International Journal of Mathematics and Computer Research
ISSN: 2320-7167
Volume 13 Issue 01 January 2025, Page no. – 4762-4772
Index Copernicus ICV: 57.55, Impact Factor: 8.316
DOI: 10.47191/ijmcr/v13i1.07

# A SURVEY OF DIMENSION OF POSETS

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**Abstract.** In 1930, E. Szpilrajn proved that any order relation on a set $X$ can be extended to linear order on $X$. It also follows that order relation is the intersection of its linear extensions. In 1941 Dushnik and Miller introduced the concept of dimension of a poset $(X, P)$, denoted $Dim(X, P)$, as the smallest positive integer $n$ for which there exist linear extensions $L_1, L_2, ..., L_n$ of $P$ so that $P = L_1 \cap L_2 \cap ... \cap L_n$. In this paper, we do literature survey of the research articles on dimension theory of posets. This survey provides an overview of key concepts, results, and challenges in dimension theory of posets. The survey also examines advanced topics like fractional dimension, geometric representations, and topological considerations. Open problems and future research directions, emphasizing the interdisciplinary potential of dimension theory in mathematics and computer science.

*Key words and Phrases*: Poset, Chain, Linear extension, Dimension.
MSC Classification 2020: 06A05, 06A06.

## 1. INTRODUCTION

In 1930, Szpilrajn [1] proved that any order relation on a set $X$ can be extended to linear order on $X$. It also follows that order relation is the intersection of its linear extensions. If $R$ is family of linear orders (chains) whose intersection is the order relation $\leq$, then $R$ is a realizer of $\leq$. In 1941 Dushnik and Miller [2] defined dimension of an ordered set $P = (x; \leq)$ to be the minimum number of linear extension whose intersection is the ordering $\leq$. In 1951 T. Hiraguchi [5] show that a poset is d-irreducible if it has dimension d and removable of any element lowers its dimension. In 1970 Baker [7] showed that a finite lattice is planar exactly when its dimension does not exceed 2. In 1974 Trotter Jr. [10] proved that the dimension of a crown is $2(n + k)/(k + 2)$. In 1977 Trotter and Moore [15] proved that the dimension of a bounded planar poset is at most two. They also proved the dimension of a planar poset having a greatest lower bound is at most three. In 1989 Schnyder [23] proved that each planar graph has dimension at most three.

## 2. PRELIMINARIES

**Definition 2.1.** *A partially ordered set(or poset) is a set $P$ of elements together with a binary relation $\leq$ on $P$ which is reflexive, antisymmetric and transitive.*

**Definition 2.2.** *A lattice is a poset in which every pair of elements has a least upper bound and a greatest lower bound.*

**Definition 2.3.** *Two elements $a, b \in P$ are comparable if either $a \leq b$ or $b \leq a$. Two elements of $P$ are said to be incomparable if they are not comparable. If $x$ and $y$ are incomparable in $P$ then we denote it by $x \parallel y$.*

**Definition 2.4.** *A chain(or linear order) is a lattice in which any two elements are comparable.*

**Definition 2.5.** *The incomparable pair $< a, b >$ is called a critical pair if $x < b$ implies $x < a$, and $x > a$ implies $x > b$.*

**Definition 2.6.** *Let $(P, \leq)$ be an ordered set. Then $\sqsubseteq$ is called a linear extension of $\leq$ if and only if $\sqsubseteq$ is a total order and it contains $\leq$.*

**Definition 2.7.** *Let $(P, \leq)$ be an ordered set. Then a family $\{\leq_i\}_{i\in I}$ of linear orders $\leq_i$ on $P$ is called a realizer of $\leq$ if and only if $\leq = \bigcap_{i\in I} \leq_i$.*

**Definition 2.8.** *Let $(P, \leq)$ be an ordered set. Then $(P, \leq)$ is of (linear) dimension $k$ and we write $Dim(P, \leq) = k$ if and only if $k \in \mathbb{N}$ is the smallest natural number such that there is a realizer for $\leq$ that has $k$ orders.*

