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On some Schläfli type modular equations

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Abstract

On page 90 of his first notebook Ramanujan has recorded many Schläfli type modular equations. This article aims to give new and simple proof of Schälfli type modular equation of degree 11 which was recorded by Ramanujan and a degree 23 modular equation. We use parametrization method to prove these modular equations.

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1 Introduction

The ordinary hyper-geometric series or the Gauss series is denoted by ${}_2F_1(a, b; c; z)$ and is defined by

$$ {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, $$

where a, b, c and z are complex numbers with $|z| < 1$ and $c \neq 0, -1, -2, \cdots$. It is interesting to record that

$$ {}_2F_1\left[\frac{1}{2}, \frac{1}{2}; 1; z\right] = \frac{\pi}{2} \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - z^2 \sin^2 \phi}}, \quad 0 < z < 1. $$

The complete elliptic integral of first kind is denoted by $K(k)$ and is defined as

$$ K(k) = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}, \quad 0 < z < 1. $$

We call k as the modulus of K. $\sqrt{1-k^2}$ is called as the complementary modulus of $K(k)$ and is usually denoted by k'.

Set $\alpha = k^2$, $\beta = l_1^2$, $\gamma = l_2^2$ and $\delta = l_3^2$. Let $k' = \sqrt{1-k^2}$, $l_1' = \sqrt{1-l_1^2}$, $l_2' = \sqrt{1-l_2^2}$ and $l_3' = \sqrt{1-l_3^2}$. Suppose that the equality

$$ n\frac{K(k')}{K(k)} = \frac{K(l_1')}{K(l_1)} $$

holds for some positive integer n. The relation between α and β induced by the above is called a modular equation of degree n. The multiplier connecting α and β, denoted by m is defined as

$$m = \frac{z_1}{z_n} = \frac{K(\sqrt{\alpha})}{K(\sqrt{\beta})} = \frac{{}_2F_1\left[\frac{1}{2},\frac{1}{2};1;\alpha\right]}{{}_2F_1\left[\frac{1}{2},\frac{1}{2};1;\beta\right]},$$

and we also say that β has degree n over α.

L. Schläfli has discovered seven modular equations in his paper [5]. Ramanujan also recorded these modular equations in his first notebook [3, pp.] and also in his second notebook [4, pp.]. These are the modular equations of degrees 3, 5, 7, 11, 13, 17, and 19. B. C. Berndt [1] has given proof of the modular equations of degrees 3, 5, and 7. For a proof of remaining modular equations, one may refer to [8], [2]. In fact, they used to deduce these modular equations by elimination techniques, which involve very complicated algebraic manipulation, which can be done only by using computer software like Maple.

In this article, we give new and simple proof of Schläfli type modular equations of degrees 11 and 23. Schläfli type modular equations of degree 11 were recorded by Ramanujan in [4] and Schläfli type modular equations of degree 23 were established by Vasuki K. R. [8]. In the next section, we prove Schläfli type modular equations of degrees 11 and 23 using parametrization.

2 Schläfli type modular equations of degrees 11 and 23

In his second notebook [3, p. 90], Ramanujan has recorded the following Schläfli type modular equation of degree 11:

Theorem 2.1. *[3, p. 90] Let β be of degree 11 over α,*

$$P := 2^{\frac{1}{6}}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/24}$$

, and

$$Q := \left\{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right\}^{1/24}.$$

Then

$$Q^6 + \frac{1}{Q^6} = 2\sqrt{2}\left(\frac{2}{P^5} - \frac{11}{P^3} + \frac{22}{P} - 22P + 11P^3 - 2P^5\right).$$

Proof: Let

$$a = \{16\alpha\beta(1-\alpha)(1-\beta)\}^{\frac{1}{12}}.$$

We suppose that $\alpha + \beta > 1$.

