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On Ramanujan's modular equations of degree three

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Abstract

The object of this article is to give new and simple proof of all but one Ramanujan's modular equations of degree 3.

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1 Introduction

The ordinary hypergeometric series or the Gauss series is denoted by ${}_2F_1(a,b;c;z)$ and is defined by

$$ {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, \quad |z| < 1, $$

where $(a)_n = a(a+1)(a+2)\cdots(a+n-1),\ n \geq 1.$ a, b, c and z are complex numbers with $|z| < 1$ and $c \neq 0, -1, -2, \cdots$. Suppose that the relation

$$ n\frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-\alpha)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;\alpha)} = \frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-\beta)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;\beta)}, $$

holds for some positive integer n and $0 < \alpha, \beta < 1$. A relation between α and β induced by the above relation is called a modular equation of degree n, and in such case we say that β is of degree n over α. The multiplier m of degree n relating α and β is defined by

$$ m = \frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;\alpha)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;\beta)}. $$

The theory of modular equations started with the discovery of a modular equation of degree 3, by Legendre [6] in 1825. Shortly thereafter Jacobi established modular equations of degree 3 and 5 in [4] and [5]. Also, many contributions were made towards the theory of modular equations by many mathematicians including Schr*ö*ter, Schl*ä*fli, Klein, Russel, Weber and many more. For references to the literature and early works on modular equations one may refer to [3].

From the literature, we learn that S. Ramanujan's contributions in the area of modular equations are immense. Infact, Ramanujan recorded most of his modular equations in Chapters 18-20 of his second notebook [7]. B. C. Berndt has proved all these modular equations and can be found in [2, p. 232]. The main techniques which be employed to prove Ramanujan's modular equations are either the theory of theta functions, or parametrization or the theory of modular forms. For details one may refer [1] and [8].
The following modular equation of degree 3 was first discovered by Legendre [6]:

$$(\alpha\beta)^{\frac{1}{4}} + \{(1-\alpha)(1-\beta)\}^{\frac{1}{4}} = 1. \tag{1.1}$$

Ramanujan also recorded the above modular equation of degree 3 on page 230 of his second notebook[7]. For a proof of the above one may refer to [1] and [2, p. 232]. The main objective of this article, is to give simple and alternative proof of the modular equations of degree 3, recorded by Ramanujan. In fact, we prove the following:

Theorem 1.1. *From [7, p. 230], we have*

1.

$$\left(\frac{\alpha^3}{\beta}\right)^{\frac{1}{8}} - \left(\frac{(1-\alpha)^3}{1-\beta}\right)^{\frac{1}{8}} = 1 = \left(\frac{(1-\beta)^3}{1-\alpha}\right)^{\frac{1}{8}} - \left(\frac{\beta^3}{\alpha}\right)^{\frac{1}{8}} \tag{1.2}$$

2.

$$m = 1 + 2\left(\frac{\beta^3}{\alpha}\right)^{\frac{1}{8}}; \qquad \frac{3}{m} = 1 + 2\left(\frac{(1-\alpha)^3}{1-\beta}\right)^{\frac{1}{8}} \tag{1.3}$$

3.

$$m^2\left\{\left(\frac{\alpha^3}{\beta}\right)^{\frac{1}{8}} - \alpha\right\} = \left(\frac{\alpha^3}{\beta}\right)^{\frac{1}{8}} - \beta \tag{1.4}$$

4.

$$m = \frac{1 - 2\left(\dfrac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}\right)^{\frac{1}{8}}}{1 - 2(\alpha\beta)^{\frac{1}{4}}} = \left\{1 + 4\left(\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}\right)^{\frac{1}{8}}\right\}^{\frac{1}{2}}$$
$$\frac{3}{m} = \frac{2\left(\dfrac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}\right)^{\frac{1}{8}} - 1}{1 - 2(\alpha\beta)^{\frac{1}{4}}} = \left\{1 + 4\left(\frac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}\right)^{\frac{1}{8}}\right\}^{\frac{1}{2}} \tag{1.5}$$

5.

$$m^2 = \left(\frac{\beta}{\alpha}\right)^{\frac{1}{2}} + \left(\frac{1-\beta}{1-\alpha}\right)^{\frac{1}{2}} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{\frac{1}{2}}$$
$$\frac{9}{m^2} = \left(\frac{\alpha}{\beta}\right)^{\frac{1}{2}} + \left(\frac{1-\alpha}{1-\beta}\right)^{\frac{1}{2}} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{\frac{1}{2}} \tag{1.6}$$