**Definition 2.9.** *The fractional dimension, denoted by $fdim(P)$, and defined as the limit of $t(k)/k$ as $k \to \infty$. Note that $fdim(P)$ is also the infimum of the set $\{t(k)/k\}$, and so, in particular, $fdim(P) \leq t(1)/1 = dim(P)$.*

**Definition 2.10.** *Poset $P = (X, \leq)$ is called an interval order if it can be represented as a set of intervals of the real line so that for $x, y \in X$ we have $x < y$ if and only if the interval corresponding to $x$ is entirely to the left of the the interval corresponding to $y$.*

**Definition 2.11.** *The pair $(\mathcal{B}, \tau)$ is a Boolean realizer when for each pair $x$, $y$ of distinct elements of $P$, $x < y$ in $P$ if and only if $\tau(q(x, y, \mathcal{B}) = 1$. The Boolean dimension of $P$, denoted $bdim(P)$, is the least positive integer $d$ for which $P$ has a Boolean realizer $(B, \tau)$ with $|\mathcal{B}| = d$.*

**Definition 2.12.** *A non-empty family $\mathcal{L}$ of partial linear extension's of a poset $P$ is called a local realizer of $P$ if the following two conditions are satisfied:*
*(1) If $x \leq y$ in $P$, there is some $L \in \mathcal{L}$ for which $x \leq y$ in $L$;*
*(2) if $(x, y) \in Inc(P)$, there is some $L \in \mathcal{L}$ for which $x > y$ in $L$.*
*The local dimension of $P$, denoted $ldim(P)$, is defined as $ldim(P) = min\{\mu(\mathcal{L}) : \mathcal{L}$ is a local realizer of $P\}$.*

## 3. Literature survey

In 1930, Szpilrajn [1] proved that any order relation on a set $X$ can be extended to linear order on $X$. It also follows that order relation is the intersection of its linear extensions. If $R$ is family of linear orders (chains) whose intersection is the order relation $\leq$, then $R$ is a realizer of $\leq$.

In 1941 Dushnik and Miller [2] defined dimension of an ordered set $P = (x; \leq)$ to be the minimum number of linear extension whose intersection is the ordering $\leq$. They discusses the properties of partial orders, including their dimensions, the relationship between partial orders and linear orders, and the representation of partial orders using intervals on a linearly ordered set.

In 1948 Horace Komm [3] proved that every finite or denumerable partial order is similar to some subset of $P_n$ (or $P'n$). The dimension of $P_n(E_n)$ is $n$, where $n$ is at least $\aleph_0$. The A-dimension of $P'_n(E_n)$ exists for every finite n, and the A-dimension of $P'_2(E_2)$ is $\aleph_0$. The A-dimension of $P_n(E_n)$ and $P'_n(E_n)$ is c, the power of the continuum, for every finite n. If $M_n$ is a subset of $E_n$ that has at most one point in common with any vertical (except for a denumerable set), then the A-dimension of $P_n(M_n)$ or $P'_n(M_n)$ is at most n. He discusses the properties, dimensions, and relationships of partial orders, including their connections to subsets of Euclidean spaces.

In 1950 R. P. Dilworth [4] proved that the every set of $k + 1$ elements of a partially ordered set P be dependent while at least one set of k elements is independent. Then P is a set sum of k disjoint chains. He also proved that let D be a finite distributive lattice. Let k(a) be the number of distinct elements in D which cover a and let k be the largest of the numbers k(a). Then D is a sublattice of a direct union of k chains and k is the smallest number for which such an imbedding holds.

In 1951 Toshio Hiraguchi [5] proved that let a poset P be decomposable to a sum $\sum_s P_s$ and $\sigma$ an element of $S$ such that $D[P_\sigma] \geq D[P_s]$ for all $s \in S$. If $D[P_\sigma] \geq D[S]$, then the set $P - P_\sigma$ is d-removable. If $D[P_\sigma] < D[S]$, then $P - P^*$ is d-removable, where $P^* = \{x_s | s \in S\}$ is a set whose element $x_s$ is an element selected arbitrarily from the set $P_s$. He also prove that if $D[P] \geq 3$, then $2D[P] \leq n[P]$. In other words, in order to define a poset of dimension $n$ a set of power $2n$ is necessary.

