[Volume 4 issue 11 November 2016]
PageNo.1670-1684 ISSN : 2320-7167

INTERNATIONAL JOURNAL OF MATHEMATICS AND COMPUTER RESEARCH

An $M/G/1$ queue with K phases of vacation with state dependent arrival rate

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Abstract

We consider an $M/G/1$ queue with K-phase of optional vacation. The vacation policy is after completion of service if there are no customers in the system, the server take vacation consisting of K-phases. After completing the K^{th} phases of vacation, the server enter into the service station independent of the number of customers in the system. The vacation periods follows general distribution. For this model the supplementary variable technique has been applied to obtain the probability generating functions of number of customers in the queue at different server states. Some particular models are obtained and a numerical study is also carried out.

Keywords: Vacation queue, Supplementary variable, Probability generating function, Performance measures.

AMS Subject classification: 60K25, 90B22 and 60K30

1. Introduction

Many real life queueing situations encountered in day-to-day as well as industrial scenario,the vacation models are the models which are the best fit. In queueing theory, the vacation perior can be considered as the period during which the server is not available as he/she has left, when the system becomes empty. In the $M/G/1$ queueing system, the concept of vacation had been first studied by Keilson and Servi (1987), they introduced the concept of modified service time which has a main role in the system with general service and vacation times. In many real life situations such as production system, bank services, computer and communication networks, we have the concept of vacation. Also for overhauling or maintanance of a system the server(system) may go to vacation.

The classic $M/G/1$ queue with various vacation policies have been well studied (see Doshi (1986, 1990), Gross and Harris (1998), Ke (1986), Takagi (1991)). The two monographes of Tian and Zhang (2006) and Takagi (1991) collected the research results of the $M/G/1$ vacation queues. Chen et al.(2009) considered a $GI/M/1$

queue with phase type of working vacations and vacation interruption where the vacation time follows a phase type distribution. Tian and Zhang (2001) treated the Geo/G/1 system with variant policies. In this system, they assumed that after serving all customers in the system, the server take a random maximum number of vacation before returning to the service station. Tian and Zhang (2002, 2003) discussed the discrete time $GI/Geo/1$ queue with server vacations and the $GI/M/1$ queue with PH vacations or setup times, respectively. Ke and Chu (2006) analysed the $M^{[X]}/G/1$ queue with modified vacation policy by stochastic decomposition property and Ke (2007) used supplementary variable technique to study an $M^{[X]}/G/1$ queue with balking under a variant vacation policy.

Ke (2003) made the contribution to the control policy of $M/G/1$ queue with server vacations, startup and breakdowns. He obtained the system characteristics of the model and obtained the total expected cost function per unit time to determine the optimal threshold of N policies at a minimum cost. Ke et al.(2010), studied the operating characteristics of an $M^{[X]}/G/1$ queueing system with N-policy and at most J vacations. In this model they assumed that the server takes at most J vacations repeatedly until atleast N customers returning from a vacation are waiting in the queue.

In this paper we consider an $M/G/1$ queue with K-phase of optional vacation. The vacation policy is after completion of service if there are no customers in the system, the server take vacation consisting of K-phases. After completing the K^{th} phases of vacation, the server enter into the service station independent of the number of customers in the system. The vacation periods follows general distribution. The mathematical description and analysis of this model is given in section 2. In section 3, we derive some operating characteristics of the model analysed in section 2. Section 4 deals with some particular models and section 5 presents some numerical results related to the model analyzed in this paper. The last section gives a conclusion.

2. The Mathematical Model and Analysis

In a single sever queueing system, the customer arrival follows a Poisson process with parameter λ and the service time is generally distributed with distribution function $B(x)$ whose Laplace stieltjes transform(LST) is $B^*(s)$. After completion of a service if there are no customers in the system, the server takes vacation consisting of K-phases with each phase respectively has time duration $V_1, V_2, V_3, ..., V_K$ and are independent random variables with distribution functions $V_i(x), i = 1, 2, ..., K$. After completing the K^{th} phase of vacation the server enter into the system independent of the number of customers in the system. That is if there are customers in the queue, the server starts service for the customer in the head of the queue other wise, the server waits ideal for new arrival. The arriving customers waiting in a queue of infinite capacity, if the service is not immediate due to server is busy or server is on vacation. For service, the first come first served rule is used.

Now the modified vacation period is

$$V = V_1 + V_2 + V_3 + ... + V_K \tag{1}$$

and the LST of V is

$$V^*(s) = \prod_{i=1}^{K} V_i^*(s) \tag{2}$$

whose mean is

$$E(V) = \sum_{j=1}^{K} E(V_j) \tag{3}$$

Now, it is assumed that the arrival rates are state dependent, ie, the arrival rate λ is defined as,

$$\lambda = \begin{cases} \lambda_0, & \text{if the arrival is during idle period} \\ \lambda_1, & \text{if the arrival is during service period} \\ \lambda_2, & \text{if the arrival is during vacation period} \end{cases}$$

The time required by a customer to complete a service cycle is $B_c = B + V$ where V is defined in equation (1). Now the LST of B_c is $B_c^*(s) = B^*(s)V^*(s)$, where $V^*(s)$ is given in equation (2) and $E(B_c) = E(B) + E(V)$, where $E(V)$ is given in equation (3).