6.

$$(\alpha\beta^5)^{\frac{1}{8}} + \left\{(1-\alpha)(1-\beta)^5\right\}^{\frac{1}{8}} = 1 - \left(\frac{\beta^3(1-\alpha)^3}{\alpha(1-\beta)}\right)^{\frac{1}{8}}$$
$$= (\alpha^5\beta)^{\frac{1}{8}} + \left\{(1-\alpha)^5(1-\beta)\right\}^{\frac{1}{8}}$$
$$= \left\{\frac{1}{2}(1 + (\alpha\beta)^{\frac{1}{2}} + \{(1-\alpha)(1-\beta)\}^{\frac{1}{2}})\right\}^{\frac{1}{2}} \tag{1.7}$$

7.

$$\{\alpha(1-\beta)\}^{\frac{1}{2}} + \{\beta(1-\alpha)\}^{\frac{1}{2}} = 2\{\alpha\beta(1-\alpha)(1-\beta)\}^{\frac{1}{8}}$$
$$= m^2\{\alpha(1-\alpha)\}^{\frac{1}{2}} + \{\beta(1-\beta)\}^{\frac{1}{2}}$$
$$= \frac{9}{m^2}\{\beta(1-\beta)\}^{\frac{1}{2}} + \{\alpha(1-\alpha)\}^{\frac{1}{2}} \tag{1.8}$$

8.

$$m(1-\alpha)^{\frac{1}{2}} + (1-\beta)^{\frac{1}{2}} = \frac{3}{m}(1-\beta)^{\frac{1}{2}} - (1-\alpha)^{\frac{1}{2}}$$
$$= 2\{(1-\alpha)(1-\beta)\}^{\frac{1}{8}}$$
$$m\alpha^{\frac{1}{2}} - \beta^{\frac{1}{2}} = \frac{3}{m}\beta^{\frac{1}{2}} + \alpha^{\frac{1}{2}} = 2(\alpha\beta)^{\frac{1}{8}} \tag{1.9}$$

9.

$$m - \frac{3}{m} = 2\left((\alpha\beta)^{\frac{1}{4}} - \{(1-\alpha)(1-\beta)\}^{\frac{1}{4}}\right)$$
$$m + \frac{3}{m} = 4\left\{\frac{1}{2}(1 + (\alpha\beta)^{\frac{1}{2}} + \{(1-\alpha)(1-\beta)\}^{\frac{1}{2}})\right\}^{\frac{1}{2}} \tag{1.10}$$

10. If $P = \{16\alpha\beta(1-\alpha)(1-\beta)\}^{\frac{1}{8}}$ *and* $Q = \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{\frac{1}{4}}$*, then*

$$Q + \frac{1}{Q} + 2\sqrt{2}\left(P - \frac{1}{P}\right) = 0 \tag{1.11}$$

11. If $P = (\alpha\beta)^{\frac{1}{8}}$ *and* $Q = \left(\frac{\beta}{\alpha}\right)^{\frac{1}{4}}$*, then*

$$Q - \frac{1}{Q} = 2\left(P - \frac{1}{P}\right). \tag{1.12}$$

B. C. Berndt proved the above modular equations by parametrization using m as a parameter [2]. N. D. Baruah and R. Berman proved the above modular equations by proving corresponding theta function identities [1]. In this article we prove these modular equations using parametrization. The parameter which we employ here is different from that of Berndt.

2 Preliminary results

Throughout this article let β is of degree 3 over α and let m be a multiplier of degree 3. We now define the parameter a as

$$a = (\alpha\beta)^{\frac{1}{4}}. \tag{2.1}$$

By definition of α and β , $a > 0$. From (1.2), we have

$$\{(1-\alpha)(1-\beta)\}^{\frac{1}{4}} = 1 - a. \tag{2.2}$$

clearly $1 - a > 0$. from (2.1) and (2.2), we have

$$\alpha\beta = a^4 \tag{2.3}$$

and

$$\alpha + \beta = 2a(2 - 3a + 2a^2). \tag{2.4}$$

From the above two equations, we find that

$$\alpha, \beta = a\left\{2 - 3a + 2a^2 \pm 2(1-a)\sqrt{1 - a + a^2}\right\}.$$

We observe that

$$2 - 3a + 2a^2 = 1 - a + (1-a)^2 > 0,$$

and

$$1-a+a^2>0.$$

Since $\alpha>\beta$, we must have

$$\alpha=a\left\{2-3a+2a^2+2(1-a)\sqrt{1-a+a^2}\right\}=a(\sqrt{1-a+a^2}+1-a)^2 \quad (2.5)$$

