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Some Likely behaviours for Axisymmetric Rising Plumes in a Quiescent uniform Ambient through Warm Discharge

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ABSTRACT

An investigation into some possible behaviours of rising plumes with a quadratic dependence relation assumption was made with the consideration that the plumes arises from a virtual sources. Our results showed that both forced and pure plumes provides buoyancy flux at the source. Meanwhile, the lazy plume rises with a significant volume flux and finite temperature with zero volume flux for both the forced plume and pure at the source. The forced plume also possesses some significant upward momentum flux and zero momentum flux for the pure plume. Determination of zero-buoyancy height and the fountain-top height are key when it comes to rising plumes. If the ambient water depth into consideration is less than the zero-buoyancy height then the warm water will definitely spread outwards as surface gravity current; otherwise, it will form a fountain.Our results showed that the upward momentum flux attains its pick at $Q = \frac{1}{2}$; and changes sign in the buoyancy force at this point while the momentum flux tends to zero with a finite final volume flux

Q_f from its maximum rise height. Meanwhile, as M_0 approaches the value $-(\frac{5}{2^5})^{\frac{2}{5}} \approx 0.475913$, so the Q_f and the Q_0 tends to the value $\frac{1}{2}$. It is worth stating also that the fountain-top height will be of great significant if the focus is on erosion of an ice cover. We therefore suggest maintaining a low source Froude number which will result to minimising impact on the lake floor or on the surface. Though, some level of carefulness is needed whenever we intend to make use of the present results to foretell the behaviour of real plumes. In particular, if laboratory experiments are required either to examine the theory or to model larger-scale flows in the environment. Lastly, with the entrainment model, it is required that the plume will be fully turbulent, a condition usually obtained with a Reynolds number Re > 2000. This can be likened to the power station warm discharge which will definitely be fully turbulent with it large volume flux. These are basic conditions on the validity of the assumptions. There are some limitations as stated in the conclusion. But then, the overall numerical results here are very good as they present us with more insight into rising plume with the quadratic dependence relation assumption.

KEYWORDS: Plumes, Warm discharge, Momentum flux, Volume flux, Zero-buoyancy, Half-width.

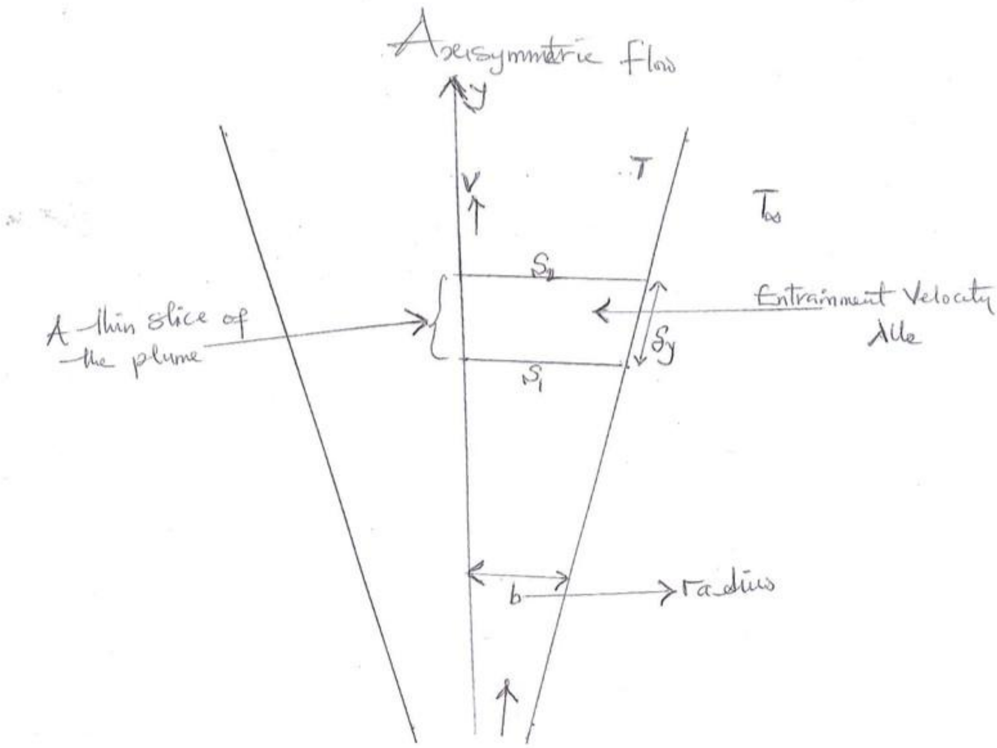

Fig. 1: The sketch of a thin slice of the plume on which to perform mass, momentum and thermal energy budgets.