Asssume that $B(0) = V_i(0) = 0, B(\infty) = V_i(\infty) = 1; i = 1, 2, ..., K$. The elapsed service time of the customer in service at time t is denoted by $\xi_0(t)$ and the elapsed vacation time of phase i is denoted by $\eta_i(t)$.

Let $Y(t)$ be the state of the server at time t and is defined as,

$$Y(t) = \begin{cases} 0, & \text{if the server is idle at time t} \\ i, & \text{if the server is at } i^{th} \text{ phase of vacation at time t; i=1, 2,..., K} \\ K+1, & \text{if the server is busy at time t} \end{cases}$$

Let the random variable $L(t)$ is defined as

$$L(t) = \begin{cases} 0, & \text{if } Y(t) = 0 \\ \eta_i(t), & \text{if } Y(t) = i; i = 1, 2, .., K \\ \xi_0(t), & \text{if } Y(t) = K+1 \end{cases}$$

and let the random variable N(t), is the number of customers in the queue. Now the following probabilities have been defined for the analysis:

$$Q(t) = Pr\{N(t) = 0, L(t) = 0\}$$
$$P_n(t, x)dx = Pr\{N(t) = n, Y(t) = K+1, x < \xi_0(t) \leq x + dx\}, n \geq 0$$
$$R_{i,n}(t, x)dx = Pr\{N(t) = n, Y(t) = i, x < \eta_i(t) \leq x + dx\}, n \geq 0, i = 1, 2, ..., K$$

where $\{(N(t), Y(t)), t \geq 0\}$ is a bivariate Markov process with state space $S = \{(0,0)\} \cup \{(K+1, j)\} \cup \{(i,j)\}, i = 1, 2, ..., K; j \geq 0$. The hazard rate function of B is $\mu(x)dx = \dfrac{dB(x)}{1 - B(x)}$ is the conditional probability of completion of a service during the time interval $(x, x + dx]$ given that the elpased service time is x. The similar quantity for V_i is $\eta_i(x)dx = \dfrac{dV_i(x)}{1 - V_i(x)}, i = 1, 2, ..., K$. In steady state,the corresponding probabilities are $Q = \lim_{t\to\infty} Q(t)$, $P_n(x) = \lim_{t\to\infty} P_n(t, x)$ and $R_{i,n}(x) = \lim_{t\to\infty} R_{i,n}(t, x)$.

The model is governed by the following differential difference equations for $x > 0$

$$\frac{d}{dx}P_0(x) + (\lambda_1 + \mu(x))P_0(x) = 0 \tag{4}$$

$$\frac{d}{dx}P_n(x) + (\lambda_1 + \mu(x))P_n(x) = \lambda_1 P_{n-1}(x), n \geq 0 \tag{5}$$

$$\frac{d}{dx}R_{i,0}(x) + (\lambda_2 + \eta_i(x))R_{i,0}(x) = 0, i = 1, 2, ..., K \tag{6}$$

$$\frac{d}{dx}R_{i,n}(x) + (\lambda_2 + \eta_i(x))R_{i,n}(x) = \lambda_2 R_{i,n-1}(x), i = 1, 2, ..., K \tag{7}$$

$$\lambda_0 Q = \int_0^\infty \eta_K(x)R_{K,0}(x)dx \tag{8}$$

The boundary conditions are

$$P_0(0) = \lambda_0 Q + \int_0^\infty \eta_K(x)R_{K,1}(x)dx + \int_0^\infty P_1(x)\mu(x)dx \tag{9}$$

$$P_n(0) = \int_0^\infty \eta_K(x)R_{K,n+1}(x)dx + \int_0^\infty P_{n+1}(x)\mu(x)dx, n \geq 1 \tag{10}$$

$$R_{1,0}(0) = \int_0^\infty \mu(x)P_0(x)dx \tag{11}$$

$$R_{1,n}(0) = 0, n \geq 1 \tag{12}$$

$$R_{i,n}(0) = \int_0^\infty R_{i-1,n}(x)\eta_{i-1}(x)dx, i = 2, 3, ..., K; n = 0, 1, 2, ... \tag{13}$$

The normalizing condition is

$$Q + P(1) + \sum_{i=1}^{K} R_i(1) = 1$$

For the analysis, the following probability generating functions have been defined $P(x, z) = \sum_{n=0}^{\infty} z^n P_n(x)$ and $R_i(x, z) = \sum_{n=0}^{\infty} z^n R_{i,n}(x), i = 1, 2, \cdots K$.

From equation (4), we have

$$P_0(x) = P_0(0)(1 - B(x))e^{-\lambda_1 x} \tag{14}$$

Multiplying equation (5) by z^n, summing from 1 to ∞ and adding equation (4), we get

$$P(x,z) = P(0,z)(1-B(x))e^{-\lambda_1(1-z)x} \tag{15}$$

Multiplying equation (7) by z^n, summing from 1 to ∞ and adding equation (6), we get

$$R_i(x,z) = R_i(0,z)(1-V_i(x))e^{-\lambda_2(1-z)x}, i = 1,2,...,K \tag{16}$$

From equation (6), we get

$$R_{i,0}(x) = R_{i,0}(0)(1-V_i(x))e^{-\lambda_2 x}, i = 1,2,...,K \tag{17}$$

Multiplying equation (10) by z^n, summing from 1 to ∞ and adding equation (9) and multiply by z, we get

$$zP(0,z) = \int_0^\infty \eta_K(x)R_K(x,z)dx - \int_0^\infty \eta_K(x)R_{K,0}(x)dx + \int_0^\infty \mu(x)P(x,z)dx$$
$$- \int_0^\infty \mu(x)P_0(x)dx + z\lambda_0 Q \tag{18}$$