and

$$\beta=a\left\{2-3a+2a^2-2(1-a)\sqrt{1-a+a^2}\right\}=a(\sqrt{1-a+a^2}-(1-a))^2. \quad (2.6)$$

From the above two identities, we have

$$1-\alpha=1-a\left\{2-3a+2a^2+2(1-a)\sqrt{1-a+a^2}\right\} \quad (2.7)$$

and

$$1-\beta=1-a\left\{2-3a+2a^2-2(1-a)\sqrt{1-a+a^2}\right\}. \quad (2.8)$$

From $(2.5)-(2.8)$ respectively, it follows that

$$\left(\frac{\alpha}{a}\right)^{\frac{1}{2}}=1-a+\sqrt{1-a+a^2}, \quad (2.9)$$

$$\left(\frac{\beta}{a}\right)^{\frac{1}{2}}=a-1+\sqrt{1-a+a^2}, \quad (2.10)$$

$$\left(\frac{1-\alpha}{1-a}\right)^{\frac{1}{2}}=-a+\sqrt{1-a+a^2}, \quad (2.11)$$

and

$$\left(\frac{1-\beta}{1-a}\right)^{\frac{1}{2}}=a+\sqrt{1-a+a^2}. \quad (2.12)$$

Let

$$x:=x(a):=(\alpha\beta)^{\frac{1}{4}}$$

and

$$y:=y(a):=\left\{(1-\alpha)(1-\beta)\right\}^{\frac{1}{4}}.$$

From (2.1) and (2.2), we have

$$x=a, \quad y=1-a. \quad (2.13)$$

Thus , we have

$$\frac{dx}{da}=1, \quad and \quad \frac{dy}{da}=-1 \quad (2.14)$$

Using the identities (2.5)−(2.8) and (2.13), (2.14), we find that

$$\alpha y\frac{dx}{da} = a(1-a)\left\{2-3a+2a^2+2(1-a)\sqrt{1-a+a^2}\right\}, \tag{2.15}$$

$$(1-\alpha)x\frac{dy}{da} = -a(1-a)\left\{1-a+2a^2-2a\sqrt{1-a+a^2}\right\}, \tag{2.16}$$

$$\beta y\frac{dx}{da} = a(1-a)\left\{2-3a+2a^2-2(1-a)\sqrt{1-a+a^2}\right\}, \tag{2.17}$$

and

$$(1-\beta)x\frac{dy}{da} = -a(1-a)\left\{1-a+2a^2+2a\sqrt{1-a+a^2}\right\}. \tag{2.18}$$

Ramanujan recorded the following relation among the multiplier m of degree n relating α and β :

$$n\frac{d\alpha}{d\beta} = \frac{\alpha(1-\alpha)}{\beta(1-\beta)}m^2. \tag{2.19}$$

For a proof of the above one may refer [**?**].
Now employing the definitions of x and y in (2.19) with $n=3$, we find that

$$\frac{3}{m^2} = -\frac{\alpha y\frac{dx}{da}+(1-\alpha)x\frac{dy}{da}}{\beta y\frac{dx}{da}+(1-\beta)x\frac{dy}{da}} \tag{2.20}$$

Utilizing (2.15)−(2.18) in the above, we find that

$$\frac{9}{m^2} = \left\{1-2a+2\sqrt{1-a+a^2}\right\}^2. \tag{2.21}$$

Which implies that

$$\frac{3}{m} = \pm(1-2a+2\sqrt{1-a+a^2}).$$

From the above, it follows that

$$m = \mp(1-2a-2\sqrt{1-a+a^2}).$$

Since $m>0$, from the above two equations, it follows that

$$m = (2a-1+2\sqrt{1-a+a^2}) \tag{2.22}$$

and

$$\frac{3}{m} = 1-2a+2\sqrt{1-a+a^2}. \tag{2.23}$$

Also from (2.21), we have

$$m^2 = (2a-1+2\sqrt{1-a+a^2}). \tag{2.24}$$

3 Proofs of modular equations of degree 3

In this section, we prove a few modular equations of degree 3.