1 INTRODUCTION

Thermoelectric power generating plants and their waste water discharge is the primary cause of most phenomenal behaviours in our environment and as well negatively impacting the aquatic ecosystem (George & Osaisai, 2023; George et al. 2023). These power plants are known for disposing their cooling waste water at a temperature approximately $10^{\circ}C$ higher than the medium (lakes, ponds, rivers, etc) into which the initial water was drawn. This implies that the disposed waste water may be less dense than that of the ambient water as we have in most lakes, ponds, rivers, etc from which this cooling water is drawn (Macqueen, 1979; Kay, 2007; George & Kay, 2017). Assuming that this waste water is being disposed at the lake floor or somewhere close to it, then it is believed that this warm water may penetrate the surrounding water upwards as a rising plume. This flow may be turbulent or laminar depending on the discharge velocity. But for sure, a flow from such power station discharge will be turbulent. But then, as the plume rises, it may also entrain surrounding water into itself which in turn will reduce its temperature gradually. But then, if the medium into which the initial water was drawn is not deep enough then this rising plume may advance quickly to the surface and spread outwards as surface gravity current and may still continue to entrain more cold water or releasing heat to the atmosphere (Kay, 2007; George & Kay, 2017).

Sometimes, there is also the possibility that the temperature of the surrounding water might be below the temperature of maximum density and in such a case, the rising plume will produce a mixture that is more dense than both the disposed waste water and that of the ambient water. This dense water from the mixture will in turn form a descending plume and sink to the bottom of the lake or pond as the case may be and spread outwards forming gravity or density current (Kay, 2007; George & Kay, 2017). Example of this later behaviour has also been observed in Lake Michigan by Hoglund and Spigarelli (1972) as also recorded in the literature by George & Kay, 2017, where the previous

(Hoglund and Spigarelli (1972)) were not really interested in the dynamics of the flow but rather in the biological implication of the spreading of the descended warm water. Whereas, analysis of the dynamical aspect of the vertical flow is our key interest here. Note that whenever water masses of different densities meet such as the case that we are considering, there might exist different stages of flow. First is a rising plume, only if the disposed waste water is less dense then the ambient water and initiated at the lake bed or somewhere close to it; followed by a descending plume (i.e., an initially rising plume may later become a fountain); a surface gravity current at the surface level if the domain into consideration is not deep enough; and lastly a density current along the floor of the lake or pond as the case may be. However, our interested here is to consider only the rising stages of the plume which is also a feature to expect whenever water masses of different densities meet. This also seem to be one of the stages as considered by Kay, 2007: though ours is with a different geometry ((axisymmetric) see figure 1). Thus, the other cases are beyond the scope of this investigation.

Here in this investigation, we hope to consider a plume rising from a multi-port diffuser, as also considered by (Kay, 2007). More details can be found in the literature by Kay, 2007 so as to gain more insight.

The ambient of consideration is assumed quiescent, uniform and unstratified with a discharge temperature $10^{\circ}C$ on the floor of a lake with a uniform ambient water temperature $0^{\circ}C$. Buoyancy reversal is one of the significant features that are associated with rising plumes. In rising plumes, entrainment result to increase in density: as both discharge and ambient water mixes further by entrainment, density difference in the mixed fluid will increase, reducing the penetrating speed of the frontal head and this will continue until its temperature attains $4^{\circ}C$ or a temperature close to it. Though, this is different from those plumes with linear mixing properties, where entrainment always reduces buoyancy. Plumes and jets with buoyancy reversal have received much attention in the past and in different situations though, with the assumption that density is a linear function of temperature. But then, Caulfield and Woods (1995); Kay (2007); have also considered the plume entrainment equations with density as a quadratic function of mixing ratio. Though, in the investigations by Caulfield and Woods (1995), mixing between plume and ambient fluid always resulted to a decrease in density. Whereas, for those by Kay (2007), entrainment always increases density as mixed fluid increases until its temperature had reached $4^{\circ}C$. Results in this case also show some similarities with those by Bloomfield and Kerr (2000) for axisymmetric fountains.

