International Journal of Mathematics and Computer Research
ISSN: 2320-7167
Volume 12 Issue 10 October 2024, Page no. –4519-4535
Index Copernicus ICV: 57.55, Impact Factor: 8.316
DOI: 10.47191/ijmcr/v12i10.10

On certain restricted partition functions

Praveenkumar[1], Jeshma Prakruthi K. S.[2], M. Manjunatha[3]

[1]Department of Studies in Mathematics, University of Mysore,
Manasagangotri, Mysuru - 570 006, INDIA.

[2,3]Department of Mathematics, P. E. S College of Engineering,
Mandya - 571 401, INDIA.

Abstract

The objective of this paper is to establish explicit formulas for certain restricted partition function in terms of divisor function.

Keywords: Theta function, Theory of partition.

2020 Mathematics Subject Classification: 11F20, 11P84, 33C20.

1 Introduction

For any positive integer n, a partition of n is an expression of the form

$$n = n_1 + n_2 + \cdots + n_k$$

with $n_1 \geq n_2 \geq n_3 \geq \cdots \geq n_k$, $k \geq 1$ and the number of partition of n is denoted by $p(n)$. We also define $p(0) = 1$.

P. A. MacMahon calculated $p(n)$ upto 200. Looking the table of $p(n)$ of MacMahon, Ramanujan conjectured that

$$p(5n+4) \equiv 0 (\text{mod } 5)$$

$$p(7n+5) \equiv 0 (\text{mod } 7)$$

$$p(11n+6) \equiv 0 (\text{mod } 11).$$

Later, all the above conjecture were proved. For details of the above proofs and history one may refer to [7].

Motivated by the above properties of partition function, H. C. Chan [6] studied and extended similar works to the restricted partition function. In fact, he define

$$\sum_{n=0}^{\infty} a(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)(1-q^{2n})}$$

and proved that $a(3n+2) \equiv 0 (\text{mod } 3)$. These are the results motivates many mathematician to work on properties of partition functions. See for example [2],[3],[10].

In this article, our aim is to establish explicit formulas in terms of divisor functions for a restricted partition function $P_{k,a,c}(n)$ defined as follows:

Let $P^e_{k,a,c}(n)$[or $P^o_{k,a,c}(n)$] denoted the number of partition of n such that

i . Parts are congruent to $\pm a$ or $\pm c$ or $\frac{k}{2} \pmod{k}$ or $0 \pmod{2k}$.

ii . Parts are congruent to $\pm a$, $\frac{k}{2} \pmod{k}$ and $0 \pmod{2k}$ are distinct.

iii . Parts are congruent to $\frac{k}{2} \pmod{k}$ and $0 \pmod{2k}$ are having two colors.

iv . Number of parts are congruent to $\frac{k}{2} \pmod{k}$ and $0 \pmod{2k}$ are even(odd).

Let

$$P_{k,a,c}(n) = P^e_{k,a,c}(n) - P^o_{k,a,c}(n).$$

In the next section we prove our main results.

Now we shall recall certain definition which are required to prove our main results. For any complex number a and q with $|q| < 1$,

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1 - aq^n).$$

For any integer n, let

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}.$$

Ramanujan's theta function $f(a,b)$ is defined by

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, \ |ab| < 1.$$

From the well known Jacobi's triple product identity, we have

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty.$$

A special case of $f(a,b)$, $\psi(q)$ is defined by

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}.$$

By the above definition of $f(a,b)$ and $\psi(q)$, one can easily see that

$$\sum_{n=0}^{\infty} P_{k,a,c}(n)q^n = \psi^2(-q^k)\frac{f(q^a,q^b)}{f(-q^c,-q^d)}, \quad \text{where} \quad a+b=c+d=2k. \tag{1.1}$$

Let $d_{x,y}(n)$ denotes the number of divisor of n which are congruent to $x(\text{mod } y)$ if n is a positive integer and $d_{x,y}(n)=0$ otherwise.

2 Main results

In this section, we establish explicit formulas in terms of divisor functions for a restricted partition function $P_{k,a,c}(n)$.

Lemma 2.1. *For each $n \in \mathbb{N}$, we have*

$$\psi^2(q^2) = \sum_{n=0}^{\infty} t_2(n)q^n$$

and

$$t_2(n) = d_{1,4}(2n+1) - d_{3,4}(2n+1),$$

where $t_2(n)$ is the number of representation of an integer n as a sum of two triangular number.

For a proof of above lemma, one may refer to [1].

Theorem 2.2. *The following identity holds:*

$$P_{6,2,1}(n) = d_{1,4}(2n+1) - d_{3,4}(2n+1) + d_{1,4}\left(\frac{2n+1}{3}\right) - d_{3,4}\left(\frac{2n+1}{3}\right).$$

Proof. From [4], we have

$$\psi^2(-q^3)\frac{f(q^2,q^4)}{f(-q,-q^5)} = \psi^2(q^2) + q\psi^2(q^6).$$

Employing (1.1) and Lemma 2.1 in the above, we find that

$$\sum_{n=0}^{\infty} P_{6,2,1}(n)q^n$$
$$= \sum_{n=0}^{\infty}\left\{d_{1,4}(2n+1) - d_{3,4}(2n+1) + d_{1,4}\left(\frac{2n+1}{3}\right) - d_{3,4}\left(\frac{2n+1}{3}\right)\right\}q^n.$$

By comparing the coefficient of q^n in the above equation, we obtain the required result. □