Proof of (1.2)**:** Using (2.1) and (2.2), we obtain that

$$\left(\frac{\alpha^3}{\beta}\right)^{\frac{1}{8}} - \left(\frac{(1-\alpha)^3}{1-\beta}\right)^{\frac{1}{8}} = \left(\frac{\alpha}{a}\right)^{\frac{1}{2}} - \left(\frac{1-\alpha}{1-a}\right)^{\frac{1}{2}}.$$

Employing (2.9) and (2.11) on the right hand side of the above, we obtain

$$\left(\frac{\alpha^3}{\beta}\right)^{\frac{1}{8}} - \left(\frac{(1-\alpha)^3}{1-\beta}\right)^{\frac{1}{8}} = 1.$$

On the same line as above, utilizing (2.1), (2.2), (2.10), and (2.12), we deduce the equality

$$\left(\frac{(1-\beta)^3}{1-\alpha}\right)^{\frac{1}{8}} - \left(\frac{\beta^3}{\alpha}\right)^{\frac{1}{8}} = 1.$$

This completes the proof of (1.2).

Proof of (1.3)**:** Using (2.1) and (2.10), we obtain that

$$\begin{aligned} 1 + 2\left(\frac{\beta^3}{\alpha}\right)^{\frac{1}{8}} &= 1 + 2\left(\frac{\beta}{a}\right)^{\frac{1}{2}} \\ &= 2a - 1 + 2\sqrt{1 - a + a^2} \\ &= m. \end{aligned}$$

Also using (2.2) and (2.11), we obtain that

$$\begin{aligned} 1 + 2\left(\frac{(1-\alpha)^3}{1-\beta}\right)^{\frac{1}{8}} &= 1 + 2\left(\frac{1-\alpha}{1-a}\right)^{\frac{1}{2}} \\ &= 1 - 2a + 2\sqrt{1 - a + a^2} \\ &= \frac{3}{m}. \end{aligned}$$

This completes the proof of (1.3).

Proof of (1.4)**:** Using (2.1), (2.9) and the definitions of α and β in (2.5) and (2.6), we find that

$$\begin{aligned} \frac{\left(\frac{\alpha^3}{\beta}\right)^{\frac{1}{8}} - \beta}{\left(\frac{\alpha^3}{\beta}\right)^{\frac{1}{8}} - \alpha} &= \frac{\left(\frac{\alpha}{a}\right)^{\frac{1}{2}} - \beta}{\left(\frac{\alpha}{a}\right)^{\frac{1}{2}} - \alpha} \\ &= \frac{1 - 3a + 3a^2 - 2a^3 + (1 + 2a - 2a^2)\sqrt{1 - a + a^2}}{1 - 3a + 3a^2 - 2a^3 + (1 - 2a + 2a^2)\sqrt{1 - a + a^2}}. \end{aligned}$$

On elementary algebraic manipulation of the right hand side of the above and using (2.24), we obtain that

$$\frac{\left(\frac{\alpha^3}{\beta}\right)^{\frac{1}{8}} - \beta}{\left(\frac{\alpha^3}{\beta}\right)^{\frac{1}{8}} - \alpha} = 8a^2 - 8a + 5 + 4(2a-1)\sqrt{1-a+a^2}$$
$$= ((2a-1) + 2\sqrt{1-a+a^2})^2$$
$$= m^2.$$

This proves (1.4).

Proof of (1.5)**:** Using (2.1) and (2.2), we find that

$$\frac{1 - 2\left(\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}\right)^{\frac{1}{8}}}{1-2(\alpha\beta)^{\frac{1}{4}}} = \frac{1 - 2\left(\frac{\beta(1-\beta)}{a(1-a)}\right)^{\frac{1}{2}}}{1-2a}.$$

Now employing (2.10) and (2.12) on the right hand side of the above and on elementary algebraic manipulation, we find that

$$\frac{1 - 2\left(\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}\right)^{\frac{1}{8}}}{1-2(\alpha\beta)^{\frac{1}{4}}} = 2a - 1 + 2\sqrt{1-a+a^2}$$
$$= m.$$

