International Journal of Mathematics and Computer Research
ISSN: 2320-7167
Volume 12 Issue 12 December 2024, Page no. – 4613-4632
Index Copernicus ICV: 57.55, Impact Factor: 8.316
DOI: 10.47191/ijmcr/v12i12.01

Stability Analysis of the Disease-Free Equilibrium State for Lymphatic Filariasis with Chemical and Biological Control on Vector

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ARTICLE INFO	ABSTRACT
Published Online: **03 December 2024** **Corresponding Author:** **Umar Saidu Bashir**	In this paper, we develop a mathematical model to analyze the transmission dynamics and control strategies for Lymphatic Filariasis (LF), incorporating both chemical and biological control measures targeting the disease vector. The model is proven to be both mathematically and epidemiologically sound. By determining the basic reproduction number (\(R_0\)), we establish the conditions for local and global stability of the disease-free equilibrium (DFE). The model highlights the impact of Wolbachia bacteria on mosquito populations and the role of drug resistance and recovery in human populations. Our results demonstrate that reducing \(R_0\) below 1 is crucial to eradicating LF from an endemic population, and thus, preventive measures, including vector control, are essential. Further research is recommended to optimize the combined use of chemical and biological controls to achieve long-term stability and disease eradication.

KEYWORDS: Lymphatic filariasis, Wolbachia, basic reproduction number, vector control, disease-free equilibrium, stability analysis

1.0 INTRODUCTIONS

Lymphatic filariasis (LF) is a neglected tropical disease (NTD) caused by parasitic filarial worms, primarily *Wuchereria bancrofti*, *Brugia malayi*, and *Brugia timori*. These parasites are transmitted to humans through the bites of infected mosquitoes, which serve as vectors. LF is characterized by the obstruction and inflammation of the lymphatic system, leading to severe and often debilitating symptoms, including lymphedema (swelling), hydrocele (fluid accumulation in the scrotum), and elephantiasis (thickening of the skin) (World Health Organization [WHO], 2021). LF is endemic in more than 70 countries, predominantly in tropical and subtropical regions of Africa, Asia, the Western Pacific, and parts of the Americas. According to the WHO, over 120 million people are infected globally, with about 40 million suffering from the severe manifestations of the disease (WHO, 2021). The disease disproportionately affects the poorest populations, leading to significant socio-economic consequences (Ottesen & Hooper, 2008). Diagnosis of lymphatic filariasis typically involves the detection of microfilariae in the blood, usually through a blood smear taken at night when the microfilariae are most abundant. Serological tests, such as antigen detection assays, can also be used to identify active infections (Becker et al., 2018).

ant filarial medications such as diethylcarbamazine (DEC) and albendazole. This approach aims to reduce the prevalence of the disease and interrupt transmission (WHO, 2021). Community-wide treatment campaigns are essential to achieve the target of eliminating LF as a public health problem. In addition to MDA, management of clinical symptoms is crucial. Patients with lymphedema may benefit from hygiene, skin care, and physical therapy, while surgical interventions can be considered for severe cases of hydrocele or lymphedema (Ottesen & Hooper, 2008).

Rojas, C. A., & Tien, C. (2016) summarizes various mathematical models developed for lymphatic filariasis, discussing methodologies, findings, and future research directions. Silumbwe, Zulu, Halwindi, Jacobs, Zgambo, Dambe, & Michelo (2017) reviews the effectiveness of mass drug administration strategies against LF, using mathematical modeling to predict outcomes under different scenarios. While not exclusively focused on lymphatic filariasis, Rogers, D. J., & Randolph, S. E. (2006) discusses the implications of climate change on the transmission of vector-borne diseases, including LF, and presents mathematical models to assess potential risks. Alonso, D., & Bansal, S. (2017) studies the uses mathematical models to evaluate the short- and long-term effects of mass drug administration on LF transmission dynamics. Other models are models are Bockarie, M. J., et al. (2009), Ferguson, N. M., & Anderson, R. M. (1999).

Lymphatic filariasis remains a significant public health challenge in many parts of the world. Efforts to eliminate the disease through MDA, health education, and improved vector control are critical to reducing the burden of LF and improving the quality of life for affected individuals.

Mwamtobe *et al*., (2017) developed a model that consider quarantine of infected-chronic and treatment of the infected acute individuals. The result shows that with quarantine and treatments, the rate of reduction of lymphatic filariasis is higher. In this work, we therefore extend his works by adding a drug resistance compartment and recovered class in the human population. We Also incorporated mosquito with wolbachia populations: Aquatic stage with wolbachia, A_w, Male mosquito with wolbachia, M_w, Female Mosquito with wolbachia, F_w.

2.0 MODEL FORMULATION

The Lymphatic Filariasis (LF) model with human population under study is divided into 6 compartments and vector (mosquito) population into 7 compartments is formulated. The model will subdivide the human population at time t, $N(t)$ into the class of susceptible individuals $S_h(t)$, the exposed class, $E_h(t)$, LF infected acute individuals, $I_{ha}(t)$, LF Infected Chronic, $I_{hc}(t)$, Drug resistance individual, $D_R(t)$ and the Recovered individuals, $R(t)$.

Such that the total human subpopulations

$$N_h(t) = S_h(t) + E_h(t) + I_{ha}(t) + I_{hc}(t) + R(t) \quad (1)$$

The model considers lymphatic filariasis, of which a wide range of mosquitoes (*anopheles, culex, Aedes*) can transmit the parasite, depending on the geographic area. the most common vector in Africa is the anopheles. The recruitment, Λ_h of individuals into the susceptible class is either by birth or immigration. Some are Exposed to lymphatic filariasis and if infected move to exposed class by the

$$\lambda_h = \frac{\beta_h \vartheta_h F_i S_h}{N_h} \quad (2)$$

with β_h being the finite probability that, in case an infectious mosquito bites a susceptible human, a worm (in the form of filarial larva) passes into the human body. ϑ_h is the mosquito biting rate (rate at which mosquitoes bite susceptible human), F_i is the infected female mosquito Population without *wolbachia*. The infectious worm moves to the lymphatic system where it develops into its next life stage. Thereafter move to infected acute class and subsequently when the sign start manifesting, they move to the infected chronic.

There is a reduction of the human population in each class through natural death at rate μ_h. We assume that the natural death rate is the same for all subpopulations of the human population.

The mosquito population $N_v(t)$ is divided into Aquatic or larva stage, $A(t)$, male mosquito without w*olbachia* $M(t)$, female mosquitoes without *wolbachia*, $F(t)$, infected female mosquito population without *Wolbachia*, F_i, Aquatic or larva stage with *wolbachia*, $A_w(t)$ male mosquito with wolbachia, $M_w(t)$ and female mosquitoes with *wolbachia*, $F_w(t)$ such that subpopulations such that

$$N_v(t) = A(t) + M(t) + F(t) + F_i(t) + A_w(t) + M_w(t) + F_w(t) \quad (3)$$

The mosquito population increases through Maturation of the larva or aquatic stage. The mosquito populations in each class are reduced by introducing *wolbachia* bacteria. Mosquitoes die naturally at rate d_w and we assume that this rate is the same throughout all subpopulation classes.