Example: For $n = 8$, we have

$P^{e}_{6,2,1}(8)$	$P^{o}_{6,2,1}(8)$	$P_{6,2,1}(8)$
8, 7+1, 5+2+1 5+1+1+1 4+2+1+1, 4+1+1+1+1, 2+1+1+1+1+1+1, 1+1+1+1+1+1+1+1, $3_r + 3_g + 1 + 1$, $3_r + 3_g + 2$.	$5 + 3_r$, $5 + 3_g$, $4 + 3_r + 1$, $4 + 3_g + 1$, $3_r + 2 + 1 + 1 + 1$, $3_g + 2 + 1 + 1 + 1$, $3_r + 1 + 1 + 1 + 1 + 1$, $3_g + 1 + 1 + 1 + 1 + 1$.	②

(Table-1)

One can easily see that

$$d_{1,4}(17) = 2, d_{3,4}(17) = 0, d_{1,4}\left(\frac{17}{3}\right) = 0 \text{ and } d_{3,4}\left(\frac{17}{3}\right) = 0.$$

From Table-1 and the above, we have

$$P_{6,2,1}(8) = d_{1,4}(17) - d_{3,4}(17) + d_{1,4}\left(\frac{17}{3}\right) - d_{3,4}\left(\frac{17}{3}\right) = 2.$$

This verifies the theorem for $n = 8$.

Theorem 2.3. *The following identity holds:*

$$P_{10,4,1}(n) - P_{10,2,3}(n-1) = d_{1,4}(2n+1) - d_{3,4}(2n+1) - d_{1,4}\left(\frac{2n+1}{5}\right) + d_{3,4}\left(\frac{2n+1}{5}\right).$$

Proof. From [5, p. 263], we have

$$\psi^2(q^5)\left[\frac{f(q^4,q^6)}{f(q,q^9)} + q\frac{f(q^2,q^8)}{f(q^3,q^7)}\right] = \psi^2(q^2) - q^2\psi^2(q^{10}).$$

By changing q to $-q$ in the above equation, we have

$$\psi^2(-q^5)\left[\frac{f(q^4,q^6)}{f(-q,-q^9)} - q\frac{f(q^2,q^8)}{f(-q^3,-q^7)}\right] = \psi^2(q^2) - q^2\psi^2(q^{10}).$$

Employing (1.1) and Lemma 2.1 in the above, we obtain

$$\sum_{n=0}^{\infty} \{P_{10,4,1}(n) - P_{10,2,3}(n-1)\} q^n$$
$$= \sum_{n=0}^{\infty} \left\{ d_{1,4}(2n+1) - d_{3,4}(2n+1) - d_{1,4}\left(\frac{2n+1}{5}\right) + d_{3,4}\left(\frac{2n+1}{5}\right)\right\} q^n.$$

By comparing the coefficient of q^n in the above equation, we obtain the required result. □

Example: For $n = 10$, we have

$P^e_{10,4,1}(10)$	$P^o_{10,4,1}(10)$	$P_{10,4,1}(10)$
1+9, 4+6, 1+1+1+1+6, 1+1+1+1+1+1+4, 1+1+1+1+1+1+1+1+1+1, $5_r + 5_g$	$1+1+1+1+1+5_r$, $1+1+1+1+1+5_g$, $1+4+5_r$, $1+4+5_g$.	②

(Table-2)

$P^e_{10,2,3}(9)$	$P^o_{10,2,3}(9)$	$P_{10,2,3}(9)$
3+3+3, 2+7.	– –	②

(Table-3)

One can easily see that

$$d_{1,4}(21) = 2, d_{3,4}(21) = 2, d_{1,4}\left(\frac{21}{5}\right) = 0 \text{ and } d_{3,4}\left(\frac{21}{5}\right) = 0.$$

From Table-2,3 and the above, we have

$$P_{10,4,1}(10) - P_{10,2,3}(9) = d_{1,4}(21) - d_{3,4}(21) - d_{1,4}\left(\frac{21}{5}\right) + d_{3,4}\left(\frac{21}{5}\right) = 0.$$

This verifies the theorem for $n = 10$.

Theorem 2.4. *The following identity holds:*

$$P_{14,6,1}(n) - P_{14,4,3}(n-1) + P_{14,2,5}(n-2)$$
$$= d_{1,4}(2n+1) - d_{3,4}(2n+1) + d_{1,4}\left(\frac{2n+1}{7}\right) - d_{3,4}\left(\frac{2n+1}{7}\right).$$

Proof. From [11], we have

$$\psi^2(q^7)\left[\frac{f(q^6,q^8)}{f(q,q^{13})} + q\frac{f(q^4,q^{10})}{f(q^3,q^{11})} + q^2\frac{f(q^2,q^{12})}{f(q^5,q^9)}\right] = \psi^2(q^2) - q^3\psi^2(q^{14}).$$

By changing q to $-q$ in the above equation, we have

$$\psi^2(-q^7)\left[\frac{f(q^6,q^8)}{f(-q,-q^{13})} - q\frac{f(q^4,q^{10})}{f(-q^3,-q^{11})} + q^2\frac{f(q^2,q^{12})}{f(-q^5,-q^9)}\right] = \psi^2(q^2)+q^3\psi^2(q^{14}).$$

Employing (1.1) and Lemma 2.1 in the above, we obtain

$$\sum_{n=0}^{\infty}\{P_{14,6,1}(n)-P_{14,4,3}(n-1)+P_{14,2,5}(n-2)\}q^n$$
$$=\sum_{n=0}^{\infty}\left\{d_{1,4}(2n+1)-d_{3,4}(2n+1)+d_{1,4}\left(\frac{2n+1}{7}\right)-d_{3,4}\left(\frac{2n+1}{7}\right)\right\}q^n.$$

By comparing the coefficient of q^n in the above equation, we obtain the required result.