On the same line as above, we find that

$$1 + 4\left(\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}\right)^{\frac{1}{8}} = 1 + 4\left(\frac{\beta(1-\beta)}{a(1-a)}\right)^{\frac{1}{2}}$$
$$= (2a - 1 + 2\sqrt{1-a+a^2})^2$$
$$= m^2.$$

Thus, we have

$$\left\{1 + 4\left(\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}\right)^{\frac{1}{8}}\right\}^{\frac{1}{2}} = m.$$

Similar to the above two proofs, by using (2.1), (2.2), (2.9), and (2.11) it is easily deduce that

$$\frac{2\left(\frac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}\right)^{\frac{1}{8}} - 1}{1-2(\alpha\beta)^{\frac{1}{4}}} = 1 - 2a + 2\sqrt{1-a+a^2}$$
$$= \frac{3}{m},$$

and

$$\left\{1+4\left(\frac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}\right)^{\frac{1}{8}}\right\}^{\frac{1}{2}} = 1-2a+2\sqrt{1-a+a^2}$$
$$= \frac{3}{m}.$$

This completes the proof of (1.5).

Proof of (1.6): Using (2.1) and (2.2), we have

$$\left(\frac{\beta}{\alpha}\right)^{\frac{1}{2}} + \left(\frac{1-\beta}{1-\alpha}\right)^{\frac{1}{2}} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{\frac{1}{2}} = \frac{\beta}{a^2} + \frac{(1-\beta)}{(1-a)^2} - \frac{\beta(1-\beta)}{(a(1-a))^2}$$
$$= \left(\frac{a-\beta}{a(1-a)}\right)^2.$$

Now employing the definition of β from (2.6) on the right hand side of the above, we obtain

$$\left(\frac{\beta}{\alpha}\right)^{\frac{1}{2}} + \left(\frac{1-\beta}{1-\alpha}\right)^{\frac{1}{2}} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{\frac{1}{2}} = (2a-1+2\sqrt{1-a+a^2})^2$$
$$= m^2.$$

On the same line as above, using (2.1), (2.2) and the definition of α in (2.5), we obtain that

$$\left(\frac{\alpha}{\beta}\right)^{\frac{1}{2}} + \left(\frac{1-\alpha}{1-\beta}\right)^{\frac{1}{2}} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{\frac{1}{2}} = \frac{\alpha}{a^2} + \frac{(1-\alpha)}{(1-a)^2} - \frac{\alpha(1-\alpha)}{(a(1-a))^2}$$
$$= \left(\frac{\alpha-a}{a(1-a)}\right)^2$$
$$= (1-2a+2\sqrt{1-a+a^2})^2$$
$$= \frac{9}{m^2}.$$

This completes the proof of (1.6).

Proof of (1.7): From (2.1) and (2.2), it follows that

$$(\alpha\beta^5)^{\frac{1}{8}}+\left\{(1-\alpha)(1-\beta)^5\right\}^{\frac{1}{8}}+\left(\frac{\beta^3(1-\alpha)^3}{\alpha(1-\beta)}\right)^{\frac{1}{8}} = (a\beta)^{\frac{1}{2}}+((1-a)\beta)^{\frac{1}{2}}+\left(\frac{\beta(1-\alpha)}{a(1-a)}\right)^{\frac{1}{2}}.$$

Employing (2.5), (2.6), (2.10), and (2.11) on the right hand side of the above and on elementary algebraic manipulation, we find that

$$(\alpha\beta^5)^{\frac{1}{8}} + \left\{(1-\alpha)(1-\beta)^5\right\}^{\frac{1}{8}} + \left(\frac{\beta^3(1-\alpha)^3}{\alpha(1-\beta)}\right)^{\frac{1}{8}} = 1.$$

From which, it follows that

$$(\alpha\beta^5)^{\frac{1}{8}} + \left\{(1-\alpha)(1-\beta)^5\right\}^{\frac{1}{8}} = 1 - \left(\frac{\beta^3(1-\alpha)^3}{\alpha(1-\beta)}\right)^{\frac{1}{8}}. \tag{3.1}$$