The corresponding mathematical equations of the above description are given by a system of Ordinary differential equations below:

$$\left.\begin{aligned}
\frac{dS_h}{dt} &= \Lambda_h + \varphi(1-p) - \frac{\beta\vartheta_h F_i S_h}{N_h} - \mu_h S_h + \omega R \\
\frac{dE_h}{dt} &= \frac{\beta\vartheta_h F_i S_h}{N_h} - (\alpha_h + \mu_h)E_h \\
\frac{dI_{ha}}{dt} &= \alpha_h E_h + \varphi p - (m + k + \mu_h + \pi)I_{ha} \\
\frac{dI_{hc}}{dt} &= kI_{ha} - (\mu_h + \gamma)I_{hc} \\
\frac{dD_R}{dt} &= mI_{ha} - (n + \mu_h)D_R \\
\frac{dR}{dt} &= \pi I_{ha} + \gamma I_{hc} + nD_R - (\omega + \mu_h)R \\
\frac{dA}{dt} &= bF - (d_m + \xi)A - \delta A \\
\frac{dF}{dt} &= (1-\eta)\delta A - d_m F - \gamma_2 F - \frac{\beta\vartheta_v (I_{ha} + I_{hc} + D_R)F}{N_v} \\
\frac{dF_i}{dt} &= \frac{\beta\vartheta_v (I_{ha} + \theta I_{hc} + \theta_1 D_R)F}{N_v} - d_m F_i \\
\frac{dA_w}{dt} &= bF_w - (d_w + \delta)A_w \\
\frac{dM_w}{dt} &= \eta\delta A_w + \gamma_1 M - d_w M_w \\
\frac{dF_w}{dt} &= (1-\eta)\delta A_w + \gamma_2 F - d_w F_w
\end{aligned}\right\} \qquad (4)$$

$$\frac{dN_h}{dt} \le \Lambda_h + \varphi - \mu_h N_h$$

$$\frac{dN_v}{dt} \le b - d_m N_v$$

In the biological-feasible regions:

$$\Omega = \left\{\begin{aligned}
&(S_h, E_h, I_{ha}, I_{hc}, D_R, R, A, M, F, F_i, \mathrm{A}_w, \mathrm{M}_w, \mathrm{F}_w) \in \Re^{13}{}_{+} : S_h \ge 0, E_h \ge 0, I_{ha} \ge 0, I_{hc} \ge 0, \\
&D_R \ge 0, R \ge 0, \mathrm{A} \ge 0, \mathrm{M} \ge 0, F \ge 0,, F_i \ge 0, A_w \ge 0, M_w \ge 0, F_w \ge 0; \\
&N_h(t) \le \frac{\Lambda_h + \varphi}{\mu_h}; N_v(t) \le \frac{be^{d_v t}}{d_v}
\end{aligned}\right\} \qquad (5)$$

Equation (5) can be shown to be positively invariant with respect to the system (4)

Table 1. Description of variables and parameters of the modified model

Parameters	Descriptions
$S_h(t)$	Susceptible human at time t
$E_h(t)$	Exposed human at time t
$I_{ha}(t)$	Infected acute human at time t

$I_{hc}(t)$	Infected chronic human at time t
$R(t)$	Recovered human at time t.
$D_R(t)$	Drug resistance class at time t.
M(t)	Population of male mosquito without *wolbachia* at t.
$F(t)$	Population of Female Mosquito without wolbachia at time t.
$F_i(t)$	Infected female mosquito population without *Wolbachia*,
$A(t)$	Population of larva or aquatic stage at time t.
$M_w(t)$	Population of male mosquito with *wolbachia* at time t.
$F_w(t)$	Population of Female mosquito with *wolbachia* at time t
A_w(t)	Population of aquatic or larva stage of mosquito with *wolbachia*
n	Recovery rates of drug resistance class due to alternative drug administration
γ	Recovery rates of chronic individuals
η	Proportion being matured to adult male
δ	Maturation rates of adult mosquito from larva or aquatic stage
Λ_h	Recruitment of human populations
ω	Rate of loss of immunity.
π	Rate at which infected acute are responding to Treatment and move to recovered class.
m	Rate of Individual that are resistant to drugs therefore move to drug resistant class.
k	Progress rate from infected acute to infected chronic
ξ	Chemical control larvicide on larva or aquatic stage
d_m	Natural death of wild mosquitoes
μ_h	Natural death of human
p	Proportion of children infected at birth
φ	Rate at which children born are infected LF at birth and born susceptible
d_w	Natural death of mosquitoes with *wolbachia*
γ_1, γ_2	Infestation rate of male and female mosquito with *wolbachia* respectively
α_h	Progress rate of human from susceptible to exposed
b	Natural reproduction rate of adult female mosquito with wolbachia

3.0 MODEL ANALYSIS

3.1 Existence of disease-free equilibrium state

At the disease-free equilibrium state, we have absence of infection. Thus, all the infected classes will be zero and the entire population will comprise of only susceptible individuals.

Theorem 3.1: A disease-free equilibrium state of the model exists at the point

$$E_0 = \left(S_h^*, E_h^*, I_{ha}^*, I_{hc}^*, D_R^*, R^*, A^*, M^*, F^*, F_i^*, \mathrm{A}_w^*, \mathrm{M}_w^*, \mathrm{F}_w^*\right)$$

$$E_0 = \left\{ \begin{array}{l} \dfrac{\Lambda_h + \varphi(1-p)}{\mu_h}, 0,0,0,0,0, \dfrac{b}{(d_m + \xi - \delta)}, \dfrac{\eta\delta b}{(\gamma_1 + d_m)(d_m + \xi + \delta)}, \dfrac{\mathrm{b}(1-\eta)\delta}{(d_m + \xi - \delta)(d_m + \gamma_2)}, 0, \\ \dfrac{-b^2\delta\gamma_2(\eta - 1)}{(\gamma_2 + d_m)(\delta + \xi + d_m)(d_w^2 + \delta d_w - b\delta + b\delta\eta)}, \dfrac{\eta\delta(d_m^2 + (\gamma_1 + \xi + \delta)d_m + \delta\gamma_1 + \xi\gamma_1)A_w + b\eta\delta\gamma_1}{d_w(\gamma_1 + d_m)(d_m + \xi + \delta)}, \\ -b\delta\gamma_2(\eta - 1)\dfrac{\delta + d_w}{(\gamma_2 + d_m)(\delta + \xi + d_m)(d_w^2 + \delta d_w - b\delta + b\delta\eta)} \end{array} \right\}$$

Proof:

At equilibrium state the rate of change of each variable is equal to zero

$$\frac{dS_h}{dt}=\frac{dE_h}{dt}=\frac{dI_{ha}}{dt}=\frac{dI_{hc}}{dt}=\frac{dD_R}{dt}=\frac{dR}{dt}=\frac{dA}{dt}=\frac{dM}{dt}=\frac{dF}{dt}=\frac{dF_i}{dt}=\frac{dA_w}{dt}=\frac{dM_w}{dt}=\frac{dF_w}{dt}=0$$

At disease-free,

$$\Rightarrow E_h=I_{ha}=I_{hc}=D_R=F_i=0$$

Hence, a disease-free equilibrium of the model exists at:

$$E_0=\left\{\begin{array}{l}\dfrac{\Lambda_h+\varphi(1-p)}{\mu_h},0,0,0,0,0,\dfrac{b}{(d_m+\xi-\delta)},\dfrac{\eta\delta b}{(\gamma_1+d_m)(d_m+\xi+\delta)},\dfrac{\mathrm{b}(1-\eta)\delta}{(d_m+\xi-\delta)(d_m+\gamma_2)},0,\\ \dfrac{-b^2\delta\gamma_2(\eta-1)}{(\gamma_2+d_m)(\delta+\xi+d_m)(d_w^2+\delta d_w-b\delta+b\delta\eta)},\dfrac{\eta\delta(d_m^2+(\gamma_1+\xi+\delta)d_m+\delta\gamma_1+\xi\gamma_1)A_w+b\eta\delta\gamma_1}{d_w(\gamma_1+d_m)(d_m+\xi+\delta)},\\ -b\delta\gamma_2(\eta-1)\dfrac{\delta+d_w}{(\gamma_2+d_m)(\delta+\xi+d_m)(d_w^2+\delta d_w-b\delta+b\delta\eta)}\end{array}\right\}$$

3.2 Invariant Region

Theorem 3.2: The closed set

$D=D_h\cup D_m\cup D_w\subset\mathbb{R}_+^6\times\mathbb{R}_+^4\times\mathbb{R}_+^3$

where

$$D_h=\left\{\left(S_h,E_h,I_{ha},I_{hc},D_R,R\right)\in\mathbb{R}_+^6:N_h(t)=\frac{\Lambda_h+\varphi_1}{N_h}\right\},$$

$$D_v=\left\{(\mathrm{A},\mathrm{M},\mathrm{F},\mathrm{F}_i,A_w,M_w,F_w)\in\mathbb{R}_+^7:N_v(t)\le\frac{b_1}{d_m}\right\}, \text{ and}$$

$$D_w=\left\{\left(A_w,M_w,F_w\right)\in\mathbb{R}_+^3:N_w(t)\le N_w(0)e^{(b-d_w)t}\right\}.$$

is positively-invariant and attracting with respect to the modified model equation given by system (4).

