx$ is satisfied for all $x, y \in S$, then the S semigroup is called the commutative semigroup.

Let us remember the definition of monoid homomorphism. From [4], a monoid homomorphism from a monoid S into a monoid T is a mapping $f : S \to T$ which preserves products: $f(xy) = f(x) f(y)$ for all $x, y \in S$ and also preserves identity elements: $f(1_S) = 1_T$.

Now, let's give the definition of semiprime semigroup that forms the basis of this paper. According to [1] and [6], the ideal I is called a semiprime ideal if $xSx \in I$ with $x \in S$ implies $x \in I$. A semigroup S is called semiprime if the zero ideal is a semiprime ideal of S. Thus, the equivalent definition can be given as follows: if $xSx = 0$ with $x \in S$ implies $x = 0$, then S is called semiprime semigroup.

Finally, let's give the definitions and theorems about the set of source of semiprimeness and $|S_S| -$ group. Definitions are referenced by [2] and [3]. For a semigroup S, $S_S = \{a \in S | aSa = 0\}$ is called the source of semiprimeness. If A

is nonzero subset of S, then $S_S(A) = \{b \in A | bSb = 0\} = S_S$. Using this definition, $|S_S| -$ group is defined as follows for semigroup S with zero such that $S \neq S_S$: S is called $|S_S| -$group if $1_S \in S$ and every element of $S - S_S$ is unit.

III. RESULTS

In this section, we will define $|S_S| -$subgroup construction for $|S_S| -$groups. As in the group theory, we define a subgroup A as a $|S_S| -$subgroup, if A is itself a $|S_S| -$group under the operation in S.

Definition 1 Let S be a $|S_S| -$group and A be a nonzero subsemigroup of S. A is called a $|S_S| -$subgroup of S, if A is itself a $|S_S| -$group under the operation in S.

The equivalent definition can be given as follows: If $1_S \in A$ and every a elements of $A - S_S$ is unit in A, then A is a $|S_S| -$subgroup of S.

First, we will give some basic properties and facts about $|S_S| -$subgroups.

1. Note that even if it exists, the identity element of subsemigroups need not be the same as the identity of the semigroup. But, as in the subgroups, it is easy to see that, the identity element of the $|S_S| -$subgroup cannot be different from the identity element of the $|S_S| -$group.
2. Any $|S_S| -$group is its own $|S_S| -$subgroup.
3. If $A = \{0\}$ then $A - S_S = \emptyset$.

Now, let us give a theorem that shows the relation between $|S_S| -$groups and $|S_S| -$subgroups.

Lemma 2 Let S be a $|S_S| -$group, A be a subsemigroup of S. If A is a $|S_A| -$group then A is a $|S_S| -$subgroup.

Proof Let A be a $|S_A| -$group and $x \in A - S_S$. From the definition of $A - S_S$, we write $x \in A$ and $x \notin S_S$. Since $S_S(A) = S_S$, we get $x \in A$ and $x \notin S_S(A)$. At the same time, using $S_A \subseteq S_S(A)$, we have $x \in A$ and $x \notin S_A$. Then $x \in A - S_A$, and so x is unit element because A is a $|S_A| -$group. Since every element of $A - S_S$ is unit element, it is concluded that A is a $|S_S| -$subgroup.

Lemma 3 Let S be a $|S_S| -$group and A be a subgroup of $S - S_S$. Then, A is a $|S_S| -$subgroup.

Proof If A is a subgroup of $S - S_S$, then $1_S \in A$ and every element of A is unit element. In this case, every element of $A - S_S$ is also unit element. So, A is a $|S_S| -$subgroup.

Now, let us give examples about $|S_S| -$subgroups. As can be seen from the examples, the above conclusions become obsolete or work in one direction when some of the hypotheses are omitted.

Example 4 Let the operation table of the semigroup S be given below.

$\cdot$	0	1	a	b	c
0	0	0	0	0	0
1	0	1	a	b	c
a	0	a	0	0	0
b	0	b	0	0	0
c	0	c	0	0	1

Now it turns out that

$$S_S = \{0, a, b\}$$

and thus

$$S - S_S = \{1, c\}$$

If the elements of $S - S_S$ are examined, c and 1 are unit elements because $c \cdot c = 1$ and $1 \cdot 1 = 1$. So, S is a $|S_S| -$group.