On the same line as above, from (2.1) and (2.2), we find that

$$(\alpha^5\beta)^{\frac{1}{8}} + \left\{(1-\alpha)^5(1-\beta)\right\}^{\frac{1}{8}} + \left(\frac{\beta^3(1-\alpha)^3}{\alpha(1-\beta)}\right)^{\frac{1}{8}} = 1.$$

This implies

$$(\alpha^5\beta)^{\frac{1}{8}} + \left\{(1-\alpha)^5(1-\beta)\right\}^{\frac{1}{8}} = 1 - \left(\frac{\beta^3(1-\alpha)^3}{\alpha(1-\beta)}\right)^{\frac{1}{8}}. \tag{3.2}$$

We observe that

$$\begin{aligned}(\alpha^5\beta)^{\frac{1}{8}} + \left\{(1-\alpha)^5(1-\beta)\right\}^{\frac{1}{8}} &= (a\alpha)^{\frac{1}{2}} + \{(1-a)(1-\alpha)\}^{\frac{1}{2}}\\ &= \sqrt{1-a+a^2}.\end{aligned}$$

Using (2.1) and (2.2) on the right hand side of the above, we find that

$$(\alpha^5\beta)^{\frac{1}{8}} + \left\{(1-\alpha)^5(1-\beta)\right\}^{\frac{1}{8}} = \left\{\frac{1}{2}(1+(\alpha\beta)^{\frac{1}{2}} + \{(1-\alpha)(1-\beta)\}^{\frac{1}{2}})\right\}^{\frac{1}{2}}. \tag{3.3}$$

The identity (1.7) follows from the equations (3.1) − (3.3).

Proof of (1.8): Using (2.5)−(2.8), we can easily deduce that

$$(\alpha(1-\beta))^{\frac{1}{2}} + (\beta(1-\alpha))^{\frac{1}{2}} = 2(a(1-a))^{\frac{1}{2}}.$$

Which is equivalent to

$$(\alpha(1-\beta))^{\frac{1}{2}} + (\beta(1-\alpha))^{\frac{1}{2}} = 2(\alpha\beta(1-\alpha)(1-\beta))^{\frac{1}{8}}. \tag{3.4}$$

From (2.9) and (2.11), we find that

$$\left\{\frac{\alpha(1-\alpha)}{a(1-a)}\right\}^{\frac{1}{2}} = (1-a+\sqrt{1-a+a^2})(-a+\sqrt{1-a+a^2})$$

and

$$\left\{\frac{\beta(1-\beta)}{a(1-a)}\right\}^{\frac{1}{2}} = (a-1+\sqrt{1-a+a^2})(a+\sqrt{1-a+a^2}).$$

Now using (2.24) and the above two equations, we find that

$$m^2\left\{\frac{\alpha(1-\alpha)}{a(1-a)}\right\}^{\frac{1}{2}} + \left\{\frac{\beta(1-\beta)}{a(1-a)}\right\}^{\frac{1}{2}} = 2.$$

From the above, it follows that

$$m^2\left\{\alpha(1-\alpha)\right\}^{\frac{1}{2}} + \left\{\beta(1-\beta)\right\}^{\frac{1}{2}} = 2(\alpha\beta(1-\alpha)(1-\beta))^{\frac{1}{8}}. \tag{3.5}$$

Similarly, we obtain

$$\frac{9}{m^2}\left\{\frac{\beta(1-\beta)}{a(1-a)}\right\}^{\frac{1}{2}} + \left\{\frac{\alpha(1-\alpha)}{a(1-a)}\right\}^{\frac{1}{2}} = 2.$$

From which, it follows that

$$\frac{9}{m^2}\left\{\beta(1-\beta)\right\}^{\frac{1}{2}} + \left\{\alpha(1-\alpha)\right\}^{\frac{1}{2}} = 2(\alpha\beta(1-\alpha)(1-\beta))^{\frac{1}{8}}. \tag{3.6}$$

from (3.4)$-$(3.6), we arrive at the required identity. (1.8)