Proof:

Considering the human population, we have

$$N_h=S_h+E_h+I_{ha}+I_{hc}+D_R+R \qquad (7)$$

Differentiating (7), we have

$$\frac{dN_h}{dt}=\frac{dS_h}{dt}+\frac{dE_h}{dt}+\frac{dI_{ha}}{dt}+\frac{dI_{hc}}{dt}+\frac{dD_R}{dt}+\frac{dR}{dt} \qquad (8)$$

Substituting the right-hand sides of equation (4) in (8), gives us

$$\frac{dN_h}{dt}\le\Lambda_h+\varphi_1-\mu_h\left(S_h+E_h+I_{ha}+I_{hc}+D_R+R\right)$$

$$\frac{dN_h}{dt}\le\Lambda_h+\varphi_1-\mu_hN_h$$

$$\frac{dN_h}{dt}+\mu_hN_h\le\Lambda_h+\varphi_1$$

$$\frac{d}{dt}\left(N_he^{\mu_ht}\right)\le\left[\Lambda_h+\varphi_1\right]e^{\mu_ht} \qquad (9)$$

Integrating (9) with respect to t we obtain

$$N_h e^{\mu_h t} \leq \frac{\left[\Lambda_h + \varphi_1\right] e^{\mu_h t}}{\mu_h} + k_a \tag{10}$$

Dividing through by $e^{\mu_h t}$, $N_h \leq \frac{\left[\Lambda_h + \varphi_1\right]}{\mu_h} + k_a e^{-\mu_h t}$

Taking the limit of $N_h \leq \frac{\left[\Lambda_h + \varphi_1\right]}{\mu_h} + k_a e^{-\mu_h t}$ as $t \to \infty$, we have

$$N_h(t) \leq \frac{\left[\Lambda_h + \varphi_1\right]}{\mu_h}$$

and thus

$$0 \leq N_h(t) \leq \frac{\left[\Lambda_h + \varphi_1\right]}{\mu_h}.$$

Hence, the invariant region for the human population is given by

$$D_h = \left\{ \left(S_h, E_h, I_{ha}, I_{hc}, D_R, R\right) \in \mathbb{R}_+^6 : N_h(t) \leq \frac{\Lambda_h + \varphi_1}{N_h} \right\}. \tag{11}$$

Similarly, considering the population of natural mosquitoes. the invariant is given by

$$D_m = \left\{ \left(A, M, F, F_i\right) \in \mathbb{R}_+^4 : N_m(t) \leq \frac{b_1}{d_m} \right\} \tag{12}$$

Similarly, considering the mosquito with Wolbachia population. the invariant region for the population is given by

$$D_w = \left\{ \left(A_w, M_w, F_w\right) \in \mathbb{R}_+^3 : N_w(t) \leq N_w(0) e^{(b - d_w)t} \right\} \tag{13}$$

Therefore, from (11), (12) and (13), the possible solutions of the system (4) will enter the positively invariant region

$$D = D_h \times D_m \times D_w. \tag{14}$$

Equation (14) defines the property by which a lymphatic filariasis remains unchanged under some transformation.

3.3 Positivity of the model solution

Lemma 3.1:

Let the initial data of the model equation given by system (4) be given as

$$\left\{ \begin{array}{l} S_h(0) > 0, E_h(0) \geq 0, I_{ha}(0) \geq 0, I_{hc}(0) \geq 0, D_R(0) \geq 0, R(0) \geq 0, A(0) \geq 0, M(0) \geq 0, F(0) \geq 0, \\ F_i(0) \geq 0, A_w(0) \geq 0, M_w(0) \geq 0, F_w(0) \geq 0 \end{array} \right\} \in \Re_+^{13}.$$

then the solution set

$\left\{ \begin{array}{l} S_h(t), E_h(t), I_{ha}(t), I_{hc}(t), D_R(t), R(t), A(t), M(t), F(t), \\ F_i(t), A_w(t), M_w(t), F_w(t) \end{array} \right\} \in \Re_+^{13}$ of the model equation given by system (4) is positive for all $t > 0$.

Proof:

Let $t_1 = \sup \left\{ \begin{array}{l} t \geq 0 : S_h(t) > 0, E_h(t) > 0, I_{ha}(t) > 0, I_{hc}(t) > 0, D_R(t) > 0, R(0) > 0, A(0) > 0, \\ M(0) > 0, F(0) > 0, F_i(0) > 0, A_w(t) > 0, M_w(t) > 0, F_w(t) > 0 \end{array} \right\}$, thus, $t_1 > 0$.

Considering equation (3.8) we have

$$\frac{dS_h}{dt} = \Lambda_h + \varphi(1-p) + \omega R - (\lambda_h + \mu_h) S_h$$

$$\frac{dS_h}{dt} \geq -(\lambda_h + \mu_h) S_h$$

Using separation of variables method, we have

$$\frac{dS_h}{S_h} \geq -(\lambda_h + \mu_h) dt$$

Integrating both sides from $t_1 = 0$ to $t = t_1$

$$\int_0^{t_1} \frac{dS_h}{S_h} \geq -\int_0^{t_1} \lambda_h(y)\,dy - \int_0^{t_1} \mu_h dt$$

$$\ln S_h(t_1) - \ln S_h(0) \geq -\int_0^{t_1} \lambda_h(y)\,dy - \mu_h t_1$$

$$\ln\left(\frac{S_h(t_1)}{S_h(0)}\right) \geq -\left(\int_0^{t_1} \lambda_h(y)\,dy + \mu_h t_1\right)$$

$$\left(\frac{S_h(t_1)}{S_h(0)}\right) \geq \mathrm{e}^{-\left(\int_0^{t_1} \lambda_h(y)dy + \mu_h t_1\right)}$$

$$S_h(t_1) \geq S_h(0)\,\mathrm{e}^{-\left(\int_0^{t_1} \lambda_h(y)dy + \mu_h t_1\right)} > 0$$

Hence, $S_h(t) > 0$.

Using similar technique, it can be shown that

$$E_h(t) \geq E_h(0)e^{-(\alpha_h + \mu_h)t} > 0,$$

$$I_{ha}(t) \geq I_{ha}(0)e^{-(\mathrm{m}+\mathrm{k}+\pi+\mu_h)t} > 0,$$

$$I_{hc}(t) \geq I_{hc}(0)e^{-(\gamma+\mu_h)t} > 0,$$

$$D_R(t) \geq D_R(0)e^{-(\mathrm{n}_1+\mu_h)t} > 0,$$

$$R(\mathrm{t}) \geq R(0)e^{-(\omega+\mu_h)t} > 0,$$

$$A(\mathrm{t}) = A(0)e^{-(d_m+\xi+\delta)t} > 0,$$

$$M(\mathrm{t}) = M(0)e^{-(\gamma_1+d_m)t} > 0,$$

$$F(\mathrm{t}) = F(0)e^{-\left(\int_0^{t_1} \lambda_v(y)dy + (\gamma_2+d_m)t\right)} > 0,$$

$$F_i(\mathrm{t}) = F_i(0)e^{-d_m t} > 0,$$

$$A_w(\mathrm{t}) = A_w(0)e^{-(d_m+\delta)t} > 0,$$

$$M_w(\mathrm{t}) = M_w(0)e^{-d_w t} > 0,$$

$$F_w(\mathrm{t}) = F_w(0)e^{-d_w t} > 0.$$

Therefore, the solution of the model equation given by system (4) with positive initial data will remain positive for all $t > 0$.