From equation (17), we have

$$\int_0^\infty R_{i,0}(x)\eta_i(x)dx = R_{i,0}(0)V_i^*(\lambda_2), i = 1,2,...,K \tag{19}$$

From equation (16), we get

$$\int_0^\infty R_i(x,z)\eta_i(x)dx = R_i(0,z)V_i^*(\lambda_2(1-z)), i = 1,2,...,K \tag{20}$$

From equation (14), we have

$$\int_0^\infty P_0(x)\mu(x)dx = P_0(0)B^*(\lambda_1) \tag{21}$$

From equation (15), we get

$$\int_0^\infty P(x,z)\mu(x)dx = P(0,z)B^*(\lambda_1(1-z)) \tag{22}$$

Using equations (19),(20),(21) and (22) in (18), we have

$$[z - B^*(R)]P(0,z) = z\lambda_0 Q + R_K(0,z)V_K^*(T) - R_{K,0}(0)V_K^*(\lambda_2) - P_0(0)B^*(\lambda_1) \tag{23}$$

where $R = \lambda_1(1-z)$ and $T = \lambda_2(1-z)$

Multiplying equation (12) by z^n, summing from $n = 1$ to ∞ and adding with equation (11), we get

$$R_1(0,z) = P_0(0)B^*(\lambda_1) \tag{24}$$

Multiplying equation (13) by z^n and summing from $n = 0$ to ∞, we get

$$R_i(0,z) = \prod_{l=1}^{i-1} V_l^*(T)B^*(\lambda_1)P_0(0), i = 2,3,...,K \tag{25}$$

Put $n = 0$ in equation (13), we get

$$R_{i,0}(0) = \prod_{l=1}^{i-1} V_l^*(\lambda_2) B^*(\lambda_1) P_0(0), i = 2, 3, ..., K \tag{26}$$

From equation (23), we get

$$[z - B^*(R)]P(0, z) = z\lambda_0 Q + R_K(0, z)V_K^*(T) - R_{K,0}(0)V_K^*(\lambda_2) - P_0(0)B^*(\lambda_1) \tag{27}$$

From equation (8), we get

$$\lambda_0 Q = V_K^*(\lambda_2) R_{K,0}(0)$$
$$R_{K,0}(0) = \frac{\lambda_0 Q}{V_K^*(\lambda_2)} \tag{28}$$

Substituting equation (28) in (27), we get

$$P(0, z) = \frac{B^*(\lambda_1)P_0(0)}{(z - B^*(R))}\Big\{(z - 1)\prod_{l=1}^{K} V_l^*(\lambda_2) + \prod_{l=1}^{K} V_l^*(T) - 1\Big\} \tag{29}$$

Now

$$\left.\begin{aligned} P(z) = \int_0^\infty P(x, z)dx = P(0, z)\frac{[1 - B^*(R)]}{R} \\ \text{and } \ R_i(z) = \int_0^\infty R_i(x, z)dx = R_i(0, z)\frac{[1 - V_i^*(T)]}{T}, i = 1, 2, ..., K. \end{aligned}\right\} \tag{30}$$

To find the unknown probability $P_0(0)$ we use the normalizing condition

$$Q + P(1) + \sum_{i=1}^{K} R_i(1) = 1,$$

we get

$$P_0(0) = \frac{\lambda_0(1 + \lambda_1 B^{*'}(0))}{B^*(\lambda_1)C_1} \tag{31}$$

where

$$C_1 = (1 + (\lambda_1 - \lambda_0)B^{*'}(0))\prod_{l=1}^{K} V_l^*(\lambda_2) - \lambda_0[1 + (\lambda_1 - \lambda_2)B^{*'}(0)]E(V)$$

and substituting equation (29) in (28),we get

$$Q = \frac{(1 + \lambda_1 B^{*'}(0))\prod_{l=1}^{K} V_l^*(\lambda_2)}{C_1} \tag{32}$$

Equations in (30), together with (24), (25), (28), (29), (31) and (32) gives the probability generating function of number of customers in the queue with server is busy and the server is on the i^{th} phase of vacation $(i = 1, 2, ..., K)$ respectively.

3. Some operating characteristics

In this section we derive the operating characteristics mean and variance number of customers in the queue when the server is busy and mean and variance number of customers in the queue when the server is on the $i^{th}(i = 1, 2, ..., K)$ phase of vacation.