Let $A = \{0, 1, a\}$. It is clear that A is a subsemigroup of S. Since

$$A - S_S = \{1\}$$

and 1 is identity element, A is a $|S_S| -$subgroup.

Example 5 Let the operation table of the semigroup S be given below.

$\cdot$	0	1	a	b	c
0	0	0	0	0	0
1	0	1	a	b	c
a	0	a	c	1	b
b	0	b	1	c	a
c	0	c	b	a	1

It is easy to see that $S_S = \{0\}$, and thus

$$S - S_S = \{1, a, b, c\}.$$

Since every element of $S - S_S$ is unit element, S is a $|S_S| -$group.

Let $= \{0, 1, c\}$. It is clear that A is a subsemigroup of S. If we investigate the elements of $A - S_S$, we see that they are unit elements and thus A is a $|S_S| -$subgroup. Also, for $S_A = \{0\}$ we get

$$A - S_A = \{1, c\}.$$

So, A is also $|S_A| -$group.

Example 6 Consider the $|S_{\mathbb{Q}}| -$ group $(\mathbb{Q},\cdot)$. $\mathbb{Z}$ is a subsemigroup of $\mathbb{Q}$. However, there are non-unit elements $\mathbb{Z} - S_{\mathbb{Q}} = \mathbb{Z} - \{0\}$. So, $\mathbb{Z}$ is not $|S_{\mathbb{Q}}| -$subgroup of $\mathbb{Q}$.

Example 7 Consider the $|S_{\mathbb{Q}}| -$ group $(\mathbb{Z}_{25},\cdot)$. It is easy to see that

$$S_{\mathbb{Z}_{25}} = \{\overline{0}, \overline{5}, \overline{10}, \overline{15}, \overline{20}\}$$

The set $A = \{\overline{0}, \overline{1}, \overline{3}, \overline{5}, \overline{17}\}$ is a subsemigroup of $\mathbb{Z}_{25}$. For $S_A = \{\overline{0}, \overline{5}\}$, every element of

$$A - S_A = \{\overline{1}, \overline{3}, \overline{17}\}$$

is unit element. This means that A is also a $|S_A| -$group.

Also, since every element of

$$A - S_{\mathbb{Z}_{25}} = \{\overline{1}, \overline{3}, \overline{17}\}$$

is unit element, A is a $|S_A| -$group.

Next, we will give properties of intersection and union of $|S_S| -$subgroups. Also, we will adapt some subgroup theorems to $|S_S| -$subgroups. Before getting down into results, let us mention that the wellknown properties in semigroup theory and group theory. For subsemigroups A and B, the set of intersection is a subsemigroup. However, for the union of this subsemigroups to be a subsemigroup, one subsemigroup must cover another. On the other hand, when the semigroup homomorphism f and subsemigroup A are given, $f(A)$ is a subsemigroup. Similarly, $f^{-1}(B)$ is a subsemigroup for subsemigroup B. These properties are exactly provided for subgroups too.

Lemma 8 Let S be a $|S_S|$ −group and $A_1, A_2, A_3, \dots, A_n$ are $|S_S|$ −subgroup of S. Then the following holds true:

1. $\bigcap_{i=1}^{n} A_i$ is a $|S_S|$ −subgroup.
2. If $A_1 \subseteq A_2$, then $A_1 \cup A_2$ is a $|S_S|$ −subgroup.

1. Since the sets $A_1, A_2, A_3, \dots, A_n$ are semigroup, $\bigcap_{i=1}^{n} A_i$ is a subsemigroup. Also, we note that $1_S \in \bigcap_{i=1}^{n} A_i$ for $|S_S|$ −subgroups $A_1, A_2, A_3, \dots, A_n$. Let $x \in (\bigcap_{i=1}^{n} A_i) - S_S$. Then, we write

$$x \in \bigcap_{i=1}^{n} (A_i - S_S).$$

Therefore, we obtain $x \in A_i - S_S$ for every subsemigroup A_i. This means that x is a unit element. So, $\bigcap_{i=1}^{n} A_i$ is a $|S_S|$ −subgroup.