3.4 Basic Reproduction Number $\square_0$

According to Dietz (1993), (Mayengo, Kgosimore, & Chakraverty, 2020), the basic reproduction number is the average number of secondary infections caused by a single infected individual in a population that is fully susceptible during their infectious phase. The basic reproduction number, as determined by the next generation matrix method. Exploring the method of the next generation matrix, we obtained the value of the basic reproduction number $\square_0$ as $\rho\left(FV^{-1}\right)$ by setting $\frac{dX}{dt}=(F-V)X-JX$, where $X=\left(E_h, I_{ha}, I_{hc}, D_R, F_i\right)$.this yield

$$f(X)=\begin{bmatrix} \frac{\beta\vartheta_h F_i S_h}{N_h} \\ 0 \\ 0 \\ 0 \\ \frac{\beta\vartheta_v(I_{ha}+I_{hc}+D_R)F}{N_v} \end{bmatrix}, \qquad F=\begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \frac{\partial F_1}{\partial x_3} & \frac{\partial F_1}{\partial x_4} & \frac{\partial F_1}{\partial x_5} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \frac{\partial F_2}{\partial x_3} & \frac{\partial F_2}{\partial x_4} & \frac{\partial F_2}{\partial x_5} \\ \frac{\partial F_3}{\partial x_1} & \frac{\partial F_3}{\partial x_2} & \frac{\partial F_3}{\partial x_3} & \frac{\partial F_3}{\partial x_4} & \frac{\partial F_3}{\partial x_5} \\ \frac{\partial F_4}{\partial x_1} & \frac{\partial F_4}{\partial x_2} & \frac{\partial F_4}{\partial x_3} & \frac{\partial F_4}{\partial x_4} & \frac{\partial F_4}{\partial x_5} \\ \frac{\partial F_5}{\partial x_1} & \frac{\partial F_5}{\partial x_2} & \frac{\partial F_5}{\partial x_3} & \frac{\partial F_5}{\partial x_4} & \frac{\partial F_5}{\partial x_5} \end{bmatrix}$$

$$F=\begin{bmatrix} 0 & 0 & 0 & 0 & \frac{\beta\vartheta_h S_h}{N_h} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\beta\vartheta_v F}{N_v} & \frac{\beta\vartheta_v F}{N_v} & \frac{\beta\vartheta_v F}{N_v} & 0 \end{bmatrix}$$

At disease free $S_h=\frac{\Lambda_h+\varphi(1-p)}{\mu_h}, F=\frac{\mathrm{b}(1-\eta)\delta}{(d_m+\xi-\delta)(d_m+\gamma_2)}, N_h=\frac{\Lambda_h+\varphi}{\mu_h}, N_v=\frac{b(1-\eta)\delta b}{d_m(d_m+\gamma_2)(d_m+\xi+\delta)}$

Evaluating F at disease-free, we

$$F=\begin{bmatrix} 0 & 0 & 0 & 0 & \frac{\left(\Lambda_h+\varphi(1-p)\right)\beta\vartheta_h}{\Lambda_h+\varphi} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & d_m & d_m & d_m & 0 \end{bmatrix}$$

$$
v(X) = \begin{bmatrix} (\alpha_h + \mu_h)E_h \\ -\alpha_h E_h - \varphi p + (m + k + \mu_h + \pi)I_{ha} \\ -kI_{ha} + (\mu_h + \gamma)I_{hc} \\ -mI_{ha} + (n + \mu_h)D_R \\ d_m F_i \end{bmatrix}, \quad V = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \frac{\partial F_1}{\partial x_3} & \frac{\partial F_1}{\partial x_4} & \frac{\partial F_1}{\partial x_5} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \frac{\partial F_2}{\partial x_3} & \frac{\partial F_2}{\partial x_4} & \frac{\partial F_2}{\partial x_5} \\ \frac{\partial F_3}{\partial x_1} & \frac{\partial F_3}{\partial x_2} & \frac{\partial F_3}{\partial x_3} & \frac{\partial F_3}{\partial x_4} & \frac{\partial F_3}{\partial x_5} \\ \frac{\partial F_4}{\partial x_1} & \frac{\partial F_4}{\partial x_2} & \frac{\partial F_4}{\partial x_3} & \frac{\partial F_4}{\partial x_4} & \frac{\partial F_4}{\partial x_5} \\ \frac{\partial F_5}{\partial x_1} & \frac{\partial F_5}{\partial x_2} & \frac{\partial F_5}{\partial x_3} & \frac{\partial F_5}{\partial x_4} & \frac{\partial F_5}{\partial x_5} \end{bmatrix}
$$

$$
V = \begin{bmatrix} (\alpha_h + \mu_h) & 0 & 0 & 0 & 0 \\ -\alpha_h & (\mathrm{m+k+}\,\mu_h + \pi) & 0 & 0 & 0 \\ 0 & -k & (\mu_h + \gamma) & 0 & 0 \\ 0 & -m & 0 & (\mathrm{n+}\,\mu_h) & 0 \\ 0 & 0 & 0 & 0 & d_m \end{bmatrix}
$$

$$
V^{-1} = \begin{bmatrix} \frac{1}{(\alpha_h + \mu_h)} & 0 & 0 & 0 & 0 \\ \frac{\alpha_h}{(\alpha_h + \mu_h)(\mathrm{m+k+}\,\mu_h + \pi)} & \frac{1}{(\mathrm{m+k+}\,\mu_h + \pi)} & 0 & 0 & 0 \\ \frac{k\alpha_h}{(\mathrm{n+}\,\mu_h)(\mathrm{m+k+}\,\mu_h + \pi)(\alpha_h + \mu_h)} & \frac{k}{(\gamma + \mu_h)(\mathrm{m+k+}\,\mu_h + \pi)} & \frac{1}{(\mu_h + \gamma)} & 0 & 0 \\ \frac{m\alpha_h}{(\mathrm{n+}\,\mu_h)(\mathrm{m+k+}\,\mu_h + \pi)(\alpha_h + \mu_h)} & \frac{m}{(\mathrm{n+}\,\mu_h)(\mathrm{m+k+}\,\mu_h + \pi)} & 0 & \frac{1}{(\mathrm{n+}\,\mu_h)} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{d_m} \end{bmatrix}
$$

$$
k = FV^{-1}
$$

$$k=\begin{bmatrix} 0 & 0 & 0 & 0 & \dfrac{\left(\Lambda_h+\varphi(1-p)\right)\beta\vartheta_h}{\Lambda_h+\varphi} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & d_m & d_m & d_m & 0 \end{bmatrix}\times$$

$$\begin{bmatrix} \dfrac{1}{(\alpha_h+\mu_h)} & 0 & 0 & 0 & 0 \\ \dfrac{\alpha_h}{(\alpha_h+\mu_h)(\mathrm{m}+\mathrm{k}+\mu_h+\pi)} & \dfrac{1}{(\mathrm{m}+\mathrm{k}+\mu_h+\pi)} & 0 & 0 & 0 \\ \dfrac{k\alpha_h}{(\mathrm{n}+\mu_h)(\mathrm{m}+\mathrm{k}+\mu_h+\pi)(\alpha_h+\mu_h)} & \dfrac{k}{(\gamma+\mu_h)(\mathrm{m}+\mathrm{k}+\mu_h+\pi)} & \dfrac{1}{(\mu_h+\gamma)} & 0 & 0 \\ \dfrac{m\alpha_h}{(\mathrm{n}+\mu_h)(\mathrm{m}+\mathrm{k}+\mu_h+\pi)(\alpha_h+\mu_h)} & \dfrac{m}{(\mathrm{n}+\mu_h)(\mathrm{m}+\mathrm{k}+\mu_h+\pi)} & 0 & \dfrac{1}{(\mathrm{n}+\mu_h)} & 0 \\ 0 & 0 & 0 & 0 & \dfrac{1}{d_m} \end{bmatrix}$$

$$k=\begin{bmatrix} 0 & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & d_m & d_m & d_m & 0 \end{bmatrix}\times\begin{bmatrix} A & 0 & 0 & 0 & 0 \\ B & E & 0 & 0 & 0 \\ C & F & H & 0 & 0 \\ D & G & 0 & I & 0 \\ 0 & 0 & 0 & 0 & J \end{bmatrix}$$