Mean number of customers in the queue when the server is busy is

$$L_b = \frac{\lambda_0 C_2}{2(1 + \lambda_1 B^{*\prime}(0))C_1}$$

Variance of number of customers in the queue when the server is busy is

$$V_b = \frac{\lambda_0[2C_1C_3 - 3\lambda_0 C_2^2]}{12(1 + \lambda_1 B^{*\prime}(0))^2 C_1^2}$$

Mean number of customers in the queue when the server is on vacation is

$$L_v = \frac{\lambda_0\lambda_2(1 + \lambda_1 B^{*'}(0))E(V^2)}{2C_1}$$

Variance of number of customers in the queue when the server is on vacation is

$$V_v = \frac{\lambda_0\lambda_2(1 + \lambda_1 B^{*'}(0))}{12C_1^2}\Big\{6C_1E(V^2) - 3\lambda_0\lambda_2(1 + \lambda_1 B^{*'}(0))E(V^2)^2 + 4\lambda_2 C_1 E(V^3)\Big\}$$

where

$$E(V^2) = \sum_{i=1}^{K} E(V_i^2) + 2\sum_{n=1}^{K-1} E(V_n)\sum_{i=n+1}^{K} E(V_i)$$

$$E(V^3) = \sum_{i=1}^{K} E(V_i^3) + 3\Big[\sum_{n=1}^{K-1} E(V_n^2)\sum_{i=n+1}^{K} E(V_i) + \sum_{n=1}^{K-1} E(V_n)\sum_{i=n+1}^{K} E(V_i^2)\Big] + 6\sum_{m=1}^{K-2} E(V_m)\sum_{j=m+1}^{K-1} E(V_j)\sum_{n=j+1}^{K} E(V_n)$$

$$C_2 = \lambda_1 B^{*''^2}(0)\Big(\prod_{l=1}^{K} V_l^*(\lambda_2) + \lambda_2 E(V)\Big) - \lambda_2^2 B^{*'}(0)(1 + \lambda_1 B^{*'}(0))E(V^2)$$

$$C_3 = [3\lambda_1 B^{*''}(0)(1 + \lambda_1 B^{*'}(0) + \lambda_1^2 B^{*''}(0)) - 2\lambda_1^2(1 + \lambda_1 B^{*'}(0))B^{*'''}(0)] \times \Big(\prod_{l=1}^{K} V_l^*(\lambda_2) + \lambda_2 E(V)\Big) - 2\lambda_2^3 B^{*'}(0)(1 + \lambda_1 B^{*'}(0))^2 E(V^3) + 3\lambda_2^2(1 + \lambda_1 B^{*'}(0))(\lambda_1 B^{*''}(0) - (1 + \lambda_1 B^{*'}(0))B^{*'}(0))E(V^2)$$

4. Some particular cases

In this section, we present five particular cases by assuming particular form to the parameters and/or particular probability distribution to service time and/or vacation time.

Case 1: Now we take $\lambda_0 = \lambda_1 = \lambda_2 = \lambda$

$$Q = \frac{(1+\lambda B^{*'}(0)) \prod_{l=1}^{K} V_l^*(\lambda)}{\prod_{l=1}^{K} V_l^*(\lambda) + \lambda E(V)}$$

$$L_b = \frac{\lambda^2 D_1}{2(1+\lambda B^{*'}(0))(\prod_{l=1}^{K} V_l^*(\lambda) + \lambda E(V)}$$

$$V_b = \frac{\lambda^2[2(\prod_{l=1}^{K} V_l^*(\lambda) + \lambda E(V))D_2 - 3\lambda^2 D_1^2]}{12(1+\lambda B^{*'}(0))^2(\prod_{l=1}^{K} V_l^*(\lambda) + \lambda E(V))^2}$$

$$L_v = \frac{\lambda^2(1+\lambda B^{*'}(0))E(V^2)}{2(\prod_{l=1}^{K} V_l^*(\lambda) + \lambda E(V))}$$

$$V_v = \frac{\lambda^2(1+\lambda B^{*'}(0))}{12(\prod_{l=1}^{K} V_l^*(\lambda) + \lambda E(V))^2}\Big\{(\prod_{l=1}^{K} V_l^*(\lambda) + \lambda E(V))[6E(V^2) + 4\lambda E(V^3)] - 3\lambda^2(1+\lambda B^{*'}(0))E(V^2)^2\Big\}$$

where

$$D_1 = B^{*''}(0)(\prod_{l=1}^{K} V_l^*(\lambda) + \lambda E(V)) - \lambda B^{*'}(0)(1+\lambda B^{*'}(0))E(V^2)$$

$$D_2 = [3B^{*''}(0)(1+\lambda B^{*'}(0) + \lambda^2 B^{*''}(0)) - 2\lambda(1+\lambda B^{*'}(0))B^{*'''}(0)] \times (\prod_{l=1}^{K} V_l^*(\lambda) + \lambda E(V)) - 2\lambda^2 B^{*'}(0)(1+\lambda B^{*'}(0))^2 E(V^3) + 3\lambda(1+\lambda B^{*'}(0))(\lambda B^{*''}(0) - (1+\lambda B^{*'}(0))B^{*'}(0))E(V^2)$$

Cases 2: The service time and vacation time follows exponential distribution. That is, $B(x) = 1 - e^{\mu(x)}$, $B^*(s) = \frac{\mu}{s+\mu}$, $B^{*'}(0) = \frac{-1}{\mu}$, $B^{*''}(0) = \frac{2}{\mu^2}$, $B^{*'''}(0) = \frac{-6}{\mu^3}$, $V_i(x) = 1 - e^{-\eta_i(x)}$, $V_i^*(s) = \frac{\eta_i}{s+\eta_i}$, $V_i^{*'}(0) = \frac{-1}{\nu_i}$, $V_i^{*''}(0) = \frac{2}{\nu_i^2}$, $V_i^{*'''}(0) = \frac{-6}{\nu i^3}$, $i = 1, 2, \cdots, K$.