2. Assume that $A_1 \subseteq A_2$. Since A_1 and A_2 are subsemigroup and $A_1 \subseteq A_2$, $A_1 \cup A_2$ is a semigroup and $1_S \in A_1 \cup A_2$. Let $x \in A_1 \cup A_2 - S_S$. From definition of the set $A_1 \cup A_2 - S_S$, we obtain $x \in A_1 \cup A_2$ and $x \notin S_S$. Using $A_1 \subseteq A_2$ in this expression, we get $x \in A_2 - S_S$. This means that x is unit element. So, $A_1 \cup A_2$ is a $|S_S|$ −subgroup.

Lemma 9 Let S be a $|S_S|$ −group, T be a $|S_T|$ −group and $f: S \rightarrow T$ be a monoid homomorphism such that $(0_S) = 0_T$. If $(x) \in S_T$, then $x \in S_S$. Conversely, if f is surjective and $x \in S_S$, then $f(x) \in S_T$.

Proof If we assume that $x \notin S_S$, then we obtain $f(x)$ is a non-unit element. In the other hand, since x is a unit element, there exist $y \in S$ such that $y = 1_S$. In this equation we get $(xy) = f(1_S) = 1_T$. Then, $f(x)f(y) = 1_T$ for homomorphism f. Similarly, using same process on equation $yx = 1_S$, we get $f(y)f(x) = 1_T$. This means that $f(x)$ is a unit element. So, we get $f(x) \notin S_T$.

Conversely, if $\in S_S$, then $xax = 0_S$ for all $a \in S$. Using this equation, we obtain $f(xax) = f(0_S)$. Since f is surjective, we write $(x)Tf(x) = 0_T$. This means that $f(x) \in S_T$.

Theorem 10 Let S be a $|S_S|$ −group, T be a $|S_T|$ −group and $f: S \rightarrow T$ be a monoid homomorphism such that $f(0_S) = 0_T$. Then the following holds true:

1. If f is surjective and A is a $|S_S|$ −subgroup, then $f(A)$ is a $|S_T|$ −subgroup.
2. If B is a $|S_T|$ −subgroup, then $f^{-1}(B)$ is a $|S_S|$ −subgroup.

Proof 1. Since A is a subsemigroup of semigroup S, $f(A)$ is a subsemigroup of semigroup T . Also, $1_T = f(1_S) \in f(A)$ is provided for $1_S \in A$. Let $y \in f(A) - S_T$. Then, there exist $x \in A$ such that $y = f(x) \notin S_T$. From Lemma 9, we get $x \notin S_S$. In this case, since A is a $|S_S|$ −subgroup, x is unit in A. Thus, there exist $z \in A$ such that $xz = 1_S$, and so$f(x)^{-1} = f(x^{-1}) = f(z) \in f(A)$ is provided for $z \in A$. This means that $y = f(x)$ is unit in $f(A)$. So, $f(A)$ is a $|S_T|$ −subgroup.

2. Since B is a subsemigroup of semigroup T, $f^{-1}(B)$ is a subsemigroup of semigroup S. Also, element 1_S is in the $f^{-1}(B)$ since f $(1_S) = 1_S \in B$. Let $\in f^{-1}(B) - S_S$. From definition of the set $f^{-1}(B) - S_S$, we write $x \in f^{-1}(B)$ and $x \notin S_S$. Also, from Lemma 9, we get $f(x) \notin S_T$. Thus, $f(x)$ and x is unit in S. This means that there exist $y \in S$ such that $xy = yx = 1_S$ and $f(x)f(y) = f(y)f(x) = 1_T$. Also, since B is a $|S_T|$ −subgroup, $f(x)$ is unit in B. Then, $f(y) \in B$. From this expression, we get $y \in f^{-1}(B)$ and $f(y) \notin S_T$. From Lemma 9, we write $y \in f^{-1}(B)$ and $y \notin S_S$. So, we get $y \in f^{-1}(B) - S_S$. Since every element in $f^{-1}(B) - S_S$ is unit, we obtain that $f^{-1}(B)$ is a $|S_S|$ −subgroup.

Finally, let us give a general example of the results obtained in the paper.