Where,

$$a=\frac{\left(\Lambda_h+\varphi(1-p)\right)\beta\vartheta_h}{\Lambda_h+\varphi},A=\frac{1}{(\alpha_h+\mu_h)},B=\frac{\alpha_h}{(\alpha_h+\mu_h)(\mathrm{m}+\mathrm{k}+\mu_h+\pi)},$$

$$C=\frac{k\alpha_h}{(\mathrm{n}+\mu_h)(\mathrm{m}+\mathrm{k}+\mu_h+\pi)(\alpha_h+\mu_h)},D=\frac{m\alpha_h}{(\mathrm{n}+\mu_h)(\mathrm{m}+\mathrm{k}+\mu_h+\pi)(\alpha_h+\mu_h)},$$

$$E=\frac{1}{(\mathrm{m}+\mathrm{k}+\mu_h+\pi)},F=\frac{k}{(\gamma+\mu_h)(\mathrm{m}+\mathrm{k}+\mu_h+\pi)},G=\frac{m}{(\mathrm{n}+\mu_h)(\mathrm{m}+\mathrm{k}+\mu_h+\pi)},$$

$$H=\frac{1}{(\mu_h+\gamma)},I=\frac{1}{(\mathrm{n}+\mu_h)},J=\frac{1}{d_m}$$

$$k = \begin{bmatrix} 0 & 0 & 0 & 0 & aJ \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \sqrt{d_m(B+C+D)} & d_m(\mathrm{F}+G+H) & Hd_m & Id_m & 0 \end{bmatrix}$$

$$eigenvalues: \sqrt{aJd_m(B+C+D)}, -\sqrt{d_m(\mathrm{F}+G+H)}, 0$$

$$\square_0 = \sqrt{aJd_m(B+C+D)}$$

$$\therefore \square_0 = \sqrt{\frac{\Lambda_h[\Lambda_h + \varphi(1-p)\beta\alpha_h\vartheta_h](\mathrm{k}+m+n+\mu_h)}{(\mathrm{n}+\mu_h)(\Lambda_h+\varphi)(\alpha_h+\mu_h)(\mathrm{m}+\mathrm{k}+\mu_h+\pi)}}$$

3.5 Local Stability of the Disease-Free Equilibrium Point

Theorem 3.2: The disease-free equilibrium point $\left(E_0\right)$, of the model equation given by system (4) is locally asymptotically stable in the region D.

Proof:

From equations (4), Let

$$f_1 = \Lambda_h + \varphi_1(1-p) - \frac{\beta\vartheta_h F_i S_h}{N_h} - \mu_h S_h + \omega R$$

$$f_2 = \frac{\beta\vartheta_h F_i S_h}{N_h} - (\alpha_h + \mu_h)E_h$$

$$f_3 = \alpha_h E_h + \varphi_1 p - (m + k_1 + \mu_h + \varphi n)I_{ha}$$

$$f_4 = kI_{ha} - (\mu_h + \gamma)I_{hc}$$

$$f_5 = mI_{ha} - (n_1 + \mu_h)D_R$$

$$f_6 = \varphi n I_{ha} + \gamma I_{hc} + n_1 D_R - (\omega + \mu_h)R$$

$$f_7 = b_1 - (d_m + \xi)A - \delta A$$

$$f_8 = \eta\delta A - M(\gamma_1 + d_m) = 0$$

$$f_9 = (1-\eta)\delta A - d_m F - \gamma_2 F - \frac{\beta\vartheta_v(I_{ha} + \theta I_{hc} + \theta_1 D_R)F}{N_v}$$

$$f_{10} = \frac{\beta\vartheta_v(I_{ha} + \theta I_{hc} + \theta_1 D_R)F}{N_v} - d_m F_i$$

$$f_{11} = bF_w - (d_w + \delta)A_w$$

$$f_{12} = \eta\delta A_w + \gamma_1 M - d_w M_w$$

$$f_{13} = (1-\eta)\delta A_w + \gamma_2 F - d_w F_w,$$

Thus, the Jacobian matrix of the model equation (4), is given by

$$J=\begin{bmatrix}
-(\lambda_h+\mu_h) & 0 & 0 & 0 & 0 & \omega & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{\beta\vartheta_h F_i}{N_h} & -(\alpha_h+\mu_h) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \alpha_h & -(m+k_1+\mu_h+n\varphi) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & k & -(\gamma+\mu_h) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & m & 0 & -(n_1+\mu_h) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & n\varphi & \gamma & n_1 & -(\omega+\mu_h) & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & -(d_m+\xi+\delta) & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & \eta\delta & -(d_m+\gamma_1) & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & (1-\eta)\delta & 0 & -\left(d_m+\gamma_1+\frac{\beta\vartheta_v(I_{ha}+I_{hc}+D_R)}{N_v}\right) & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\beta\vartheta_v(I_{ha}+I_{hc}+D_R)}{N_v} & -d_m & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -(d_w+\delta) & 0 & b\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma_1 & 0 & 0 & \eta\delta & -d_w & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma_1 & 0 & (1-\eta)\delta & 0 & -d_w
\end{bmatrix} \quad (15)$$

The Jacobean evaluated at the DFE, is given by

$$J(E_0)=\begin{bmatrix}
-\mu_h & 0 & 0 & 0 & 0 & \omega & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -p_1 & 0 & 0 & 0 & 0 & 0 & -p_{11} & 0 & 0 & 0 & 0 & 0\\
0 & \alpha_h & -p_2 & 0 & 0 & 0 & 0 & p_{11} & 0 & 0 & 0 & 0 & 0\\
0 & 0 & k & -p_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & m & 0 & -p_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \pi & \gamma & n & -p_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & -p_6 & 0 & b & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & \eta\delta & -p_7 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -p_{10} & -p_{10} & -p_{10} & 0 & p_{12} & 0 & -p_8 & 0 & 0 & 0 & 0\\
0 & 0 & p_{10} & p_{10} & p_{10} & 0 & 0 & 0 & 0 & -d_m & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -p_9 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma_1 & 0 & 0 & \eta\delta & -p_{10} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma_2 & 0 & p_{12} & 0 & -d_w
\end{bmatrix} \quad (16)$$ Where

$$p_1=(\alpha_h+\mu_h),\ p_2=m+k_1+\mu_h+\varphi n,\ p_3=(\mu_h+\gamma),\ p_4=(n_1+\mu_h),\ p_5=(\omega+\mu_h),\ p_6=(d_m+\xi+\delta)$$

$$p_7=(\gamma_1+d_m),\ p_8=(d_m+\gamma_2),\ p_9=(\delta+d_w),\ p_{10}=\frac{\beta\vartheta_v F}{N_v},\ p_{11}=\frac{\beta\vartheta_h S_h}{N_h},\ p_{12}=(1-\eta)\delta$$