$$Q = \frac{(\mu - \lambda_1) \prod_{l=1}^{K} \frac{\eta_l}{\lambda_2 + \eta_l}}{D_3}$$

$$L_b = \frac{\lambda_0 D_6}{(\mu - \lambda_1) D_3}$$

$$V_b = \frac{\lambda_0 [D_3 D_7 - \lambda_0 D_6^2]}{(\mu - \lambda_1)^2 D_3^2}$$

$$L_v = \frac{\lambda_0 \lambda_2 (\mu - \lambda_1) D_4}{D_3}$$

$$V_v = \frac{\lambda_0 \lambda_2 (\mu - \lambda_1)[D_3 D_4 - 2\lambda_2 D_3 D_5 - \lambda_0 \lambda_2 (\mu - \lambda_1) D_4^2]}{D_3^2}$$

where

$$D_3 = (\mu + (\lambda_0 - \lambda_1)) \prod_{l=1}^{K} \frac{\eta_l}{\lambda_2 + \eta_l} + \lambda_0 (\mu + (\lambda_2 - \lambda_1)) \sum_{i=1}^{K} \frac{1}{\nu_i}$$

$$D_4 = \sum_{n=1}^{K} \frac{1}{\nu_n^2} + \sum_{n=1}^{K-1} \frac{1}{\nu_n} \sum_{i=n+1}^{K} \frac{1}{\nu_i}$$

$$D_5 = \sum_{n=1}^{K} \frac{1}{\nu_n^3} + \sum_{n=1}^{K-1} \frac{1}{\nu_n^2} \sum_{i=n+1}^{K} \frac{1}{\nu_i} + \sum_{n=1}^{K-1} \frac{1}{\nu_n} \sum_{i=n+1}^{K} \frac{1}{\nu_i^2} + \sum_{m=1}^{K-2} \frac{1}{\nu_m} \sum_{j=m+1}^{K-1} \frac{1}{\nu_j} \sum_{n=j+1}^{K} \frac{1}{\nu_n}$$

$$D_6 = \lambda_1 (\prod_{l=1}^{K} \frac{\eta_l}{\lambda_2 + \eta_l} + \lambda_2 \sum_{i=1}^{K} \frac{1}{\nu_i}) + \lambda_2^2 (\mu - \lambda_1) D_4$$

$$D_7 = \lambda_1 (\mu + \lambda_1) (\prod_{l=1}^{K} \frac{\eta_l}{\lambda_2 + \eta_l} + \lambda_2 \sum_{i=1}^{K} \frac{1}{\nu_i}) + \lambda_2^2 (\mu^2 - \lambda_1^2) D_4 - 2\lambda_2^3 (\mu - \lambda_1)^2 D_5$$

Case 3: The service time follows exponential distribution.
That is, $B(x) = 1 - e^{\mu x}$, $B^*(s) = \frac{\mu}{s + \mu}$, $B^{*'}(0) = \frac{-1}{\mu}$, $B^{*''}(0) = \frac{2}{\mu^2}$, $B^{*'''}(0) = \frac{-6}{\mu^3}$.

$$Q = \frac{(\mu - \lambda_1)(\prod_{l=1}^{K} V_l^*(\lambda_2))}{D_8}$$

$$L_b = \frac{\lambda_0 D_9}{2(\mu - \lambda_1) D_8}$$

$$V_b = \frac{\lambda_0 [2 D_8 D_{10} - 3\lambda_0 D_9^2]}{12(\mu - \lambda_1)^2 D_8^2}$$

$$L_v = \frac{\lambda_0\lambda_2(\mu-\lambda_1)E(V^2)}{2D_8}$$

$$V_v = \frac{\lambda_0\lambda_2(\mu-\lambda_1)}{12D_8^2}\Big\{6D_8E(V^2)+4\lambda_2D_8E(V^3)-3\lambda_0\lambda_2(\mu-\lambda_1)E(V^2)^2\Big\}$$

where

$$D_8 = (\mu+(\lambda_0-\lambda_1))\prod_{l=1}^{K}V_l^*(\lambda_2)-\lambda_0(\mu+(\lambda_2-\lambda_1))\sum_{i=1}^{K}V_i^{*'}(0)$$

$$D_9 = 2\lambda_1(\prod_{l=1}^{K}V_l^*(\lambda_2)+\lambda_2E(V))+\lambda_2^2(\mu-\lambda_1)E(V^2)$$

$$D_{10} = 6\lambda_1(\mu+\lambda_1)(\prod_{l=1}^{K}V_l^*(\lambda_2)+\lambda_2E(V))+3\lambda_2^2(\mu^2-\lambda_1^2)E(V^2)$$
$$+2\lambda_2^3(\mu-\lambda_1)^2E(V^3)$$

<u>**Case 4**</u>: The vacation time follows exponential distribution.
That is, $V_i(x) = 1-e^{-\eta_i(x)}$, $V_i^*(s) = \frac{\eta_i}{s+\eta_i}$, $V_i^{*'}(0) = \frac{-1}{\nu_i}$, $V_i^{*''}(0) = \frac{2}{\nu_i^2}$, $V_i^{*'''}(0) = \frac{-6}{\nu_i^3}$, $i = 1,2,\cdots,K$.