In 1955 Toshio Hiraguchi [6] gave the result for if $P$ be an order defined on set $A$ and $B$ a subset of $A$. Then there exists a right (left) linear extensions of $P$ with respect to $B$, if and only if $B$ is a linear(P) subset of $A$. He also gave the result for let $P$ be an order defined on a set $A$ and $B$ and $B'$ two linear(P) subsets of $A$ such that $B$ is order-disjoint (P) upwards (downwards) to $B'$ Then there exists a linear extension of $P$ which is both right(left) with respect to $B$ and left (right) with respect to $B'$.

In 1970 Baker, et al. [7] proved the result for suppose $\prec$ is a partial order on a finite set $X$ and $\sim$ is its incomparability relation. Then all of the following statements are equivalent:(a) $D(\prec) < 2$. (b) There is a conjugate partial order $\prec^*$ on X. (c) There is a nonseparating linear extension of $\prec$. (d) $(X, \sim)$ is a comparability graph. (e) Every odd $\sim$ -cycle has a triangular chord. (f) $(X, \prec)$ has no comparability cycle. (g) $L(X, \prec)$ has a planar Hasse diagram. (h) $\prec$ is realizable as the partial order of inclusion on a set of intervals in some linear order. (i) $\prec$ satisfies the weak interval condition.

In 1973 Kenneth P. Bogart [8] provides a new proof of Hiraguchi's Theorem, which states that the dimension of a partially ordered set with n elements is at most the greatest integer less than or equal to $\frac{n}{2}$. The key lemma in the new proof is that a partially ordered set either has a cover of rank 0 or a pair of covers with elements of one incomparable with elements of the other. The new proof uses an inductive approach, verifying the theorem for small sets and then reducing larger sets to smaller ones by removing certain structures. In 1973 Kenneth P. Bogart and Willim T. Trotter, Jr. [9] The only "maximal dimensional" partially ordered sets with 2n elements and dimension n, then it is isomorphic to the set of $n-1$ element subsets and element subsets of a set, are the Dushnik-Miller example and the six element chevron ordering. The theory of partially ordered sets of dimension n is not finitely axiomatizable in first order logic, in contrast with the finite axiomatization of distributive lattices of dimension n. They provide a detailed characterization of the maximal dimensional posets, involving an examination of covering pairs and the use of several key lemmas.

In 1974 Willim T. Trotter Jr. [10] introduced a concept of dimension of a crown is $\frac{2(n+k)}{k+2}$. He also proved that for each $n > 3$ there exist infinitely many nonisomorphic irreducible posets of dimension n. In 1974 David Kelly and Ivan Rival [11] proved that every finite lattice is either dismantlable or contains a crown, but not both. A modular lattice of finite length is dismantlable if and only if it has breadth 2 (or equivalently, it contains no crown of order 6). A finite distributive lattice is dismantlable if and only if it is planar.

In 1975 Willim T. Trotter [12] introduces new inequalities involving the dimensions of posets and their subsets, particularly focusing on the relationships between dimension, width, and the presence of maximal elements or antichains. For example, he establishes that $(\dim(X, P) \leq |X - A|)$ where A is an antichain, and $(\dim(X, P) \leq 1 + \text{width}(X - E))$ where E is the set of maximal elements. These inequalities not only extend existing knowledge but also provide a framework for analyzing the structural properties of posets in various configurations. In 1975 Kelly and Rival [13] proved that a finite lattice is planar if and only if it does not contain any lattice in $L$ as a subposet. Moreover, $L$ is the minimum such list; that is, if $F$ is a set of lattices such that the first assertion remains true with $L$ replaced by $F$, then $L \subseteq F$.

In 1977 Trotter and Moore [15] proved that the dimension of a bounded planar poset is at most two. They proved the dimension of a planar poset having a greatest lower bound is at most three. They also proved that the dimension of a tree is at most three. They gave a new concept called "stable dimension" is introduced as a generalization of the dimension of a partially ordered set (poset). The main theorem shows that the stable dimension is equal to the maximum number of elements in a pair of antichains (sets of incomparable elements) of the poset, where one set lies above the other. The stable dimension can be used to find improved bounds on the dimension of the poset compared to previous results.