□

Example: For $n=12$, we have

$P^e_{14,6,1}(12)$	$P^o_{14,6,1}(12)$	$P_{14,6,1}(12)$
1+1+1+1+8, 1+1+1+1+1+1+6, 1+1+1+1+1+1+1+1+1.	$1+1+1+1+1+7_r$, $1+1+1+1+1+7_g$.	①

(Table-4)

$P^e_{14,4,3}(11)$	$P^o_{14,4,3}(11)$	$P_{14,4,3}(11)$
11	$4+7_r$ $4+7_g$	(-1)

(Table-5)

$P^e_{14,2,5}(10)$	$P^o_{14,2,5}(10)$	$P_{14,2,5}(10)$
5+5	–	①

(Table-6)

One can easily see that

$$d_{1,4}(25)=3, d_{3,4}(25)=0, d_{1,4}\left(\frac{25}{7}\right)=0 \text{ and } d_{3,4}\left(\frac{25}{7}\right)=0.$$

From Table-4,3,6 and the above, we have

$$P_{14,6,1}(12)-P_{14,4,3}(11)+P_{14,2,5}(10)=d_{1,4}(25)-d_{3,4}(25)+d_{1,4}\left(\frac{25}{7}\right)-d_{3,4}\left(\frac{25}{7}\right)=3.$$

This verifies the theorem for $n=12$.

Theorem 2.5. *The following identity holds:*

$$\psi^2(q^2)-q^4\psi^2(q^{18})=\psi^2(q^9)\left[\frac{f(q^8,q^{10})}{f(q,q^{17})}+q\frac{f(q^6,q^{12})}{f(q^3,q^{15})}+q^2\frac{f(q^4,q^{14})}{f(q^5,q^{13})}+q^3\frac{f(q^2,q^{16})}{f(q^7,q^{11})}\right].$$

Proof. In [9, p. 216], Ramanujan recorded following identity:

$$\psi^2(q^2) = \sum_{k=0}^{\infty} \frac{q^k}{1+q^{2k+1}}.$$

Which implies

$$\begin{aligned}\psi^2(q^2) = &\sum_{k=0}^{\infty} \frac{q^{9k}}{1+q^{18k+1}} + \sum_{k=0}^{\infty} \frac{q^{9k+1}}{1+q^{18k+3}} + \sum_{k=0}^{\infty} \frac{q^{9k+2}}{1+q^{18k+5}} + \sum_{k=0}^{\infty} \frac{q^{9k+3}}{1+q^{18k+7}} \\ &+ \sum_{k=0}^{\infty} \frac{q^{9k+4}}{1+q^{18k+9}} + \sum_{k=0}^{\infty} \frac{q^{9k+5}}{1+q^{18k+11}} + \sum_{k=0}^{\infty} \frac{q^{9k+6}}{1+q^{18k+13}} + \sum_{k=0}^{\infty} \frac{q^{9k+7}}{1+q^{18k+15}} \\ &+ \sum_{k=0}^{\infty} \frac{q^{9k+8}}{1+q^{18k+17}}.\end{aligned}$$

This implies

$$\begin{aligned}\psi^2(q^2) - q^4\psi^2(q^{18}) = &\sum_{k=-\infty}^{\infty} \frac{q^{9k}}{1+q^{18k+1}} + \sum_{k=-\infty}^{\infty} \frac{q^{9k+1}}{1+q^{18k+3}} + \sum_{k=-\infty}^{\infty} \frac{q^{9k+2}}{1+q^{18k+5}} \\ &+ \sum_{k=-\infty}^{\infty} \frac{q^{9k+3}}{1+q^{18k+7}}.\end{aligned} \tag{2.1}$$

The following is the famous Ramanujan's ${}_1\psi_1$ summation formula [5, p. 34]:

$$\sum_{k=-\infty}^{\infty} \frac{(a;q)_k}{(b;q)_k} z^k = \frac{(az;q)_\infty (q/az;q)_\infty (q;q)_\infty (b/a;q)_\infty}{(z;q)_\infty (b/az;q)_\infty (b;q)_\infty (q/a;q)_\infty}, \quad |b/a| < |z| < 1. \tag{2.2}$$

Setting $b = aq$ and $z = q^9$ in the above equation and then replacing q by q^{18}, we obtain

$$\sum_{k=-\infty}^{\infty} \frac{q^{9k}}{1-aq^{18k}} = \frac{f(-aq^9, -q^9/a)(q^{18};q^{18})_\infty^3}{f(-q^9,-q^9) f(-a,-q^{18}/a)}. \tag{2.3}$$

Which implies

$$\sum_{k=-\infty}^{\infty} \frac{q^{9k}}{1-aq^{18k}} = \psi^2(q^9) \frac{f(-aq^9, -q^9/a)}{f(-a,-q^{18}/a)}. \tag{2.4}$$

Employing (2.4) in (2.1) with $a = -q, -q^3, -q^5$, and $-q^7$, we obtain the required result. □

Theorem 2.6. *The following identity holds:*

$$P_{18,8,1}(n)-P_{18,6,3}(n-1)+P_{18,4,5}(n-2)-P_{18,2,7}(n-3)$$
$$=d_{1,4}(2n+1)-d_{3,4}(2n+1)-d_{1,4}\left(\frac{2n+1}{9}\right)+d_{3,4}\left(\frac{2n+1}{9}\right).$$