$$Q = \frac{(1+\lambda_1B^{*'}(0))\prod_{l=1}^{K}\frac{\eta_l}{\lambda_2+\eta_l}}{D_{11}}$$

$$L_b = \frac{\lambda_0D_{12}}{2(1+\lambda_1B^{*'}(0))D_{11}}$$

$$V_b = \frac{\lambda_0[2D_{11}D_{13}-3\lambda_0D_{12}^2]}{12(1+\lambda_1B^{*'}(0))^2D_{11}^2}$$

$$L_v = \frac{\lambda_0\lambda_2(1+\lambda_1B^{*'}(0))D_4}{D_{11}}$$

$$V_v = \frac{\lambda_0\lambda_2(1+\lambda_1B^{*'}(0))[D_4D_{11}-2\lambda_2D_5D_{11}-\lambda_0\lambda_2(1+\lambda_1B^{*'}(0))D_4^2]}{D_{11}^2}$$

where

$$D_{11} = (1+(\lambda_1-\lambda_0)B^{*'}(0))\prod_{l=1}^{K}\frac{\eta_l}{\lambda_2+\eta_l}+\lambda_0(1+(\lambda_1-\lambda_2)B^{*'}(0))\sum_{i=1}^{K}\frac{1}{\nu_i}$$

$$D_{12} = \lambda_1B^{*''}(0)(\prod_{l=1}^{K}\frac{\eta_l}{\lambda_2+\eta_l}+\lambda_2\sum_{i=1}^{K}\frac{1}{\nu_i})-2\lambda_2^2B^{*'}(0)(1+\lambda_1B^{*'}(0))D_4$$

$$D_{13} = [3\lambda_1B^{*''}(0)(1+\lambda_1B^{*'}(0)+\lambda_1^2B^{*''}(0))-2\lambda_1^2B^{*'''}(0)(1+\lambda_1B^{*'}(0))]$$
$$\times(\prod_{l=1}^{K}\frac{\eta_l}{\lambda_2+\eta_l}+\lambda_2\sum_{i=1}^{K}\frac{1}{\nu_i})+12\lambda_2^3B^{*'}(0)(1+\lambda_1B^{*'}(0))^2D_5$$
$$+6\lambda_2^2(1+\lambda_1B^{*'}(0))D_4[\lambda_1B^{*''}(0)-B^{*'}(0)(1+\lambda_1B^{*'}(0))]$$

Case 5: Now $\lambda_0 = \lambda_1 = \lambda_2 = \lambda$ and $K = 1$ (An $M/G/1$ queue with single vacation)

$$Q = \frac{(1+\lambda B^{*'}(0))V_1^*(\lambda)}{V_1^*(\lambda) - \lambda V_1^{*'}(0)}$$

$$L_b = \frac{\lambda^2 D_{14}}{2(1+\lambda B^{*'}(0))(V_1^*(\lambda) - \lambda V_1^{*'}(0))}$$

$$V_b = \frac{\lambda^2[2(V_1^*(\lambda) - \lambda V_1^{*'}(0))D_{15} - 3\lambda^2 D_{14}^2]}{12(1+\lambda B^{*'}(0))^2(V_1^*(\lambda) - \lambda V_1^{*'}(0))^2}$$

$$L_v = \frac{\lambda^2 V_1^{*''}(0)(1+\lambda B^{*'}(0))}{2(V_1^*(\lambda) - \lambda V_1^{*'}(0))}$$

$$V_v = \frac{1}{12(V_1^*(\lambda) - \lambda V_1^{*'}(0))^2}\Big\{\lambda^2(1+\lambda B^{*'}(0))[6V_1^{*''}(0)(V_1^*(\lambda) - \lambda V_1^{*'}(0))$$
$$- 4\lambda V_1^{*'''}(0)(V_1^*(\lambda) - \lambda V_1^{*'}(0)) - 3\lambda^2 V_1^{*''^2}(0)(1+\lambda B^{*'}(0))]\Big\}$$

where

$$D_{14} = B^{*''}(0)(V_1^*(\lambda) - \lambda V_1^{*'}(0)) - \lambda B^{*'}(0)(1+\lambda B^{*'}(0))V_1^{*''}(0)$$

$$D_{15} = [3B^{*''}(0)(1+\lambda B^{*'}(0) + \lambda^2 B^{*''}(0)) - 2\lambda B^{*'''}(0)(1+\lambda B^{*'}(0))]$$
$$\times (V_1^*(\lambda) - \lambda V_1^{*'}(0)) + 2\lambda^2 B^{*'}(0)(1+\lambda B^{*'}(0))^2 V_1^{*'''}(0)$$
$$+ 3\lambda(1+\lambda B^{*'}(0))[\lambda B^{*''}(0) - B^{*'}(0)(1+\lambda B^{*'}(0))]V_1^{*''}(0)$$

5. Numerical results

In this section, We present some numerical results in order to illustrate the effect of various parameters on the performance measures of the models in section 4. The effect of the parameters arrival rate, service rate, vacation rate and the number of phases of vacation on the system performance measures (i) the mean number of customers when the server is busy (L_b), (ii) the mean number of customers in the queue when the server is on vacation (L_v), (iii) the variance of the number of customers in the queue when the server is busy (V_b) and (iv) the variance of the number of customers in the queue when the server is on vacation (V_v) have been numerically analysed. Figures $1-4$ represents the graph of mean number of customers when $K = 1, 3, 5$ and 7 by varying the service rate. Tables $5-8$ shows the variance of number of customers. In all the figures, it is clear that the mean number of customers in the queue when the server is busy is decreasing function with respect to service rate where as its counter part are increasing functions as expected. The variance value with respect to server busy decreases as the service rate increases but in the case of variance with respect to vacation we encounters the contary concept that is variance increases. For this analysis the values of $\lambda_0 = 0.6$, $\lambda_1 = 0.7$, $\lambda_2 = 0.8$ are fixed.