Proof of (1.9)**:** As in the proof of (1.8), we find that

$$m\left(\frac{1-\alpha}{1-a}\right)^{\frac{1}{2}} + \left(\frac{1-\beta}{1-a}\right)^{\frac{1}{2}} = 2, \tag{3.7}$$

$$\frac{3}{m}\left(\frac{1-\beta}{1-a}\right)^{\frac{1}{2}} - \left(\frac{1-\alpha}{1-a}\right)^{\frac{1}{2}} = 2, \tag{3.8}$$

$$m\left(\frac{\alpha}{a}\right)^{\frac{1}{2}} - \left(\frac{\beta}{a}\right)^{\frac{1}{2}} = 2, \tag{3.9}$$

and

$$\frac{3}{m}\left(\frac{\beta}{a}\right)^{\frac{1}{2}} + \left(\frac{\alpha}{a}\right)^{\frac{1}{2}} = 2. \tag{3.10}$$

Using the fact that $a = (\alpha\beta)^{\frac{1}{4}}$ and $(1-a) = \{(1-\alpha)(1-\beta)\}^{\frac{1}{4}}$ in (3.7)$-$(3.10), we obtain the required identity (1.9).

Proof of (1.10)**:** Adding (2.22) and (2.23), we obtain that

$$\begin{aligned} m+\frac{3}{m} &= 4\sqrt{1-a+a^2} \\ &= 4\sqrt{\frac{1}{2}(2-2a+2a^2)} \\ &= 4\left\{\frac{1}{2}(1+(\alpha\beta)^{\frac{1}{2}}+\{(1-\alpha)(1-\beta)\}^{\frac{1}{2}})\right\}^{\frac{1}{2}}. \end{aligned}$$

Also subtracting (2.23) from (2.22), we have

$$\begin{aligned} m-\frac{3}{m} &= 2(2a-1) \\ &= 2\left((\alpha\beta)^{\frac{1}{4}}-\{(1-\alpha)(1-\beta)\}^{\frac{1}{4}}\right). \end{aligned}$$

This completes the proof of (1.10).

Proof of (1.11)**:** Employing (2.5)−(2.8), we deduce that

$$Q+\frac{1}{Q}=\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{\frac{1}{4}}-\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{\frac{1}{4}}$$

$$Q+\frac{1}{Q}=\frac{2(1-2a+2a^2)}{(a(1-a))^{\frac{1}{2}}}. \tag{3.11}$$

Also, we observe that

$$P-\frac{1}{P}=\frac{2a(1-a)-1}{\{2a(1-a)\}^{\frac{1}{2}}}.$$

From (3.11) and the above identity, we obtain

$$Q+\frac{1}{Q}+2\sqrt{2}\left(P-\frac{1}{P}\right)=0.$$

This completes the proof of (1.11).

Proof of (1.12)**:** We have

$$\begin{aligned} \frac{Q-\dfrac{1}{Q}}{P-\dfrac{1}{P}} &= \frac{\left(\dfrac{\beta}{\alpha}\right)^{\frac{1}{4}}-\left(\dfrac{\alpha}{\beta}\right)^{\frac{1}{4}}}{\sqrt{a}-\dfrac{1}{\sqrt{a}}} \\ &= \frac{\beta^{\frac{1}{2}}-\alpha^{\frac{1}{2}}}{\sqrt{a}(a-1)}. \end{aligned}$$

Employing (2.9) and (2.10) on the right hand side of the above, we find that

$$\frac{Q-\dfrac{1}{Q}}{P-\dfrac{1}{P}}=2.$$

The identity (1.12), follows from the above equation.

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References

[1] N. D. Baruah and R. Barman, Certain theta function identities and Ramanujan's modular equations of degree 3, *Indian J. Math.*, **48(3)**(2006), 113-133.

[2] B. C. Berndt, *Ramanujan's Notebooks, Part III*, Springer-Verlag, New York, 1991.

[3] M. Hanna, The modular equations, *Proc. London Math. Soc.*, **s2-28**(1928), 46-52.

[4] C. G. J. Jacobi, *Fundamenta Nova Theoriae Functionum Ellipticarum*, Sumptibus Fratrum Bornträger, Regiomonti, 1829.

[5] C. G. J. Jacobi, *Gesammelte Werke*, Erster Band, G. Reimer, Berlin, 1881.

[6] A. M. Legendre, *Traité des Fonctions Elliptiques*, **t. 1**, Huzard-Courcier, Paris, (1825).

[7] *S. Ramanujan Notebook (volume 2)*, Tata institute of fundamental Research, Bombay, 1957.

[8] L. -C. Shen, On the modular equations of degree 3, *Proc. Amer. Math. Soc.*, **122(4)**, 1994, 1101-1114.