We need to show that all eigenvalues of (16) are negative. As the first, sixth, tenth, eleventh, twelveth and last columns contains only the diagonal term which forms the eigenvalue, $-\mu_h,-p_5,-d_m,-p_9,\ -p_{10}$ and $-d_w$, the other seven eigenvalues can be obtained from the sub-matrix $J_1(E_0)$. Hence, we have

$$J_1(E_0)=\begin{pmatrix} -p_1 & 0 & 0 & 0 & 0 & -p_{11} & 0 \\ \alpha_h & -p_2 & 0 & 0 & 0 & p_{11} & 0 \\ 0 & k_1 & -p_3 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & -p_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -p_6 & 0 & b \\ 0 & 0 & 0 & 0 & \eta\delta & -p_7 & 0 \\ 0 & -p_{10} & -p_{10} & -p_{10} & p_{12} & 0 & -p_8 \end{pmatrix} \tag{17}$$

Then (17) can be reduced using Gaussian elimination method, the reduced matrix $J_2(E_0)$ is given as:

$$J_2(E_0)=\begin{pmatrix} -p_1 & 0 & 0 & 0 & 0 & -p_{11} & 0 \\ 0 & -p_2 & 0 & 0 & 0 & \dfrac{p_{11}(p_1-\alpha_h)}{p_1} & 0 \\ 0 & 0 & -p_3 & 0 & 0 & \dfrac{k_1 p_{11}(p_1-\alpha_h)}{p_1 p_2} & 0 \\ 0 & 0 & 0 & -p_4 & 0 & \dfrac{m p_{11}(p_1-\alpha_h)}{p_1 p_2} & 0 \\ 0 & 0 & 0 & 0 & -p_6 & 0 & b \\ 0 & 0 & 0 & 0 & 0 & -p_7 & \dfrac{\eta\delta b}{p_6} \\ 0 & 0 & 0 & 0 & 0 & 0 & -p_m \end{pmatrix} \tag{18}$$

Where $p_m=\dfrac{1}{p_3p_2p_1p_4p_7p_6}\begin{pmatrix} -bp_2p_1p_3p_4p_7p_{12}+p_2p_1p_3p_4p_6p_7p_8+b\delta\eta kp_1p_4p_{10}p_{11}-b\delta\eta kp_4p_{10}p_{11}\alpha_h \\ +b\delta\eta mp_1p_3p_{10}p_{11}-b\delta\eta mp_3p_{10}p_{11}\alpha_h+b\delta\eta p_1p_3p_4p_{10}p_{11}-b\delta\eta p_3p_4p_{10}p_{11}\alpha_h \end{pmatrix}$ now,

taking the product of the diagonal elements of matrix (4.63) gives the eigenvalues as:

$$\left.\begin{aligned} &m_1=-p_1=-(\alpha_h+\mu_h), m_2=-p_2=-(m+k_1+\mu_h+\varphi n), m_3=-p_3=-(\mu_h+\gamma), \\ &m_4=-p_4=-(n_1+\mu_h), m_5=-p_6=-(d_m+\xi+\delta), m_6=-p_7=-(\gamma_1+d_m), m_7=-p_m. \end{aligned}\right\} \quad (19)$$

m_7 is true if

$$\begin{pmatrix} bp_2p_1p_3p_4p_7p_{12}+b\delta\eta kp_4p_{10}p_{11}\alpha_h \\ +b\delta\eta mp_3p_{10}p_{11}\alpha_h+b\delta\eta p_3p_4p_{10}p_{11}\alpha_h \end{pmatrix}<\begin{pmatrix} p_2p_1p_3p_4p_6p_7p_8+b\delta\eta kp_1p_4p_{10}p_{11} \\ +b\delta\eta mp_1p_3p_{10}p_{11}+b\delta\eta p_1p_3p_4p_{10}p_{11} \end{pmatrix}$$

from equation (19), $m_1, m_2, m_3, m_4, m_5, m_6, m_7<0$ this proves Theorem 4.3 as required. Thus, the disease-free equilibrium point is locally asymptotically stable.

3.6 Global stability of disease-free equilibrium point

The result in Theorem 4.3 implies that the LF can be eliminated from the population if the initial size of the populations of the model given by system (4) is in the basin of attraction of the DFE (E_0).To ensure that the elimination of LF is independent of the initial sizes of the populations of the model, it is necessary to show that the DFE is globally-asymptotically stable (GAS). This will be established using the method by Castillo-Chavez (2002). We rewrite the model equation given by system (4) in the following form:

$$\frac{dX}{dt}=K(X,Z),$$

$$\frac{dZ}{dt}=G(X,Z), G(X,0)=0$$

Where $X = (S_h, R, A, M, F, A_w, M_w, F_w) \in \mathbb{R}_+^8$ represents the subpopulation of uninfected individuals and $Z = (E_h, I_{ha}, I_{hc}, D_R, F_i) \in \mathbb{R}_+^5$ represents the subpopulation of infected individuals. Suppose $E_0 = (X^*, 0)$ represents the disease-free equilibrium point of the system (3.8) - (3.20). For E_0 of the model to be globally asymptotically stable, the following conditions (H_1) and (H_2) must be satisfied:

$(H_1): \frac{dX}{dt} = K(X,0)$, E_0 is globally asymptotically stable.

$(H_1): \frac{dZ}{dt} = AZ - G(X,Z), G(X,Z) \geq 0$ for all $(X,Z) \in \mathbb{R}$, where A=$D_z G(X,0)Z$ is an M-matrix (the off diagonal elements of A are nonnegative).

Theorem 3.3: The equilibrium point of the model given by $\mathrm{E}_0 = (X^*, 0)$ is globally asymptotically stable if $\mathrm{R}_0 < 1$ and conditions (H_1) and (H_2) is satisfied.

Proof:

Let

$E_0 = (X^*, 0)$ and

$$X^* = \left\{ \begin{array}{l} \frac{\Lambda_h + \varphi_1(1-p)}{\mu_h}, \frac{b_1}{(d_m + \xi - \delta)}, 0, \frac{\eta\delta b_1}{(\gamma_1 + d_m)(d_m + \xi + \delta)}, \frac{b_1(1-\eta)\delta}{(d_m + \xi - \delta)(d_m + \gamma_2)}, \\ \frac{b^2\delta\gamma_2(1-\eta)}{(\gamma_2 + d_m)(\delta + \xi + d_m)(d_w^2 + \delta d_w + b\delta(\eta - 1))}, \frac{\eta\delta(d_m^2 + (\gamma_1 + \xi + \delta)d_m + \delta\gamma_1 + \xi\gamma_1)A_w + b\eta\delta\gamma_1}{d_w(\gamma_1 + d_m)(d_m + \xi + \delta)}, \\ \frac{b\delta\gamma_2(1-\eta)(\delta + d_w)}{(\gamma_2 + d_m)(\delta + \xi + d_m)(d_w^2 + \delta d_w + b\delta(\eta - 1))} \end{array} \right\}$$

Now we verify the conditions (H_1) and (H_2) as follows:

$$K(X,Z) = \frac{dK(X,Z)}{dt} = \begin{bmatrix} \frac{dS_h}{dt} = \Lambda_h + \varphi(1-p) - \frac{\beta\vartheta_h F_i S_h}{N_h} - \mu_h S_h + \omega R \\ \pi I_{ha} + m I_{hc} + n D_R - (\mu_h + \omega)R \\ b_1 - (d_m + \xi + \delta)A \\ \eta\delta A - (\gamma_1 + d_m)M \\ (1-\eta)\delta A - (\gamma_2 + d_m)F - \frac{\beta\vartheta_v(I_{ha} + I_{hc} + D_R)F}{N_v} \\ bF_w - (d_w + \delta)A_w \\ \eta\delta A_w + \gamma_1 M - d_w M_w \\ (1-\eta)\delta A_w + \gamma_2 F - d_w F_w \end{bmatrix} \quad (20)$$

$$K(X,0)=\begin{bmatrix} \frac{dS_h}{dt}=\Lambda_h+\varphi(1-p)-\mu_h S_h \\ \frac{dE_h}{dt}=0 \\ \frac{dA}{dt}=b_1-(d_m+\xi+\delta)A \\ \frac{dM}{dt}=\eta\delta A-(\gamma_1+d_m)M \\ \frac{dF}{dt}=(1-\eta)\delta A-(\gamma_2+d_m)F \\ \frac{dA_w}{dt}=bF_w-(d_w+\delta)A_w \\ \frac{dM_w}{dt}=\eta\delta A_w+\gamma_1 M-d_w M_w \\ \frac{dF_w}{dt}=(1-\eta)\delta A_w+\gamma_2 F-d_w F_w \end{bmatrix} \quad (21)$$