In 1978 I. Rabinovitch [16] investigated the structure of semiorders and interval orders, and gave various characterizations. Then by using these ideas he proved that the dimension of a semiorder is at most 3, characterize semiorders of dimension 3 and height 2, and proved Hiraguchi's result, let $(X, P)$ be a poset with $|X| \geq 4$. Then $d(p) \leq \frac{1}{2}|X|$.

In 1981 Babai and Duffus [17] proved that for each positive integer $k$ there is an integer $M(k)$ such that if $L$ is a finite modular lattice with $dim(L) < k$ and the order of the automorphism group $Aut(L)$ is divisible by a prime $p > M(k)$ then there is a cover-preserving embedding of $M_p$ in $L$.

In 1982 Bennett [18] gave necessary and sufficient conditions on a lattice L which guarantee it's being the lattice of faces of the n-dimensional cube. In 1982 Felsner, et al. [19] proved that the decision problem $(3DH2)$ for dimension is equivalent to deciding for the existence of bipartite triangle containment representations $(BTCon)$. This problem then allows a reduction from a class of planar satisfiability problems $(P-3-CON-3-SAT(4))$ which is known to be NP-hard.

In 1983 Kelly [20] proved that planar posets have arbitrary finite dimension. He has presented two new families of irreducible posets and proved that finite dismantlable lattices have arbitrary finite dimension. He also introduced the dimension product construction in. He proved that $P \otimes 2$, the dimension product of a 3-irreducible poset P and a 2-element chain, is 4-irreducible. In 1983 Spinrad and Valdes [21] gave proof for the recognition of two dimensional partial orders: the problem of determining whether the dimension of a given partial order is less than or equal to two. Determining whether a partial order has dimension one is a trivial problem since it must be a total order. Determining the dimension of a partial order is NP-complete for dimension greater than two.

In 1989 Schnyder [23]proved that each planar graph has dimension at most three. In 1991 Erdos, et al. [24] proved that for every $\in > 0$, there exist $\delta, c > 0$ so that if $(\log^{1+\in} n)/n < p \leq 1/\log n$, then $dim(P) > \delta pn \log pn$ for almost all $P \in \Omega(n,p)$ and if $1/\log n \leq p < 1 - n^{-1+\in}$, then $dim(P) > n - cn/p \log n$ for almost all $P \in \Omega(n,p)$. They also studied the space $L(n)$ of all labelled ordered sets on $n$ points and showed that there exist positive constants $c_1$, $c_2$ so that $n/4 - c_1 n/\log n < dim(P) < n/4 - c_2 n/\log n$ for almost all $P \in L(n)$.

In 1992 Brightwell and Scheinerman [25] proved that for any graph G we have $fdim[P(G)] \leq 3$ with equality holding if and only if G contains a triangle. They also proved the fractional dimension of $S_n^k$ is $2n/(k+1)$.

In 1994 Felsner and Trotter Jr. [26] proved that when $w \geq 3$, the fractional dimension of a poset $P$ of width $w$ is less than $w$ unless $P$ contains $S_w$. If $P$ is a poset containing an antichain $A$ and at most $n$ other points, where $n \geq 3$, they show that the fractional dimension of $P$ is less than $n$ unless $P$ contains $S_n$. If $P$ contains an antichain $A$ such that all antichains disjoint from $A$ have size at most $w \geq 4$, then the fractional dimension of $P$ is at most $2w$, and this bound is best possible.

In 1998 Bayoumi, et al. [27] described an algorithm for solving the problem of recognizing the dimension of a poset $P$, $dim(P)$. In 1999 Hosten and Morris Jr. [28] proved that the order dimension of the complete graph on $n$ vertices is the smallest integer $t$ for which there are n antichains in the subset lattice of $[t-1]$ that do not contain $[t-1]$ or two sets whose union is $[t-1]$.