In [4, p. 243], Ramanujan has recorded the following modular equation:

$$(\alpha\beta)^{\frac{1}{4}} + \{(1-\alpha)(1-\beta)\}^{\frac{1}{4}} = 1 - 2a. \tag{2.1}$$

From [10], we have

$$(\alpha\beta)^{\frac{1}{2}} + \{(1-\alpha)(1-\beta)\}^{\frac{1}{2}} = 1 - 4a + 4a^2 - a^3, \tag{2.2}$$

$$\alpha\beta + (1-\alpha)(1-\beta) = s, \tag{2.3}$$

$$\alpha\beta - (1-\alpha)(1-\beta) = \frac{\sqrt{4s^2 - a^{12}}}{2}, \tag{2.4}$$

Thus we have

$$\alpha = \frac{2 + \sqrt{4s^2 - a^{12}} + \sqrt{4 + 4s^2 - a^{12} - 8s}}{4}, \tag{2.5}$$

$$\beta = \frac{2 + \sqrt{4s^2 - a^{12}} - \sqrt{4 + 4s^2 - a^{12} - 8s}}{4}, \tag{2.6}$$

$$1 - \alpha = \frac{2 - \sqrt{4s^2 - a^{12}} - \sqrt{4 + 4s^2 - a^{12} - 8s}}{4}, \tag{2.7}$$

$$1 - \beta = \frac{2 - \sqrt{4s^2 - a^{12}} + \sqrt{4 + 4s^2 - a^{12} - 8s}}{4} \tag{2.8}$$

where

$$s = 1 - 8a + 24a^2 - 34a^3 + 24a^4 - 8a^5 + \frac{a^6}{2}$$

From (2.5)–(2.8), we have

$$\frac{\beta(1-\beta)}{\alpha(1-\alpha)} = \frac{(G+H)^2}{G^2 - H^2},$$

where $G = 2a^{12} - 8s^2 + 8s$ and $H = 2\sqrt{(4s^2 - a^{12})(4 + 4s^2 - a^{12} - 8s)}$. Equivalently, we have

$$\left\{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right\}^{\frac{1}{2}} = \pm\frac{G+H}{\sqrt{G^2 - H^2}}.$$

Since $\alpha, \beta, 1-\alpha$, and $1-\beta$ are positive ($G - H > 0$ and $G + H > 0$), we have

$$\left\{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right\}^{\frac{1}{2}} = \frac{G+H}{\sqrt{G^2 - H^2}}. \tag{2.9}$$

Similarly, we find that

$$\left\{\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right\}^{\frac{1}{2}} = \frac{G-H}{\sqrt{G^2 - H^2}}. \tag{2.10}$$

Adding (2.9) and (2.10), we have

$$\left\{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right\}^{\frac{1}{2}} + \left\{\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right\}^{\frac{1}{2}} = \frac{2G}{\sqrt{G^2 - H^2}}.$$

From the above, we find that

$$\left(\left\{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right\}^{\frac{1}{4}}+\left\{\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right\}^{\frac{1}{4}}\right)^2=\frac{2G}{\sqrt{G^2-H^2}}+2.$$

Which implies

$$\left(\left\{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right\}^{\frac{1}{4}}+\left\{\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right\}^{\frac{1}{4}}\right)^2=\frac{8(2a^5-11a^4+22a^3-22a^2+11a-2)^2}{a^5}.$$

Using the fact, $a=P^2$ in the above and then taking square root on both side of the above, we obtain the required result.

Theorem 2.2. *Let β be of degree 23 over α,*

$$P:=2^{\frac{1}{6}}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/24}$$

, and

$$Q:=\left\{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right\}^{1/24}.$$

Then

$$\begin{aligned}Q^{12}+\frac{1}{Q^{12}}&=32\sqrt{2}\left(P^{11}+\frac{1}{P^{11}}\right)-736\left(P^{10}+\frac{1}{P^{10}}\right)+4048\sqrt{2}\left(P^{9}+\frac{1}{P^{9}}\right)\\&-28704\left(P^{8}+\frac{1}{P^{8}}\right)+74336\sqrt{2}\left(P^{7}+\frac{1}{P^{7}}\right)-300840\left(P^{6}+\frac{1}{P^{6}}\right)\\&+495328\sqrt{2}\left(P^{5}+\frac{1}{P^{5}}\right)-1362336\left(P^{4}+\frac{1}{P^{4}}\right)+1592336\sqrt{2}\left(P^{3}+\frac{1}{P^{3}}\right)\\&-3200864\left(P^{2}+\frac{1}{P^{2}}\right)+2787232\sqrt{2}\left(P+\frac{1}{P}\right)-4223122.\end{aligned}$$