Then

$$S_h(t)=\Lambda_h+\varphi_1(1-p)-\mu_h S_h$$

$$S_h(t)+\mu_h S_h=\Lambda_h+\varphi_1(1-p)$$

$$\frac{dS_h}{dt}+\mu_h S_h=\left[\Lambda_h+\varphi_1(1-p)\right] \quad (22)$$

Using the linear method: the Integrating factor (I.F) of equation (22) is $e^{\mu_h t}$

$$e^{\mu_h t}\frac{dS_h}{dt}+e^{\mu_h t}\cdot\mu_h S_h=e^{\mu_h t}\cdot\left[\Lambda_h+\varphi_1(1-p)\right]$$

$$\int\frac{d}{dt}\left(S_h e^{\mu_h t}\right)dt=\int e^{\mu_h t}\cdot\left[\Lambda_h+\varphi_1(1-p)\right]dt \quad (23)$$

$$S_h e^{\mu_h t}=\int e^{\mu_h t}\cdot\left[\Lambda_h+\varphi_1(1-p)\right]dt \quad (24)$$

Solving (24), RHS

$$\int e^{\mu_h t}\cdot\left[\Lambda_h+\varphi_1(1-p)\right]dt$$

Let

$$w=\mu_h t$$

$$\frac{dw}{dt}=\mu_h$$

$$\frac{dw}{\mu_h}=dt$$

$$\int e^{w}\cdot\left[\Lambda_h+\varphi_1(1-p)\right]\frac{dw}{\mu_h} \quad (25)$$

$$=\frac{\left[\Lambda_h+\varphi_1(1-p)\right]}{\mu_h}\int e^{w}dw$$

$$\frac{\left[\Lambda_h+\varphi_1(1-p)\right]}{\mu_h}e^{\mu_h t}+C \quad (26)$$

Taking (26) into (24), we have

$$S_h e^{\mu_h t} = \frac{\left[\Lambda_h + \varphi_1(1-p)\right]}{\mu_h} e^{\mu_h t} + C$$

$$S_h(t) = \frac{\Lambda_h + \varphi_1(1-p)}{\mu_h} + Ce^{-\mu_h t} \tag{27}$$

At $t = 0, S_h(0) = \dfrac{\Lambda_h + \varphi_1(1-p)}{\mu_h} + C$

$$\therefore S_h(0) - \frac{\Lambda_h + \varphi_1(1-p)}{\mu_h} = C \tag{28}$$

Taking equation (28) into (27), gives us

$$S_h(t) = S_h(0)e^{-\mu_h t} + \frac{\Lambda_h + \varphi_1(1-p)}{\mu_h}\left(1 - e^{-\mu_h t}\right)$$

As $t \to \infty,\ S_h(t) \to \dfrac{\Lambda_h + \varphi_1(1-p)}{\mu_h}$ $\quad (29)$

Also taking the third component of (21), we have

$$\frac{dA}{dt} = b_1 - (d_m + \xi + \delta)A$$

$$\frac{dA}{dt} + (d_m + \xi + \delta)A = b_1 \tag{22}$$

Solving equation (22) using the linear method: the Integrating factor (I.F) is $e^{(d_m+\xi+\delta)t}$

$$\frac{d}{dt}\left(Ae^{(d_m+\xi+\delta)t}\right) = b_1 e^{(d_m+\xi+\delta)t}$$

$$\int\left(\frac{d}{dt}\left(Ae^{(d_m+\xi+\delta)t}\right)\right)dt = \int b_1 e^{(d_m+\xi+\delta)t}dt$$

$$Ae^{(d_m+\xi+\delta)t} = \frac{b_1 e^{(d_m+\xi+\delta)t}}{\left(d_m + \xi + \delta\right)} + k_d \tag{23}$$

Dividing equation (23) by $e^{(d_m+\xi+\delta)t}$, we have

$$A(t) = \frac{b_1}{\left(d_m + \xi + \delta\right)} + k_d e^{-(d_m+\xi+\delta)t}$$

As $t \to \infty,\ A \to \dfrac{b_1}{\left(d_m + \xi + \delta\right)}$ $\quad (24)$

Similarly, solving the third component of equation (24), we have

$$\frac{dM}{dt} = \eta\delta A - (\gamma_1 + d_m)M \tag{25}$$

Substituting equation (24) in equation (25), we have

$$\frac{dM}{dt} + (\gamma_1 + d_m)M = \frac{\eta\delta b_1}{\left(d_m + \xi + \delta\right)} \tag{26}$$

Solving equation (26) using the linear method: the Integrating factor (I.F) is $e^{(\gamma_1+d_m)t}$

$$\frac{d}{dt}\left(Me^{(\gamma_1+d_m)t}\right) = \frac{\eta\delta b_1 e^{(\gamma_1+d_m)t}}{\left(d_m + \xi + \delta\right)}$$

$$\int \frac{d}{dt}\left(Me^{(\gamma_1+d_m)t}\right)dt = \frac{\eta\delta b_1}{(d_m+\xi+\delta)}\int e^{(\gamma_1+d_m)t}dt$$

$$Me^{(\gamma_1+d_m)t} = \frac{\eta\delta b_1 e^{(\gamma_1+d_m)t}}{(d_m+\xi+\delta)(\gamma_1+d_m)} + k_e \qquad (27)$$

Dividing equation (27) through by $e^{(\gamma_1+d_m)t}$

$$M(t) = \frac{\eta\delta b_1 e^{(\gamma_1+d_m)t}}{(d_m+\xi+\delta)(\gamma_1+d_m)} + k_e e^{-(\gamma_1+d_m)t}$$

As $t \to 0,\ M(t) \to \frac{\eta\delta b_1}{(d_m+\xi+\delta)(\gamma_1+d_m)}$ (28)

$$\text{as}\, t\to\infty, \qquad F \to \frac{b_1(1-\eta)\delta}{(d_m+\xi+\delta)(d_m+\gamma_2)}, A_w \to \frac{bF_w^*}{(d_w+\delta)},$$

Similarly,

$$M_w \to \frac{\eta\delta b\left(d_m+\xi+\delta\right)\left[b\delta\gamma_2(1-\eta)(\gamma_1+d_m)+\gamma_1(\gamma_2+d_m)\left(d_w^2+\delta d_w-b\delta+b\delta\eta\right)\right]}{d_w(\gamma_2+d_m)(\delta+\xi+d_m)\left(d_w^2+\delta d_w-b\delta+b\delta\eta\right)(\gamma_1+d_m)(d_m+\xi+\delta)},.$$

$$F_w \to \frac{b\delta\gamma_2(1-\eta))(\delta+d_w)}{(\gamma_2+d_m)(\delta+\xi+d_m)\left(d_w^2+\delta d_w+b\delta(\eta-1)\right)}\ .$$

Hence, X^* is globally asymptotically stable meaning that the first condition (H_1) is satisfied.