2 MODEL FORMULATION AND GOVERNING EQUATIONS WITH SCALINGS

We consider here a two-dimensional axisymmetric fountain progressing vertically from the bottom through a homogeneous and quiescent water, as illustrated in Figure 1. Entrainment and dilution taking place mainly along its sides, the buoyancy and vertical velocity v within the plume depend on both distance y above the source and radial distance b from the centreline. For a successful formulation, it is ideal to consider the entrainment of surrounding water by the plume and the non-monotonic dependence of density ρ on temperature T which are also key factors that describe a rising turbulent plume. A quadratic dependence relation of density on temperature is also assumed as consider by Kay, (2007), George & Kay (2017), George & Kay (2022) and George & Osaisai (2022) so as to have a good illustration of the real dynamics.

$$\rho_m - \rho = \beta(T - T_m)^2 \tag{1}$$

Note that the terms in Equation (1) are the same as those used in Kay (2007), George & Kay, (2022) with the same explanation. We are also adopting the already existing hypothesis on entrainment by Morton et al. (1956) that ambient fluid is entrained at a velocity proportional to the vertical velocity within the plume as also used by Kay (2007). Here in this work, the liquid properties are considered constant except for the water density, which changes with temperature and in turn results to the buoyancy force. Thus, fountains as a result of buoyancy reversal at the point where mixture attains temperature of maximum density is our main focus (Kay, 2007).

Classical models of steady plumes uses equations for conservation of mass, momentum and buoyancy (Kay, 2007). Though, conservation of buoyancy is used when the buoyancy is a linear function of some conserved quantity such as thermal energy, salinity, etc. (Kay, 2007). Meanwhile, buoyancy here in our case is a nonlinear function of temperature which is proportional to thermal energy. As such, we will also derive our equations from the conservation laws for mass, momentum and thermal energy with the nonlinearity appearing in the buoyancy forcing term in the momentum equation similar to those by Kay (2007). Density variations will be ignored except in the buoyancy term which is the difference between hydrostatic pressure gradients within and outside the plume (Boussinesq approximation).

In the following we shall use the top hat profiles, which are equivalent to replacing the mass, momentum and thermal fluxes by their mean values defined by integrating across the plume.

$$b^2 v = \int_0^{\infty} 2rv\,dr \tag{2}$$

$$b^2 v^2 = \int_0^{\infty} 2rv^2\,dr \tag{3}$$

$$b^2 vT = \int_0^{\infty} 2rvT\,dr \tag{4}$$

Surrounding water of homogeneous temperature T_∞ and density ρ_∞ is entrained into the plume at a velocity u_e assumed to be proportional to the vertical velocity (Morton et al., 1956):

$$u_e = \lambda|v| \tag{5}$$

The constant of entrainment λ have an acceptable value of about 0.08 for top-hat profiles (Turner, 1973). Though, value of this constant $\lambda = 0.1$ whenever in use. But then, it is usually scaled out in many numerical calculations (Kay, 2007).

As earlier stated, a two-dimensional axisymmetric fountain progressing vertically (upward) is of interest positioning the vertical coordinate y and the vertical velocity v in the direction of motion. Having that an alteration in direction does not really suggest a change from entrainment to detrainment, there is the need to consider $|v|$ as a factor in the equations that describes entrainment. With all these, we strongly believe that this investigation will bring forth valuable information on some key parameters such as maximum height of an upward plume and height at which buoyancy reversal occurs.

Considering the mass conservation budget over the thin slice of the plume as shown in Figure 1. we have that

Entrainment Mass Flux = Mass Flux out flow of S_2 - Mass Flux in flow S_1

$$\textit{Mass Flux in flow } S_1 = \rho\pi b^2 v \tag{6}$$

$$\textit{Mass Flux out flow of } S_2 = (\rho + \delta\rho)\pi(b + \delta b)^2(v + \delta v) \tag{7}$$

$$\textit{Entrainment Mass Flux} = 2\lambda\pi b v \rho_\infty \delta y \tag{8}$$

From equation (6) - (8), after using the Boussinesq approximation (that density variations will be ignored except in the buoyancy term) we have equation for volume flux as follows:

Entrainment volume Flux = volume Flux out flow of S_2 - volume Flux in flow S_1

$$2\lambda\pi b v \delta y = \pi(b + \delta b)^2(v + \delta v) - \pi b^2 v \tag{9}$$

After expanding further, and divide through by δy and taking limit as $\delta b \to 0$ we have

$$\frac{d(b^2 v)}{dy} = 2\lambda b v \tag{10}$$

In like manner we determine the vertical momentum flux over the same thin slice. So that we can write

Upward buoyancy force = Momentum Flux out flow of S_2 - Momentum Flux in flow S_1. Note here that, we will make use of the transformed buoyancy term "$(\rho_\infty - \rho) = \beta[(T - T_m)+(T_\infty - T_m)](T - T_\infty)$" derived from equation (1) above. Also, the upper and lower signs refer to upward and downward direction plumes respectively (Kay, 2007). Thus,

$$\frac{\mp\pi b^2 \not{g}}{\rho_m}[(T - T_m) + (T_\infty - T_m)](T - T_\infty)\delta y = \pi(b + \delta b)^2(v + \delta v)^2 - \pi b^2 v^2 \tag{11}$$