Proof: Let the parameter a be defined by

$$a=2^{\frac{2}{3}}\{\alpha\beta(1-\alpha)(1-\beta)\}^{\frac{1}{24}}.$$

By the definition of a, it is clear that a is positive. On page 248 of his second notebook [4], Ramanujan recorded

$$(\alpha\beta)^{\frac{1}{8}}+\{(1-\alpha)(1-\beta)\}\}^{\frac{1}{8}}=1-a. \tag{2.11}$$

Berndt proved this by using one of the Schröter’s formula [1, p. 69, Eqn. (36.11)]. Schröter has recorded (2.11) in [6] and [7].

From [9], we have

$$\alpha=\frac{16+A\sqrt{p}+B\sqrt{qr}}{32}, \tag{2.12}$$

$$\beta=\frac{16+A\sqrt{p}-B\sqrt{qr}}{32}, \tag{2.13}$$

$$1-\alpha=\frac{16-A\sqrt{p}-B\sqrt{qr}}{32}, \tag{2.14}$$

and

$$1-\beta=\frac{16-A\sqrt{p}+B\sqrt{qr}}{32} \tag{2.15}$$

where

$$\begin{aligned}
A&=(1-a)(2-4a+a^2)(2-4a+2a^2-a^3)(4-8a+6a^2-4a^3+a^4),\\
B&=a(2-a)(2-a+a^2)(4-4a+4a^2-a^3),\\
p&=1-2a+a^2-a^3,\\
q&=8-4a+4a^2-a^3,
\end{aligned}$$

and

$$r=8-20a+28a^2-25a^3+14a^4-5a^5+a^6.$$

From (2.12)$-$(2.15), we have

$$\frac{\beta(1-\beta)}{\alpha(1-\alpha)}=\frac{(16^2-A^2p-B^2qr+2AB\sqrt{pqr})^2}{(16^2-A^2p-B^2qr)^2-(2AB\sqrt{pqr})^2}$$

Equivalently, we have

$$\left\{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right\}^{\frac{1}{2}}=\pm\frac{16^2-A^2p-B^2qr+2AB\sqrt{pqr}}{\sqrt{(16^2-A^2p-B^2qr)^2-(2AB\sqrt{pqr})^2}}$$

Since $\alpha,\beta,1-\alpha$, and $1-\beta$ are positive, we have

$$\left\{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right\}^{\frac{1}{2}}=\frac{16^2-A^2p-B^2qr+2AB\sqrt{pqr}}{\sqrt{(16^2-A^2p-B^2qr)^2-(2AB\sqrt{pqr})^2}} \tag{2.16}$$

Similarly, we find that

$$\left\{\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right\}^{\frac{1}{2}}=\frac{16^2-A^2p-B^2qr-2AB\sqrt{pqr}}{\sqrt{(16^2-A^2p-B^2qr)^2-(2AB\sqrt{pqr})^2}} \tag{2.17}$$

Adding (2.16) and (2.17), we have

$$\begin{aligned}
\left\{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right\}^{\frac{1}{2}}+\left\{\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right\}^{\frac{1}{2}}&=a^{11}-23a^{10}+253a^9-1794a^8+9292a^7-37605a^6\\
&+123832a^5-340584a^4+796168a^3-1600432a^2+2787232a-4223122\\
&+\frac{5574464}{a}-\frac{6401728}{a^2}+\frac{6369344}{a^3}-\frac{5449344}{a^4}+\frac{3962624}{a^5}-\frac{2406720}{a^6}\\
&+\frac{1189376}{a^7}-\frac{459264}{a^8}+\frac{129536}{a^9}-\frac{23552}{a^{10}}+\frac{2048}{a^{11}}
\end{aligned}$$

Using the fact that $a=2^{\frac{1}{2}}P$ in the above, we obtain the required result.

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