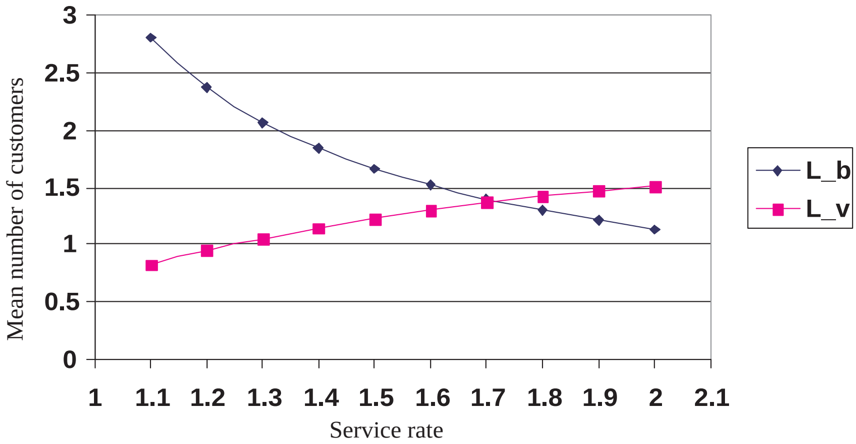

Figure 1: Mean number of customers for K=1

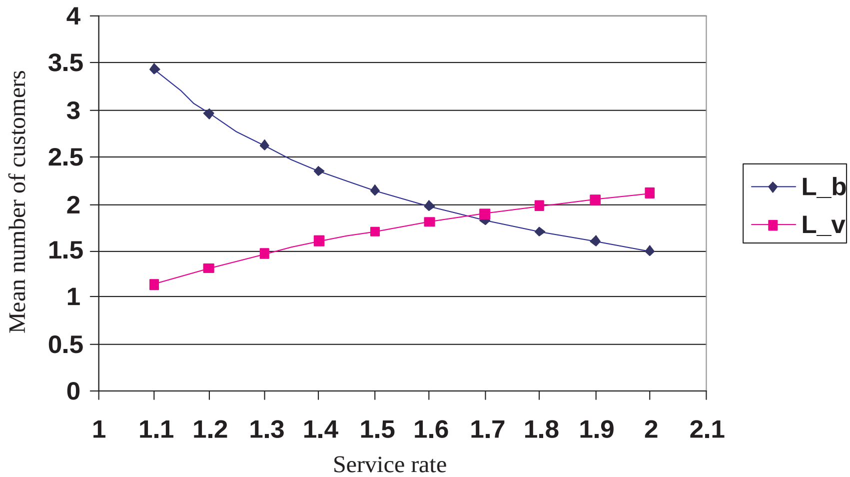

Figure 2: Mean number of customers for K=3

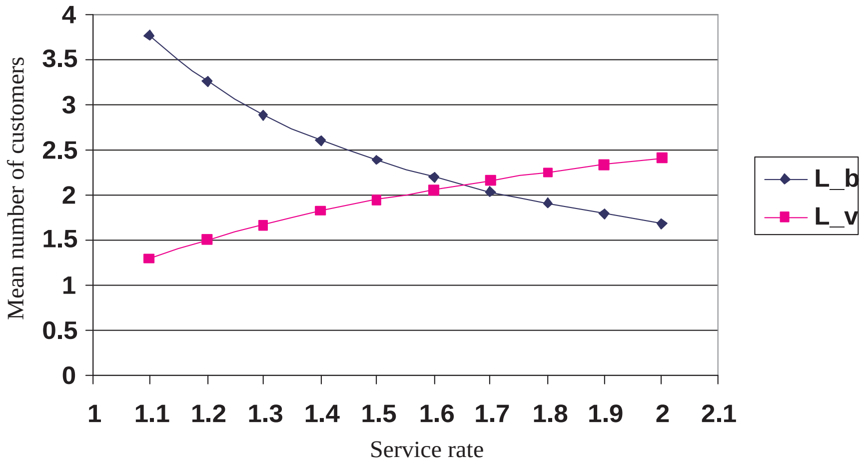

Figure 3: Mean number of customers for K=5

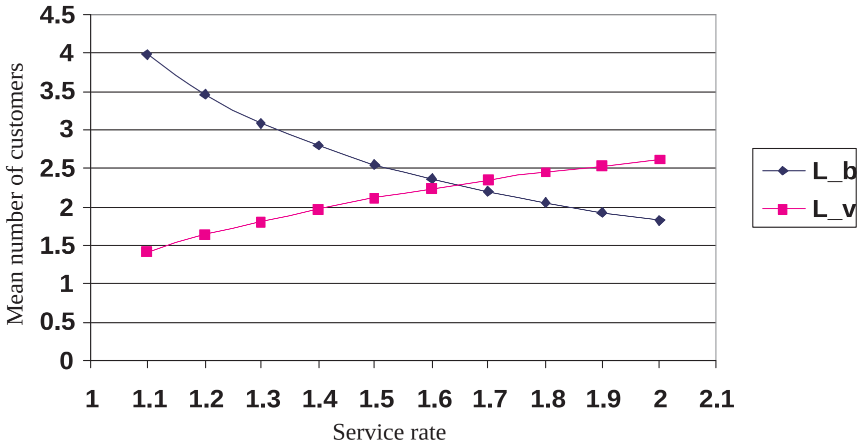

Figure 4: Mean number of customers for K=7

Table 5: Variance for $K = 1$

μ	V_b	V_v
1.1	13.5135	4.5233
1.2	11.4613	5.0948
1.3	10.1056	5.5586
1.4	9.1171	5.9414
1.5	8.3492	6.2618
1.6	7.7267	6.5332
1.7	7.2067	6.7657
1.8	6.7627	6.9667
1.9	6.3771	7.1419
2.0	6.0378	7.2959

Table 6: Variance for $K = 3$

μ	V_b	V_v
1.1	17.3232	6.6830
1.2	15.1104	7.4809
1.3	13.5996	8.1161
1.4	12.4637	8.6301
1.5	11.5569	9.0522
1.6	10.8042	9.4032
1.7	10.1624	9.6982
1.8	9.6044	9.9488
1.9	9.1124	10.1634
2.0	8.6736	10.3487

Table 7: Variance for $K = 5$

μ	V_b	V_v
1.1	19.5701	8.0106
1.2	17.2697	8.9438
1.3	15.6725	9.6803
1.4	14.4533	10.2712
1.5	13.4673	10.7521
1.6	12.6396	11.1485
1.7	11.9273	11.4788
1.8	11.3031	11.7568
1.9	10.7489	11.9928
2.0	10.2519	12.1948

Table 8: Variance for $K = 7$

μ	V_b	V_v
1.1	21.2769	9.0384
1.2	18.9135	10.0755
1.3	17.2534	10.8895
1.4	15.9729	11.5389
1.5	14.9280	12.0645
1.6	14.0444	12.4952
1.7	13.2792	12.8521
1.8	12.6052	13.1506
1.9	12.0042	13.4025
2.0	11.4631	13.6168

6. Conclusion

In the foregoing analysis an $M/G/1$ queue with K-phase vacation has been considered. For this model the queue length distribution and the mean queue length are obtained. An extensive numerical work has been carried out to observe the nature of the operating characteristics.

References

1. Doshi,B.T., *Queueing systems with vacations - a survey,* Queueing System, 1,29-66, 1986.
2. Doshi,B.T., *Single server queues with vacations,* In:Takagi,H.(ed.), Stochastic Analysis of the computer and communication systems, 217-264, North-Holland/Elsevier, Ansterdam, 1990.
3. Gross,D. and Harris,C.M., *Fundamentals of queueing theory,* 3^{rd} edn, Wiley, New York, 1998.