Proof. From previous theorem, we have

$$\left[\frac{f(q^8,q^{10})}{f(q,q^{17})}+q\frac{f(q^6,q^{12})}{f(q^3,q^{15})}+q^2\frac{f(q^4,q^{14})}{f(q^5,q^{13})}+q^3\frac{f(q^2,q^{16})}{f(q^7,q^{11})}\right]\psi^2(q^9)=\psi^2(q^2)-q^4\psi^2(q^{18}).$$

Changing q to $-q$ in the above equation, we have

$$\left[\frac{f(q^8,q^{10})}{f(-q,-q^{17})}-q\frac{f(q^6,q^{12})}{f(-q^3,-q^{15})}+q^2\frac{f(q^4,q^{14})}{f(-q^5,-q^{13})}-q^3\frac{f(q^2,q^{16})}{f(-q^7,-q^{11})}\right]\psi^2(-q^9)$$
$$=\psi^2(q^2)-q^4\psi^2(q^{18}).$$

Employing (1.1) and Lemma 2.1 in the above, we obtain

$$\sum_{n=0}^{\infty}\{P_{18,8,1}(n)-P_{18,6,3}(n-1)+P_{18,4,5}(n-2)-P_{18,2,7}(n-3)\}q^n$$
$$=\sum_{n=0}^{\infty}\left\{d_{1,4}(2n+1)-d_{3,4}(2n+1)-d_{1,4}\left(\frac{2n+1}{9}\right)+d_{3,4}\left(\frac{2n+1}{9}\right)\right\}q^n.$$

By comparing the coefficient of q^n in the above equation, we obtain the required result. □

Example: For $n=14$, we have

$P^e_{18,8,1}(14)$	$P^o_{18,8,1}(14)$	$P_{18,8,1}(14)$
1+1+1+1+10, 1+1+1+1+1+1+8, 1+1+1+1+1+1+1+1+1 +1+1+1+1+1+1.	$1+1+1+1+1+9_r$ $1+1+1+1+1+9_g$,	①

(Table-7)

$P^e_{18,6,3}(13)$	$P^o_{18,6,3}(13)$	$P_{18,6,3}(13)$
–	–	⓪

(Table-8)

$P^e_{18,4,5}(12)$	$P^o_{18,4,5}(12)$	$P_{18,4,5}(12)$
$-$	$-$	(0)

(Table-9)

$P^e_{18,2,7}(11)$	$P^o_{18,2,7}(11)$	$P_{18,2,7}(11)$
11	$2+9_r$, $2+9_g$.	(-1)

(Table-10)

One can easily see that

$$d_{1,4}(29)=2, d_{3,4}(29)=0, d_{1,4}\left(\frac{29}{9}\right)=0 \text{ and } d_{3,4}\left(\frac{29}{9}\right)=0.$$

From Table-7,8,9,10 and the above, we have

$$P_{18,8,1}(14)-P_{18,6,3}(13)+P_{18,4,5}(12)-P_{18,2,7}(11)$$
$$=d_{1,4}(29)-d_{3,4}(29)-d_{1,4}\left(\frac{29}{9}\right)+d_{3,4}\left(\frac{29}{9}\right)=2.$$

This verifies the theorem for $n=14$.

Theorem 2.7. *The following identity holds:*

$$P_{22,10,1}(n)-P_{22,8,3}(n-1)+P_{22,6,5}(n-2)-P_{22,4,7}(n-3)+P_{22,2,9}(n-4)$$
$$=d_{1,4}(2n+1)-d_{3,4}(2n+1)+d_{1,4}\left(\frac{2n+1}{11}\right)-d_{3,4}\left(\frac{2n+1}{11}\right).$$

Proof. From [8], we have

$$\left[\frac{f(q^{10},q^{12})}{f(q,q^{21})}+q\frac{f(q^8,q^{14})}{f(q^3,q^{19})}+q^2\frac{f(q^6,q^{16})}{f(q^5,q^{17})}+q^3\frac{f(q^4,q^{18})}{f(q^7,q^{15})}+q^4\frac{f(q^2,q^{20})}{f(q^9,q^{13})}\right]\psi^2(q^{11})$$
$$=\psi^2(q^2)-q^5\psi^2(q^{22}).$$

By changing q to $-q$ in the above, we have

$$\left[\frac{f(q^{10},q^{12})}{f(-q,-q^{21})}-q\frac{f(q^8,q^{14})}{f(-q^3,-q^{19})}+q^2\frac{f(q^6,q^{16})}{f(-q^5,-q^{17})}-q^3\frac{f(q^4,q^{18})}{f(-q^7,-q^{15})}+q^4\frac{f(q^2,q^{20})}{f(-q^9,-q^{13})}\right]$$
$$\times\psi^2(-q^{11})=\psi^2(q^2)+q^5\psi^2(q^{22}).$$

Employing (1.1) and Lemma 2.1 in the above, we obtain

$$\sum_{n=0}^{\infty}\{P_{22,10,1}(n)-P_{22,8,3}(n-1)+P_{22,6,5}(n-2)-P_{22,4,7}(n-3)+P_{22,2,9}(n-4)\}\,q^n$$
$$=\sum_{n=0}^{\infty}\left\{d_{1,4}(2n+1)-d_{3,4}(2n+1)+d_{1,4}\left(\frac{2n+1}{11}\right)-d_{3,4}\left(\frac{2n+1}{11}\right)\right\}q^n.$$

By comparing the coefficient of q^n in the above equation, we obtain the required result.