In 2013 Baym and West [29] studied the k-dimension of products of finite orders. For $k \in o(ln\ n)$, the value $2dim_k(P) - dim_k(P \times P)$ is unbounded when $P$ is an n-element antichain, and $2dim_2(mP) - dim_2(mP \times mP)$ is unbounded when $P$ is a fixed poset with unique maximum and minimum. For products of the "standard" orders $S_m$ and $S_n$ of dimensions $m$ and $n$, $dim_k(S_m \times S_n) = m + n - min\{2, k-2\}$. For higher-order products of "standard" orders, $dim_2(\Pi_{i=1}^{t} S_{n_i}) = \sum n_i$ if each $n_i \geq t$. In 2013 Bosek, et al. [30] they analyzed the on-line dimension of partially ordered sets as a value of a two-person game between Algorithm and Spoiler. The game is played in rounds. Spoiler presents an on-line order of width at most $w$, one point at a time. Algorithm maintains its realizer, i.e., the set of $d$ linear extensions which intersect to the presented order. Algorithm may not change the ordering of the previously introduced elements in the existing linear extensions. The value of the game $val(w)$ is the least $d$ such that Algorithm has a strategy against Spoiler presenting any order of width at most $w$. For interval orders Hopkins [22] showed that $val(w) \leq 4w - 4$. They analyze the on-line dimension of semi-orders i.e., interval orders admitting a unit-length representation. For up-growing semi-orders of width $w$ they prove a matching lower and upper bound of $w$. In the general (not necessarily up-growing) case we provide an upper bound of $2w$.

In 2015 Felsner, et al. [31] they continued the study of conditions that bound the dimension of posets with planar cover graphs. They showed that if P is poset with a planar comparability graph, then the dimension of P is at most four. They also showed that if P has an outerplanar cover graph, then the dimension of P is at most four. Finally, if P has an outerplanar cover graph and the height of P is two, then the dimension of P is at most three. These three inequalities are all best possible.

In 2017 Joret, et al. [32] proved that every poset whose cover graph has treewidth at most 2 has dimension at most 1276. In 2018 Kim, et al.[33] proved that the maximum local dimension of a poset on n points is $\Theta(n/log\ n)$. They also proved that the local dimension of the n-dimensional Boolean lattice is $\Omega(n/log\ n)$. In 2018 Scoot and Wood[34] proved that the boxicity of a graph G is the minimum integer d such that G is the intersection graph of d-dimensional axis-aligned boxes. They proved that every graph with maximum degree $\Delta$ has boxicity at most $\Delta log^{1+o(1)}\Delta$, which is also within a $logo(1)\Delta$ factor of optimal. They also show that the maximum boxicity of graphs with Euler genus g is $\Theta(\sqrt{g\ log\ g})$.

In 2020 Barrera-Cruz, et al. [35] proved that the Boolean dimension of a poset is bounded in terms of the tree-width of its cover graph, independent of its height. They show that the local dimension of a poset cannot be bounded in terms of the tree-width of its cover graph, independent of height. They also prove that the local dimension of a poset is bounded in terms of the path-width of its cover graph. In 2024 Ashok Bhavale [36] proved that the dimension of a dismantlable lattice is at most three.

## 4. OPEN PROBLEMS

In 1955 Toshio Hiraguchi [6] raised the following problems.

(1) Let $P$ be an order defined on a set $A$ and $a$ a maximal(P) element of $A$. If there exists one and only one element $b$ such that $(b : a) \in P$, then $a$ is d-removable. It may be proved easily that if, moreover, either no element other than $b$ exceeds $(P)a$ or the suborder $P(A - a)$ is d-irreducible, then $a$ is d-removable.
(2) It is not possible to define a d-irreducible order on a set whose cardinality is an odd integer.

In 1976 Trotter [14] raised the following problems.

(1) Determine whether doubly irreducible posets exist.
(2) Give a forbidden subposet characterization of $Dim(X) \leq W(X)$.
(3) Determine condition on $X$ which insure that $DimS(X) = 1 + Dim(X)$.
(4) For each $n \geq 1$, does there exist an n-dimensional poset $X$ for which $X \times X$ is also n-dimensional?

In 1977 Trotter and Moore [15] raised the following problems.

(1) Determining the maximum possible dimension of a planar poset.
(2) Constructing a planar poset with dimension greater than four.
(3) Exploring the connection between the authors' results on the dimension of trees and previous work.