4. Hai-Yan Chen, Ji-Hong Li and Nai-Shuo Tian, *The $GI/M/1$ queue with phase-type working vacations and vacation interruptions,* J.Appl.Math.Comput.,30,121-141, 2009.

5. Ke,J.C., *The analysis of general input queue with N policy and exponential vacations,* Queueing Syst.,45,135-160, 1986.
6. Ke,J.C., *The optimal control of an* $M/G/1$ *queueing system with server vacations, startup and breakdowns, Computers and Industrial Engineering,* Vol.44,567-579, 2003.
7. Ke,J.C.,and Wang,K.H., *Analysis of operating characteristics for the heterogeneous batch arrival queue with server startup and breakdowns,* Rairo oper. Res., Vol.37,157-177, 2003.
8. Ke,J.C.,and Chu,Y.K., *A modified vacation model* $M^{[x]}/G/1$ *system,* Appl. Stochastic Models. Bus. Ind.,22,1-16,2006.
9. Ke,J.C., *Operating characteristics analysis on the* $M^{[x]}/G/1$ *system with a variant vacation policy and balking,* Appl. Math. Model.,31(7),1321-1337, 2007.
10. Ke,J.C.,and Huang,K.B and Pearn,W.L., *The randomized vacation policy for a batch arrival queue,* Applied Mathematical Modeling, Vol.34,1524-1538, 2010.
11. Keilson,J. and Servi,L.D., *Dynamic of the* $M/G/1$ *vacation model,* Operations Research, Vol.35 (4), 575-582, 1987.
12. Takagi,H., *Vacation and Priority Systems Part 1. Queueing Analysis: A Foundation of Performance Evaluation,* Vol.1., North-Holland/Elsevier, Amsterdam, 1991.
13. Tian,N., and Zhang,Z.G., *Discrete time* $Geo/GI/1$ *queue with multiple adaptive vacations,* Queueing Syst., 38,219-249, 2001.
14. Tian,N., and Zhang,Z.G., *The discrete time* $GI/Geo/1$ *queue with multiple vacations,* Queueing Syst., 40,283-294, 2002.
15. Tian,N., and Zhang,Z.G., *A note on* $GI/M/1$ *queues with phase-type setup times or server vacations,* INFOR,41,341-351, 2003.
16. Tian,N., and Zhang,Z.G., *Vacation queueing models: Theory and Applications,* Springer, New York, 2006.