□

Example: For $n = 16$, we have

$P^e_{22,10,1}(16)$	$P^o_{22,10,1}(16)$	$P_{22,10,1}(16)$
12+1+1+1+1+1, 10+1+1+1+1+1+1+1+1, 1+1+1+1+1+1+1+1+1+1+1+1 +1+1+1+1.	11_r+1+1+1+1+1+1 , 11_g+1+1+1+1+1+1,	①

(Table-11)

$P^e_{22,8,3}(15)$	$P^o_{22,8,3}(15)$	$P_{22,8,3}(15)$
3+3+3+3	–	①

(Table-12)

$P^e_{22,6,5}(14)$	$P^o_{22,6,5}(14)$	$P_{22,6,5}(14)$
–	–	⓪

(Table-13)

$P^e_{22,4,7}(13)$	$P^o_{22,4,7}(13)$	$P_{22,4,7}(13)$
–	–	⓪

(Table-14)

$P^e_{22,2,9}(12)$	$P^o_{22,2,9}(12)$	$P_{22,2,9}(12)$
–	–	⓪

(Table-15)

One can easily see that

$$d_{1,4}(33) = 2, d_{3,4}(33) = 2, d_{1,4}\left(\frac{33}{11}\right) = 1 \text{ and } d_{3,4}\left(\frac{33}{11}\right) = 1.$$

From Table-11,12,13,14,15 and the above, we have

$$P_{22,10,1}(16) - P_{22,8,3}(15) + P_{22,6,5}(14) - P_{22,4,7}(13) + P_{22,2,9}(12)$$
$$= d_{1,4}(33) - d_{3,4}(33) + d_{1,4}\left(\frac{33}{11}\right) - d_{3,4}\left(\frac{33}{11}\right) = 0.$$

This verifies the theorem for $n = 16$.

Theorem 2.8. *The following identity holds:*

$$\psi^2(q^2) - q^6\psi^2(q^{26}) = \psi^2(q^{13})\times$$
$$\left[\frac{f(q^{12},q^{14})}{f(q,q^{25})} + q\frac{f(q^{10},q^{16})}{f(q^3,q^{23})} + q^2\frac{f(q^8,q^{18})}{f(q^5,q^{21})} + q^3\frac{f(q^6,q^{20})}{f(q^7,q^{19})} + q^4\frac{f(q^4,q^{22})}{f(q^9,q^{17})} + q^5\frac{f(q^2,q^{24})}{f(q^{11},q^{15})}\right].$$

Proof. In [9, p. 216], Ramanujan recorded following identity:

$$\psi^2(q^2) = \sum_{k=0}^{\infty} \frac{q^k}{1+q^{2k+1}}.$$

Which implies

$$\begin{aligned}\psi^2(q^2) &= \sum_{k=0}^{\infty}\frac{q^{13k}}{1+q^{26k+1}} + \sum_{k=0}^{\infty}\frac{q^{13k+1}}{1+q^{26k+3}} + \sum_{k=0}^{\infty}\frac{q^{13k+2}}{1+q^{26k+5}} + \sum_{k=0}^{\infty}\frac{q^{13k+3}}{1+q^{26k+7}}\\ &+ \sum_{k=0}^{\infty}\frac{q^{13k+4}}{1+q^{26k+9}} + \sum_{k=0}^{\infty}\frac{q^{13k+5}}{1+q^{26k+11}} + \sum_{k=0}^{\infty}\frac{q^{13k+6}}{1+q^{26k+13}} + \sum_{k=0}^{\infty}\frac{q^{13k+7}}{1+q^{26k+15}}\\ &+ \sum_{k=0}^{\infty}\frac{q^{13k+8}}{1+q^{26k+17}} + \sum_{k=0}^{\infty}\frac{q^{13k+9}}{1+q^{26k+19}} + \sum_{k=0}^{\infty}\frac{q^{13k+10}}{1+q^{26k+21}} + \sum_{k=0}^{\infty}\frac{q^{13k+11}}{1+q^{26k+23}}\\ &. + \sum_{k=0}^{\infty}\frac{q^{13k+12}}{1+q^{26k+25}}.\end{aligned}$$

This implies

$$\begin{aligned}\psi^2(q^2) - q^6\psi^2(q^{26}) &= \sum_{k=-\infty}^{\infty}\frac{q^{13k}}{1+q^{26k+1}} + \sum_{k=-\infty}^{\infty}\frac{q^{13k+1}}{1+q^{26k+3}} + \sum_{k=-\infty}^{\infty}\frac{q^{13k+2}}{1+q^{26k+5}}\\ &+ \sum_{k=-\infty}^{\infty}\frac{q^{13k+3}}{1+q^{26k+7}} + \sum_{k=-\infty}^{\infty}\frac{q^{13k+4}}{1+q^{26k+9}} + \sum_{k=-\infty}^{\infty}\frac{q^{13k+5}}{1+q^{26k+11}}\end{aligned} \tag{2.5}$$

Setting $b = aq$ and $z = q^{13}$ in the Ramanujan's ${}_1\psi_1$ summation formula (2.2) and then replacing q by q^{26}, we obtain

$$\sum_{k=-\infty}^{\infty}\frac{q^{13k}}{1-aq^{26k}} = \frac{f(-aq^{13},-q^{13}/a)(q^{26};q^{26})_\infty^3}{f(-q^{13},-q^{13})f(-a,-q^{26}/a)}. \tag{2.6}$$

Which implies

$$\sum_{k=-\infty}^{\infty}\frac{q^{13k}}{1-aq^{26k}} = \psi^2(q^{13})\frac{f(-aq^{13},-q^{13}/a)}{f(-a,-q^{26}/a)}. \tag{2.7}$$