In 1981 Babai and Duffus [17] raised the following problems.

(1) Find lattice varieties $V$ with the following property: for every $N$ there exists a finite group $G$ such that if $Aut(L) \cong G$ for some finite lattice $L \in V$ then $dim(L) > N$.
(2) The automorphism groups of finite modular lattices of bounded dimension do not represent every finite group.
(3) Characterize the automorphism groups of finite lattices of dimension 2.

In 1982 Felsner, et al. [19] raised the following problems.

(1) What is the complexity of deciding whether a bipartite graph of maximum degree 3 admits a BTCon representation?
(2) What is the complexity of deciding whether a planar bipartite graph admits a BTCon representation?
(3) What is the complexity of deciding whether an incidence orders of planar graphs (a subdivision of a planar graph) admits a PUTCon representation?

In 1983 Spinrad and Valdes [21] raised the following problems.

(1) Investigating the use of the modular representation developed in the paper for describing and analyzing other classes of graphs beyond two-dimensional partial orders.

(2) Exploring whether the two-dimensional representation of a graph can be updated efficiently (in linear time) when a new vertex and its edges are added.
(3) Finding a linear-time algorithm for recognizing two-dimensional partial orders and testing isomorphism, as the current algorithm runs in quadratic time.
(4) Applying the ideas developed in the paper to solve the problems of recognizing permutation graphs and transitively orientable graphs.
(5) Finding a more efficient O(n+e) algorithm for recognizing two-dimensional partial orders and testing isomorphism.
(6) Extending the algorithm to solve the related problems of recognizing permutation graphs and transitively orientable graphs.
(7) Developing an algorithm to efficiently update the two-dimensional representation of a graph when adding a new vertex and its incident edges.

In 1991 Erdos, et al. [24] raised the following problems.

(1) For $p$ satisfying $log^2pn = 0(logn)$, find improved bounds for the expected value $E(dim(P))$ of the dimension of $P$. As of now, we know $c_2pn\ log\ pn < E(dim(P)) < c_1pn\ log^2pn$.
(2) Can random methods be used to find a (possibly quite rare) ordered set $P$ with $dim(P)$ close to $\Delta(P)log^2\Delta(P)$?
(3) Alternately, can the Furedi/Kahn inequality $dim(P) < c\Delta(P)log^2\Delta(P)$ be improved by lowering the exponent on the $log\Delta(P)$ term?
(4) Is the inequality $f(n, k) > n^{1-1/k}$ best possible?
(5) As $p$ increases, when does $P$ first satisfy $dim(P) \geq k$, for $k$ = 3,4,5, . . .?
(6) Is the expected value of $dim(P)$ unimodal as $p$ increases?
(7) When does $\lim_{n\to\infty} \frac{E(dim(P))}{pn\ log\ pn}$ exist, and what is its value?
(8) How does $dim(P)$ behave when $p$ is very close to 1?
(9) Evaluate $\lim_{n\to\infty} \frac{dim(P)}{n}$ when $p = 1/log\ n$.
(10) How tight is the inequality $sdim(P) \leq dim(P)$? We suspect that this inequality is very tight for almost all $P \in \Omega(n, p)$.
(11) How does $dim(P)$ behave if we consider a bipartite model with $n$ minimal elements and $m$ maximal elements with $m > n$?
(12) How do the results change when the comparabilities in $P$ are the union of $k$ random edge disjoint matchings? This approach may prove particularly useful when $k$ is very small.

In 1992 Brightwell and Scheinerman [25] raised the following problems.

(1) Is there a polynomial time algorithm to evaluate $fdim(P)$?
(2) For a rational $p/q$, what is the asymptotic number of $n$-element ordered sets with fractional dimension at most $p/q$?
(3) Is there a sequence $(I_k)$ of interval orders with $fdim(I_k) \to 4$ as $k$ tends to infinity?

In 1994 Felsner and Trotter Jr. [26] raised the following problems.