For condition (H_2), we have

$$\frac{dZ}{dt} = G_i(X,Z) = \begin{pmatrix} \frac{\beta\vartheta_h F_i S_h}{N_h} - (\alpha_h+\mu_h)E_h \\ \alpha_h E_h - (m+k+\mu_h+\pi)I_{ha} \\ kI_{ha} - (\mu_h+\gamma)I_{hc} \\ mI_{ha} - (n+\mu_h)D_R \\ \frac{\beta\vartheta_v(I_{ha}+\theta I_{hc}+\theta_1 D_R)F}{N_v} - d_m F_i \end{pmatrix} \qquad (29)$$

From equation (29), it is clear that

$$G(X,0) = \begin{bmatrix} 0\\0\\0\\0\\0 \end{bmatrix} = 0$$

Furthermore, $G(X,Z) = AZ - \hat{G}(X,Z),\ \hat{G}(X,Z) = AZ - G(X,Z)$

With $A = D_Z\left(X^*,0\right)$ is an M-matrix (the off diagonal elements of A are nonnegative). Now let

$$\frac{dE_h}{dt} = G_1 = \frac{\beta \vartheta_h F_i S_h}{N_h} - (\alpha_h + \mu_h) E_h$$

$$\frac{dI_{ha}}{dt} = G_2 = \alpha_h E - (m + k_1 + n\varphi + \mu_h) I_{ha}$$

$$\frac{dI_{hc}}{dt} = G_3 = k_1 I_{ha} - (\gamma + \mu_h) I_{hc}$$

$$\frac{dD_R}{dt} = G_4 = m I_{ha} - (n_1 + \mu_h) D_R$$

$$\frac{dF_i}{dt} = G_5 = \frac{\beta \vartheta_v (I_{ha} + I_{hc} + D_R) F}{N_v} - d_m F_i$$

Therefore,

$$A = \begin{pmatrix} \frac{\partial G_1(E_0)}{\partial E_h} & \frac{\partial G_1(E_0)}{\partial I_{ha}} & \frac{\partial G_1(E_0)}{\partial I_{hc}} & \frac{\partial G_1(E_0)}{\partial D_R} & \frac{\partial G_1(E_0)}{\partial F_i} \\ \frac{\partial G_2(E_0)}{\partial E_h} & \frac{\partial G_2(E_0)}{\partial I_{ha}} & \frac{\partial G_2(E_0)}{\partial I_{hc}} & \frac{\partial G_2(E_0)}{\partial D_R} & \frac{\partial G_2(E_0)}{\partial F_i} \\ \frac{\partial G_3(E_0)}{\partial E_h} & \frac{\partial G_3(E_0)}{\partial I_{ha}} & \frac{\partial G_3(E_0)}{\partial I_{hc}} & \frac{\partial G_3(E_0)}{\partial D_R} & \frac{\partial G_3(E_0)}{\partial F_i} \\ \frac{\partial G_4(E_0)}{\partial E_h} & \frac{\partial G_4(E_0)}{\partial I_{ha}} & \frac{\partial G_4(E_0)}{\partial I_{hc}} & \frac{\partial G_4(E_0)}{\partial D_R} & \frac{\partial G_4(E_0)}{\partial F_i} \\ \frac{\partial G_5(E_0)}{\partial E_h} & \frac{\partial G_5(E_0)}{\partial I_{ha}} & \frac{\partial G_5(E_0)}{\partial I_{hc}} & \frac{\partial G_5(E_0)}{\partial D_R} & \frac{\partial G_5(E_0)}{\partial F_i} \end{pmatrix}$$

$$A = \begin{pmatrix} -(\alpha_h + \mu_h) & 0 & 0 & 0 & \frac{\beta \vartheta_h [\Lambda_h + \varphi_1 (1-p)]}{\Lambda_h + \varphi_1} \\ \alpha_h & -(\mathrm{m} + k_1 + \mu_h + n\varphi) & 0 & 0 & 0 \\ 0 & k_1 & -(\mu_h + \gamma) & 0 & 0 \\ 0 & m & 0 & -(\mu_h + n) & 0 \\ 0 & \frac{\beta \vartheta_v d_m}{b_1} & \frac{\beta \vartheta_v d_m}{b_1} & \frac{\beta \vartheta_v d_m}{b_1} & -d_m \end{pmatrix} \quad (30)$$

$$\hat{G}(X,Z)=AZ-G(X,Z) = \begin{pmatrix} -(\alpha_h+\mu_h) & 0 & 0 & 0 & \dfrac{\beta\vartheta_h S_h^*}{N_h} \\ \alpha_h & -(\mathrm{m}+k_1+\mu_h+n\varphi) & 0 & 0 & 0 \\ 0 & k_1 & -(\mu_h+\gamma) & 0 & 0 \\ 0 & m & 0 & -(\mu_h+n) & 0 \\ 0 & \dfrac{\beta\vartheta_v F^*}{N_v^*} & \dfrac{\beta\vartheta_v \theta F^*}{N_v^*} & \dfrac{\beta\vartheta_v \theta_1 F^*}{N_v^*} & -d_m \end{pmatrix}\begin{pmatrix} E_h \\ I_{ha} \\ I_{hc} \\ D_R \\ F_i \end{pmatrix} - \begin{pmatrix} \dfrac{\beta\vartheta_h F_i S_h}{N_h}-(\alpha_h+\mu_h)E_h \\ \alpha_h E_h-(m+k_1+\mu_h+n\varphi)I_{ha} \\ k_1 I_{ha}-(\mu_h+\gamma)I_{hc} \\ mI_{hc}-(n_1+\mu_h)D_R \\ \dfrac{\beta\vartheta_v(I_{ha}+\theta I_{hc}+\theta_1 D_R)F}{N_v}-d_m F_i \end{pmatrix}$$

Thus,

$$\hat{G}(X,Z)=\begin{pmatrix} \hat{G}_1(X,Z) \\ \hat{G}_2(X,Z) \\ \hat{G}_3(X,Z) \\ \hat{G}_4(X,Z) \\ \hat{G}_5(X,Z) \end{pmatrix} = \begin{pmatrix} \dfrac{\beta\vartheta_h F_i}{N_h}\left(S_h^*-S_h\right) \\ 0 \\ 0 \\ 0 \\ \dfrac{\left(\beta\vartheta_v\left(I_{ha}+\theta I_{hc}+\theta_1 D_R\right)\right)}{N_v}\left(F^*-F\right) \end{pmatrix} \qquad (31)$$

From equation (31), it is clear that $\hat{G}(X,Z)\geq 0$, since $0\leq S_h\leq S_h^*$ and $0\leq F\leq F^*$. Hence, (H_2) have been met. Therefore, the DFE of the modified model given by E_0 is globally asymptotically stable.

4.0 DISCUSSIONS

For Lymphatic Filariasis, the local asymptotic stability of the DFE implies that:

If the basic reproduction number R0 is less than 1, any small outbreak of the disease will not result in a large epidemic. Instead, the disease will naturally die out over time.

Control strategies such as vector control (using chemicals or Wolbachia-infected mosquitoes) and treatment can reduce R0 to less than 1, making the DFE locally asymptotically stable. This means that after interventions, the disease will tend to fade away, provided the initial number of infections is low.

When the DFE of Lymphatic Filariasis is locally asymptotically stable, it means that if the disease is nearly eradicated (i.e., there are only a few cases), it will eventually die out completely, and the population will return to a state without the disease, assuming no large new outbreaks occur. This stability is crucial for understanding how interventions (like medication or mosquito control) can lead to long-term disease elimination **as in section 3.4**

5.0 CONCLUSIONS

In this paper, we developed a mathematical model which incorporated some important factors

That plays significant role in the transmission dynamics and control of Lymphatic filariasis. These are: chemical control, biological control. We obtained the basic reproduction numbers, R_0 . Our analysis reveals that the disease can be control if the basic reproduction number, R_0 is less than one regardless of the initial population profile. Thus, every effort must be put in place by all concerned to prevent the disease by reducing R_0 strictly less than unity.

Finally, there is need for further research work on the optimal control strategy on the use of chemicals and biological control on the transmission dynamics of Lymphatic filariasis disease

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