Example 11 Consider the semigroup

$$M = \left\{ \begin{bmatrix} a & 0 & 0 \\ b & a & 0 \\ 0 & 0 & a \end{bmatrix} \middle| a, b \in \mathbb{R} \right\}.$$

M is a semigroup with identity element and zero element by multiplication operation in matrices. For this semigroup, it is not hard to see that

$$S_M = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ b & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| b \in \mathbb{R} \right\},$$

and thus

$$M - S_M = \left\{ \begin{bmatrix} a & 0 & 0 \\ b & a & 0 \\ 0 & 0 & a \end{bmatrix} \middle| a, b \in \mathbb{R}, a \neq 0 \right\}.$$

Since every element of $M - S_M$ is unit element, M is a $|S_M|$ −group.

Let define the subsemigroup A of semigroup M.

$$A = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} \middle| a \in \mathbb{R} \right\}$$

It is easy to see that

$$A - S_M = \left\{ \begin{bmatrix} a & 0 & 0 \\ b & a & 0 \\ 0 & 0 & a \end{bmatrix} \middle| a \in \mathbb{Q}, a \neq 0 \right\},$$

and so every element of $A - S_M$ is unit element. Also, it is obvious that $1_M \in A$. So, A is a $|S_M|$ −subgroup.

On the other hand, $(\mathbb{R},\cdot)$ is a semigroup with identity element and zero element. For $S_{\mathbb{R}} = \{0\}$, we get $\mathbb{R} - S_{\mathbb{R}} = \mathbb{R} - \{0\}$. Since every element of this set is unit element, $\mathbb{R}$ is a $|S_{\mathbb{R}}|$ −group. Also, it is clear that $\mathbb{Q}$ is a $|S_{\mathbb{R}}|$ −subgroup.

Now, we define the mapping $f: M \longrightarrow \mathbb{R}$ such that

$$f\left(\begin{bmatrix} a & 0 & 0 \\ b & a & 0 \\ 0 & 0 & a \end{bmatrix} \right) = a.$$

For all $A, B \in M$, the equation $f(AB) = f(A)f(B)$ is provided. Also, since

$$f(1_M) = f\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) = 1 = 1_R,$$

f is a monoid homomorphism.

First, let us investigate the set $f(A)$ for $|S_M|$ −subgroup A. Note that $f(M) = \mathbb{R}$, and so f is surjective. Using A is a $|S_M|$ −subgroup and Theorem 10, we get $f(A)$ is a $|S_{\mathbb{R}}|$ −subgroup. Indeed, it is clear that for $|S_M|$ −subgroup A, $f(A) = \mathbb{Q}$ is a $|S_{\mathbb{R}}|$ −subgroup.

Next, we will investigate $f^{-1}(\mathbb{Q})$ for $|S_{\mathbb{R}}|$ −subgroup $\mathbb{Q}$.

Using $\mathbb{Q}$ is a $|S_{\mathbb{R}}|$ $-$subgroup and Theorem 10, ve get $f^{-1}(\mathbb{Q})$ is a $|S_M|$ $-$subgroup. Indeed, every element of

$$f^{-1}(\mathbb{Q}) - S_M = \left\{ \begin{bmatrix} a & 0 & 0 \\ b & a & 0 \\ 0 & 0 & a \end{bmatrix} \middle| a \in \mathbb{Q}, b \in \mathbb{R}, a \neq 0 \right\}$$

Is unit in $f^{-1}(\mathbb{Q})$.

Acknowledgements: This work was supported by Çanakkale Onsekiz Mart University The Scientific Research Coordination Unit, Project number: FBA-2021-3726

REFERENCES

1. Adams, D. H. 1971. Semigroups with No Non-zero Nilpotent Elements. Math. Z., 123, 168-176.
2. Albayrak, B. 2024 "Inverse Semigroup with Source of Semiprimeness," International Journal of Mathematics Trends and Technology (IJMTT), 70(6), 13-19.
3. Aydın, N., Demir, Ç. and Karalarlıoğlu Camcı, D. 2018. The Source of Semiprimeness of Rings. Commun. Korean Math. Soc, 33(4), 1083-1096.
4. Clifford, A. H., Preston, G. B. 1961. The Algebraic Theory of Semigroups I. AMS, Providence.
5. Grillet, P. A. 1995. Semigroups: An Introduction to the Structure Theory. Marcel Dekker, Inc., New York.
6. Park, Y. S. and Kim, J. P. 1992. Prime and Semiprime Ideals in Semigroups. Kyungpook Mathematical Journal, 32(3), 629-632.