(1) Are the following two inequalities of Theorem 8.2 best possible? Let $P = (X, P)$ be a poset, let $A$ be an antichain with $X - A$ nonempty, and let $Q = (Y, P(Y)$. Then let $w = width(Q)$.
(1) If $w = 2$, then $fdim(P) < 1/2 + 2w$.
(2) If $w = 3$, then $fdim(P) < 1/4 + 2w$.
(2) Characterizing the relationship between fractional dimension and width of a poset, as proposed in Conjecture;
  (a) For each $\epsilon > 0$, there is an integer $w_\epsilon$ so that for every $w > w_\epsilon$, there exists a poset $P$ so that $w - \epsilon < fdim(P) < w = width(P)$.
  (b) For every positive integer $w \geq 2$, there is an $\epsilon_w > 0$ so that $fdim(P) \leq w - \epsilon_w$, for every poset $P$ with $fdim(P) < w = width(P)$.
(3) For each $t \geq 3$, let $f(t)$ be the minimum number of incomparable pairs in a poset P with $fdim(P) \geq t$. Is it true that $f(t) = t^2$?
(4) Given rational numbers p and q, what is the minimum value of the fractional dimension of $P \times Q$, where $fdim(P) = p$ and $fdim(Q) = q$?
(5) Does there exist an absolute constant $\epsilon > 0$ so that any poset with 3 or more points always contains a pair whose removal decreases the fractional dimension by at most $2 - \epsilon$?
(6) Tightening the upper bound on fractional dimension given in Theorem; If $P = (X, P)$ is a poset, then $fdim(P) \leq 1 + \Delta_D(P)$, as there are posets where the actual fractional dimension is slightly higher.

In 2015 Felsner, et al. [31] raised the following problems

(1) Is it true that for every $n \geq 3$, there exists an integer $t_n$ so that if $P$ is a poset with a planar cover graph and $dim(P) > t_n$, then $P$ contains the standard example $S_n$ as a subposet?
(2) Determine for each $n \geq 3$, the least integer mn for which there exists a poset $P$ with $m_n$ points for which $dim(P) \geq n$ and $P$ has a planar cover graph. Of course $m_3 = 6$ and $m_4 = 8$, and $m_5 \geq 12$. Again, this question can be asked for posets with planar order diagrams.
(3) Whether there exists a polynomial time algorithm that will determine whether a poset $P$ is a subposet of a poset with a planar cover graph, and again the same question for planar order diagrams.

In 2018 Kim, et al. [33] raised the following problems

(1) Let $n+1$ be a power of 2 and let $H_n = H(n, n; f_n)$ be the difference graph such that $f_n(i) = n + 1 - i$. What is the exact value of $lbc(H_n)$?
(2) Is it true that $ldim(2^n) = n$ for all $n \geq 1$?

In 2018 Scott and Wood [34] raised the following problems

(1) What is the maximum boxicity of graphs with maximum degree 4?
(2) What is the maximum boxicity of k-degenerate graphs with maximum degree $\Delta$?
(3) What is the maximum boxicity of graphs with treewidth k?
(4) What is the maximum boxicity of graphs with no $K_t$ minor?

In 2020 Barrera-Cruz, et al. [35] raised the following problems

(1) For a positive integer $w$, what is the maximum value of the Boolean dimension of a poset whose width is $w$?
(2) For a positive integer $w$, what is the maximum value of the local dimension of a poset whose width is $w$?
(3) For a non-negative integer $n$, what is the maximum value of the Boolean dimension of a poset consisting of an antichain and $n$ additional points?
(4) Is there a constant $d_0$ such that every planar poset has Boolean dimension at most $d_0$?
(5) Is there a constant $d_0$ such that every poset with a planar cover graph has Boolean dimension at most $d_0$?
(6) If a planar poset has large dimension, must it contain a large standard example?
(7) If a planar poset has large Boolean dimension, must it contain a large standard example?
(8) If a planar poset has large local dimension, must it contain a large standard example?
(9) For an integer $d \geq 4$, what is the maximum local dimension of a disconnected poset in which each component has local dimension at most $d$? Note. The answer is either $d, d+1$ or $d+2$.
(10) What is the maximum amount the Boolean dimension of a poset can drop when a single point is removed? Note. The answer is either $1, 2$ or $3$.
(11) What is the Boolean dimension and the local dimension of $2^d$?

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