Employing (2.7) in (2.5) with $a = -q, -q^3, -q^5, -q^7, -q^9$, and $-q^{11}$, we obtain the required result. □

Theorem 2.9. *The following identity holds:*

$$P_{26,12,1}(n) - P_{26,10,3}(n-1) + P_{26,8,5}(n-2) - P_{26,6,7}(n-3) + P_{26,4,9}(n-4) - P_{26,2,11}(n-5) = d_{1,4}(2n+1) - d_{3,4}(2n+1) - d_{1,4}\left(\frac{2n+1}{13}\right) + d_{3,4}\left(\frac{2n+1}{13}\right).$$

Proof. From previous theorem, we have

$$\left[\frac{f(q^{12},q^{14})}{f(q,q^{25})} + q\frac{f(q^{10},q^{16})}{f(q^3,q^{23})} + q^2\frac{f(q^8,q^{18})}{f(q^5,q^{21})} + q^3\frac{f(q^6,q^{20})}{f(q^7,q^{19})} + q^4\frac{f(q^4,q^{22})}{f(q^9,q^{17})} + q^5\frac{f(q^2,q^{24})}{f(q^{11},q^{15})}\right] \times \psi^2(q^{13}) = \psi^2(q^2) - q^4\psi^2(q^{26})$$

Changing q to $-q$ in the above equation, we have

$$\left[\frac{f(q^{12},q^{14})}{f(-q,-q^{25})} - q\frac{f(q^{10},q^{16})}{f(-q^3,-q^{23})} + q^2\frac{f(q^8,q^{18})}{f(-q^5,-q^{21})} - q^3\frac{f(q^6,q^{20})}{f(-q^7,-q^{19})} + q^4\frac{f(q^4,q^{22})}{f(-q^9,-q^{17})} - q^5\frac{f(q^2,q^{24})}{f(-q^{11},-q^{15})}\right]\psi^2(-q^{13}) = \psi^2(q^2) - q^4\psi^2(q^{26}).$$

Employing (1.1) and Lemma 2.1 in the above, we obtain

$$\sum_{n=0}^{\infty}\Big\{P_{26,12,1}(n) - P_{26,10,3}(n-1) + P_{26,8,5}(n-2) - P_{26,6,7}(n-3) + P_{26,4,9}(n-4) - P_{26,2,11}(n-5)\Big\}q^n = \sum_{n=0}^{\infty}\left\{d_{1,4}(2n+1) - d_{3,4}(2n+1) - d_{1,4}\left(\frac{2n+1}{13}\right) + d_{3,4}\left(\frac{2n+1}{13}\right)\right\}q^n.$$

By comparing the coefficient of q^n in the above equation, we obtain the required result. □

Example: For $n = 18$, we have

$P^e_{26,12,1}(18)$	$P^o_{26,12,1}(18)$	$P_{26,12,1}(18)$
12+1+1+1+1+1+1+1, 14+1+1+1+1+1, 1+1+1+1+1+1+1+1+1+1 +1+1+1+1+1+1+1+1+1.	13_r+1+1+1+1+1+1, 13_g+1+1+1+1+1+1.	①

(Table-16)

$P^e_{26,10,3}(17)$	$P^o_{26,10,3}(17)$	$P_{26,10,3}(17)$
$-$	$-$	⓪

(Table-17)

$P^e_{26,8,5}(16)$	$P^o_{26,8,5}(16)$	$P_{26,8,5}(16)$
$-$	$-$	⓪

(Table-18)

$P^e_{26,6,7}(15)$	$P^o_{26,6,7}(15)$	$P_{26,6,7}(15)$
$-$	$-$	⓪

(Table-19)

$P^e_{26,4,9}(14)$	$P^o_{26,4,9}(14)$	$P_{26,4,9}(14)$
$-$	$-$	⓪

(Table-20)

$P^e_{26,2,11}(13)$	$P^o_{26,2,11}(13)$	$P_{26,2,11}(13)$
2+11	13_r 13_g	(-1)

(Table-21)

One can easily see that

$$d_{1,4}(37) = 2, d_{3,4}(37) = 0, d_{1,4}\left(\frac{37}{13}\right) = 0 \text{ and } d_{3,4}\left(\frac{37}{13}\right) = 0.$$

From Table-16, 17, 18, 19, 20, 21 and the above, we have

$$\begin{aligned} P_{26,12,1}(18) - P_{26,10,3}(17) + P_{26,8,5}(16) \\ - P_{26,6,7}(15) + P_{26,4,9}(14) - P_{26,2,11}(13) \\ = d_{1,4}(37) - d_{3,4}(37) - d_{1,4}\left(\frac{37}{13}\right) + d_{3,4}\left(\frac{37}{13}\right) = 2. \end{aligned}$$

This verifies the theorem for $n = 18$.

Theorem 2.10. *The following identity holds:*

$$\begin{aligned} &\psi^2(q^2) - q^7\psi^2(q^{30}) \\ &= \psi^2(q^{15})\left[\frac{f(q^{14}, q^{16})}{f(q, q^{29})} + q\frac{f(q^{12}, q^{18})}{f(q^3, q^{27})} + q^2\frac{f(q^{10}, q^{20})}{f(q^5, q^{25})}\right. \\ &\left. + q^3\frac{f(q^8, q^{22})}{f(q^7, q^{23})} + q^4\frac{f(q^6, q^{24})}{f(q^9, q^{21})} + q^5\frac{f(q^4, q^{26})}{f(q^{11}, q^{19})} + q^6\frac{f(q^2, q^{28})}{f(q^{13}, q^{17})}\right]. \end{aligned}$$

Proof. In [9, p. 216], Ramanujan recorded following identity:

$$\psi^2(q^2) = \sum_{k=0}^{\infty} \frac{q^k}{1+q^{2k+1}}.$$

Which implies

$$\begin{aligned}\psi^2(q^2) &= \sum_{k=0}^{\infty} \frac{q^{15k}}{1+q^{30k+1}} + \sum_{k=0}^{\infty} \frac{q^{15k+1}}{1+q^{30k+3}} + \sum_{k=0}^{\infty} \frac{q^{15k+2}}{1+q^{30k+5}} + \sum_{k=0}^{\infty} \frac{q^{15k+3}}{1+q^{30k+7}} \\ &+ \sum_{k=0}^{\infty} \frac{q^{15k+4}}{1+q^{30k+9}} + \sum_{k=0}^{\infty} \frac{q^{15k+5}}{1+q^{30k+11}} + \sum_{k=0}^{\infty} \frac{q^{15k+6}}{1+q^{30k+13}} + \sum_{k=0}^{\infty} \frac{q^{15k+7}}{1+q^{30k+15}} \\ &+ \sum_{k=0}^{\infty} \frac{q^{15k+8}}{1+q^{30k+17}} + \sum_{k=0}^{\infty} \frac{q^{15k+9}}{1+q^{30k+19}} + \sum_{k=0}^{\infty} \frac{q^{15k+10}}{1+q^{30k+21}} + \sum_{k=0}^{\infty} \frac{q^{15k+11}}{1+q^{30k+23}} \\ &. + \sum_{k=0}^{\infty} \frac{q^{15k+12}}{1+q^{30k+25}} + \sum_{k=0}^{\infty} \frac{q^{15k+13}}{1+q^{30k+27}} + \sum_{k=0}^{\infty} \frac{q^{15k+14}}{1+q^{30k+29}}.\end{aligned}$$

This implies

$$\begin{aligned}\psi^2(q^2) - q^7\psi^2(q^{30}) &= \sum_{k=-\infty}^{\infty} \frac{q^{15k}}{1+q^{30k+1}} + \sum_{k=-\infty}^{\infty} \frac{q^{15k+1}}{1+q^{30k+3}} + \sum_{k=-\infty}^{\infty} \frac{q^{15k+2}}{1+q^{30k+5}} \\ &+ \sum_{k=-\infty}^{\infty} \frac{q^{15k+3}}{1+q^{30k+7}} + \sum_{k=-\infty}^{\infty} \frac{q^{15k+4}}{1+q^{30k+9}} + \sum_{k=-\infty}^{\infty} \frac{q^{15k+5}}{1+q^{30k+11}} \\ &+ \sum_{k=-\infty}^{\infty} \frac{q^{15k+6}}{1+q^{30k+13}}.\end{aligned} \tag{2.8}$$

Setting $b = aq$ and $z = q^{15}$ in the Ramanujan's ${}_1\psi_1$ summation formula (2.2) and then replacing q by q^{30}, we obtain

$$\sum_{k=-\infty}^{\infty} \frac{q^{15k}}{1-aq^{30k}} = \frac{f(-aq^{15}, -q^{15}/a)(q^{30}; q^{30})_\infty^3}{f(-q^{15}, -q^{15})f(-a, -q^{30}/a)}. \tag{2.9}$$

Which implies

$$\sum_{k=-\infty}^{\infty} \frac{q^{15k}}{1-aq^{30k}} = \psi^2(q^{15}) \frac{f(-aq^{15}, -q^{15}/a)}{f(-a, -q^{30}/a)}. \tag{2.10}$$

Employing (2.10) in (2.8) with $a = -q, -q^3, -q^5, -q^7, -q^9, -q^{11}$ and $-q^{13}$, we obtain the required result.

□

Theorem 2.11. *The following identity holds:*

$$\begin{aligned}P_{30,14,1}(n) - P_{30,12,3}(n-1) + P_{30,10,5}(n-2) \\ - P_{30,8,7}(n-3) + P_{30,6,9}(n-4) - P_{30,4,11}(n-5) + P_{30,2,13}(n-6) \\ = d_{1,4}(2n+1) - d_{3,4}(2n+1) + d_{1,4}\left(\frac{2n+1}{15}\right) - d_{3,4}\left(\frac{2n+1}{15}\right).\end{aligned}$$

Proof. From previous theorem, we have

$$\begin{aligned}\Bigg[\frac{f(q^{14},q^{16})}{f(q,q^{29})} + q\frac{f(q^{12},q^{18})}{f(q^3,q^{27})} + q^2\frac{f(q^{10},q^{20})}{f(q^5,q^{25})} \\ + q^3\frac{f(q^8,q^{22})}{f(q^7,q^{23})} + q^4\frac{f(q^6,q^{24})}{f(q^9,q^{21})} + q^5\frac{f(q^4,q^{26})}{f(q^{11},q^{19})} + q^6\frac{f(q^2,q^{28})}{f(q^{13},q^{17})}\Bigg]\psi^2(q^{15}) \\ = \psi^2(q^2) - q^7\psi^2(q^{30})\end{aligned}$$

By changing q to $-q$ in the above equation, we have

$$\begin{aligned}\Bigg[\frac{f(q^{14},q^{16})}{f(-q,-q^{29})} + q\frac{f(q^{12},q^{18})}{f(-q^3,-q^{27})} + q^2\frac{f(q^{10},q^{20})}{f(-q^5,-q^{25})} \\ - q^3\frac{f(q^8,q^{22})}{f(-q^7,-q^{23})} + q^4\frac{f(q^6,q^{24})}{f(-q^9,-q^{21})} - q^5\frac{f(q^4,q^{26})}{f(-q^{11},-q^{19})} + q^6\frac{f(q^2,q^{28})}{f(-q^{13},-q^{17})}\Bigg] \\ \times\psi^2(-q^{15}) = \psi^2(q^2) + q^7\psi^2(q^{30})\end{aligned}$$

Employing (1.1) and Lemma 2.1 in the above, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\Big\{P_{30,14,1}(n) - P_{30,12,3}(n-1) + P_{30,10,5}(n-2) - P_{30,8,7}(n-3) + P_{30,6,9}(n-4) \\ - P_{30,4,11}(n-5) + P_{30,2,13}(n-6)\Big\}q^n \\ = \sum_{n=0}^{\infty}\left\{d_{1,4}(2n+1) - d_{3,4}(2n+1) + d_{1,4}\left(\frac{2n+1}{15}\right) - d_{3,4}\left(\frac{2n+1}{15}\right)\right\}q^n.\end{aligned}$$

By comparing the coefficient of q^n in the above equation, we obtain the required result. □

Example: For $n = 20$, we have

$P^{e}_{30,14,1}(20)$	$P^{o}_{30,14,1}(20)$	$P_{30,14,1}(20)$
16+1+1+1+1, 14+1+1+1+1+1+1+1, 1+1+1+1+1+1+1+1+1+1+1+1 +1+1+1+1+1+1+1+1+1 .	15_r+1+1+1+1+1+1 15_g+1+1+1+1+1	①

(Table-22)

$P^e_{30,12,3}(19)$	$P^o_{30,12,3}(19)$	$P_{30,12,3}(19)$
–	–	(0)

(Table-23)

$P^e_{30,10,5}(18)$	$P^o_{30,10,5}(18)$	$P_{30,10,5}(18)$
–	–	(0)

(Table-24)

$P^e_{30,8,7}(17)$	$P^o_{30,8,7}(17)$	$P_{30,8,7}(17)$
–	–	(0)

(Table-25)

$P^e_{30,6,9}(16)$	$P^o_{30,6,9}(16)$	$P_{30,6,9}(16)$
–	–	(0)

(Table-26)

$P^e_{30,4,11}(15)$	$P^o_{30,4,9}(15)$	$P_{30,4,9}(11)$
4+11	15_r , 15_g.	(-1)

(Table-27)

$P^e_{30,2,13}(14)$	$P^o_{30,2,13}(14)$	$P_{30,2,13}(14)$
–	–	(0)

(Table-28)

One can easily see that

$$d_{1,4}(41) = 2, d_{3,4}(41) = 0, d_{1,4}\left(\frac{41}{15}\right) = 0 \text{ and } d_{3,4}\left(\frac{41}{15}\right) = 0.$$

From Table-22, 23, 24, 25, 26, 27, 28 and the above, we have

$$\begin{aligned} P_{30,14,1}(20) - P_{30,12,3}(19) + P_{30,10,5}(18) - P_{30,8,7}(17) \\ + P_{30,6,9}(16) - P_{30,4,11}(15) + P_{30,2,13}(14) \\ = d_{1,4}(41) - d_{3,4}(41) + d_{1,4}\left(\frac{41}{15}\right) - d_{3,4}\left(\frac{41}{15}\right) = 2. \end{aligned}$$

This verifies the theorem for $n = 20$.

Acknowledgment: The first author is supported by grant No.09/0119(15441/2022-EMR-I) (Dated:11.09.2024) by the funding agency CSIR, INDIA, under CSIR-SRF. The author is grateful to the funding agency.

References

[1] C. Adiga and K. R. Vasuki, *On sums of triangular numbers*, Math. Student, **70**(1-4), (2001) 185-190.

[2] Z. Ahmed, N. D. Baruah, and M. G. Dastidar, New congruences modulo 5 for the number of 2-color partitions, *J. Number Theory*, **157**(2015), 184–198.

[3] N. D. Baruah and B. K. Sarmah, Identities and congruences for the general partition and Ramanujan's tau functions, *Indian J. Pure Appl. Math.*, **44**(5) (2013), 643–671.

[4] N. D. Baruah and R. Barman, Certain theta- function identities and modular equation equations of degree 3, *Indian J. Math.*, **48**(2001), 185-190.

[5] B. C. Berndt, *Ramanujan notebooks, Part III*, Springer-Verlag, New York, 1991.

[6] H. C. Chan, Ramanujan's cubic continued fraction and an analog of his "most beautiful identity", *Int. J. Number Theory*, **3**(2010), 673-680.

[7] M. D. Hirschhorn, *The Power of q - A Personal Journey*, Springer-2017.

[8] Mahadevaswamy, *Ramanujan's modular equations of degree seven and eleven*, Ph.D thesis submitted to the University of Mysore, September -2020.

[9] S. Ramanujan, *Notebooks (2 volumes)*, Tata institute of fundamental Research, Bombay, 1957.

[10] N. Saikia and C. Boruah, Arithmetic properties of partition 5 and 7 tuples with odd parts distinct, *Palestine Journal of Mathematics*, **7**(1)(2018), 141–150.

[11] K. R Vasuki and R. G. Veresha, On Ramanujan's modular equation of degree 7, *J. Number Theory*, **153**(2015), 304-308.