After expanding further also, and divide through by δy and taking limit as $\delta b \to 0$ and $(\delta v)^2 \to 0$ we have

$$\frac{d(b^2 v^2)}{dy} = \mp\frac{b^2 \not{g}}{2\rho_m}[2T_m - T - T_\infty](T - T_\infty) \tag{12}$$

Lastly, we determine the Thermal flux and we have that:

Entrainment Thermal Flux = Thermal Flux out flow of S_2 - Thermal Flux in flow S_1

$$2\lambda\pi b v T_\infty \delta y = \pi(b + \delta b)^2(v + \delta v)T - \pi b^2 v T \tag{13}$$

$$2\lambda b v T_\infty \delta y = (b^2 + 2b\delta b + (\delta b)^2)(v + \delta v)T - b^2 v T \tag{14}$$

After expanding further as usual, and divide through by δy and taking limit as $\delta b \to 0$ we have

$$\frac{d(b^2 v T)}{dy} = 2\lambda b v T_\infty \tag{15}$$

Hence, the governing Volume, Momentum and Thermal Flux are:

$$\frac{d(b^2 v)}{dy} = 2\lambda b v \tag{16}$$

$$\frac{d(b^2 v^2)}{dy} = \mp\frac{b^2 \not{g}}{2\rho_m}[2T_m - T - T_\infty](T - T_\infty) \tag{17}$$

$$\frac{d(b^2 v T)}{dy} = 2\lambda b v T_\infty \tag{18}$$

Combining equation (16) and (18) gives:

$$\frac{d(b^2 v T)}{dy} = T_\infty \frac{d(b^2 v)}{dy} \implies \frac{d}{dy}[b^2 v(T - T_\infty)] = 0 \tag{19}$$

Upon integration we have

$$b^2 v(T - T_\infty) = F = Constant \tag{20}$$

$$T = T_\infty + \frac{F}{b^2 v} \tag{21}$$

Equation (21) is the temperature in the plume and F is the relative thermal flux which is conserved because of the unstratified ambient condition. Using equation (21), we can rewrite (17) as follows:

$$\frac{d(b^2 v^2)}{dy} = \mp\frac{\not{g}}{2\rho_m}\frac{F}{v}[2T_m - 2T_\infty - \frac{F}{b^2 v}] \tag{22}$$

From equation (16) If we let

$$q = b^2 v \tag{23}$$

and from equation (17) if we let

$$m = b^2 v^2 \tag{24}$$

From (23) and (24) we can write

$$v = \frac{q}{b^2} \quad and \quad v = \frac{m^{\frac{1}{2}}}{b} \tag{25}$$

From equation (25) if we make b the subject we have:

$$b = \frac{q}{m^{\frac{1}{2}}} \tag{26}$$

which is the plume width. In like manner we can write for plume velocity from (23) and (24) as follows:

$$v = \frac{m}{q} \tag{27}$$

From here we can rewrite the volume and momentum flux as dependent variables rather than having the in terms of width and velocity. Then equation (16) and (22) becomes:

$$\frac{dq}{dy} \quad = \quad 2\lambda m^{\frac{1}{2}} \tag{28}$$

$$\frac{dm}{dy} = \mp \frac{\beta g}{2\rho_m} \frac{Fq}{m} [2(T_m - T_\infty) - \frac{F}{q}] \tag{29}$$

Now, we can consider the scaling parameters of this given problem where temperature scale we have to be ($T_m - T_\infty$), the conserved thermal flux F and the Buoyancy scale we have to be

$$g_m = \frac{\beta g}{\rho_m}(T_m - T_\infty)^2 \tag{30}$$

Also, we have the volume flux scale through the combination of both the conserved thermal flux and the temperature as follows:

$$q_T = \frac{F}{(T_m - T_\infty)} \tag{31}$$

Substituting (30) and (31) into (29) we have

$$\frac{dm}{dy} = \mp \frac{\beta g}{2\rho_m} \frac{q_T(T_m - T_\infty)q}{m} [2(T_m - T_\infty) - \frac{q_T(T_m - T_\infty)}{q}]$$

$$\frac{dm}{dy} = \mp \frac{q_T q}{2m} [2g_m - \frac{q_T g_m}{q}]$$

$$\frac{dm}{dy} = \mp \frac{q_T q g_m}{2m} [2 - \frac{q_T}{q}] \tag{32}$$

our new set of equation becomes (28) and (32) we can define dimensionless variables for the problem as follows:

$$q = Qq_T, \quad y = (\frac{q_T^2}{\lambda^4 g_m})^{\frac{1}{5}} Y, \quad m = (\frac{q_T^6 g_m^2}{\lambda^2})^{\frac{1}{5}} M, \quad \phi = \frac{T - T_\infty}{(T_m - T_\infty)}, \quad b = (\frac{\lambda q_T^2}{g_m})^{\frac{1}{5}} B, \quad v = (\frac{q_T g_m^2}{\lambda^2})^{\frac{1}{5}} V \tag{33}$$

where we are scaling out the entrainment coefficient λ so that our results are independent of its numerical value. The thermal flux equation (20) yields a relation between dimensionless temperature and volume flux as

$$\phi = \frac{1}{Q} \tag{34}$$

Using also (33) on (28) and (32) so as to have the equation of motion in dimensionless form as follows:

$$\frac{dQ}{dY} \quad = \quad 2M^{\frac{1}{2}} \tag{35}$$

$$\frac{dM}{dY} = \mp \frac{2Q - 1}{2M} \tag{36}$$

Combining (35) and (36) to remove Y we have:

$$\frac{dM}{dQ} = \mp \frac{2Q - 1}{4M^{\frac{3}{2}}} \tag{37}$$

Upon integration we have the solution to be

$$M^{\frac{5}{2}} = M_0^{\frac{5}{2}} \mp \frac{5}{8}(Q^2 - Q) \tag{38}$$

Where M_0 is the value of M at Q = 0. We can also determine the dimensionless variables for (26) and (27) which are the plume width and velocity respectively. For plume dimensionless width we have

$$B = \frac{Q}{M^{\frac{1}{2}}} \tag{39}$$

For plume dimensionless velocity we also have

$$V = \frac{M}{Q} \tag{40}$$

There is also need for us to relate M_0 to conditions at the physical source. The plumes behaviour is governed by dimensionless temperature and Froude number at the source (Kay, 2007), defined as

$$\phi_s = \frac{T_s - T_\infty}{(T_m - T_\infty)}, \quad Fr_s = \frac{v_s}{(g_m b_s)^{\frac{1}{2}}} \tag{41}$$

where b_s, v_s and T_s are the width, velocity and temperature at the physical source. note that we shall always define Froude numbers with respect to the constant buoyancy scale g_m rather than the buoyancy of the plume, so that the Froude number is simply a dimensionless velocity. Given positive, finite values of φ_s and Fr_s, we can find the corresponding co-ordinates in Q-M space as:

$$Q_s = \frac{1}{\phi_s} \tag{42}$$

$$M_s = \left(\frac{Fr_s^4 \lambda^2}{\phi_s^6}\right)^{\frac{1}{5}} \tag{43}$$

Substituting (42) and (43) into equation
(38) we have

$$M_0 = \left(\frac{5}{8\phi_s^3}\right)^{\frac{2}{5}} \left[\frac{8\lambda Fr_s^2}{5} \pm (\phi_s - \phi_s^2)\right]^{\frac{2}{5}} \tag{44}$$

Where, any point Q_s,M_s on a trajectory in Q-M space can be regarded as a possible physical source for a plume with

$$\phi_s = \frac{1}{Q_s}, \quad Fr_s = \frac{M_s^{\frac{5}{4}}}{\lambda^{\frac{1}{2}} Q_s^{\frac{3}{2}}} \tag{45}$$

3 NUMERICAL RESULTS

3.1 Rising plume

We have considered here a rising plume from the solution in (38) and plotted for $M_0 = -0.14$,0 and 0.14 for upward motion as shown in Fig. 2. Where the dashed curve represent a lazy plume, solid curve represent pure plume while the dotted curve represent a forced plume as also described in the literature by Hunt & Kaye (2005) and Kay, (2007). The overall results as shown on this paper are very similar to those by Kay, (2007) for a symmetric flow. As also recorded by the author the results here also showed that pure plume arises from a virtual source and also provides buoyancy flux at the source but then zero momentum and volume flux. The lazy plume also emanates from a virtual source with a significant volume flux and finite temperature. But then, it showed that the upwards momentum flux for the lazy plume is comparatively small than the other plumes. Meanwhile, the forced plume possesses an upward momentum flux together with buoyancy flux at its virtual source. Volume flux for the forced plume is zero at the source which in turn give rise to an infinite temperature as given in Equation (34). It is worth noting that plume motion here in these plots propagate from left to right having that volume flux Q increase as a result of entrainment. The source term (M_0) for the lazy plume indicates that there should be an initial downward momentum flux at a source which we did not capture in our plots. This is also similar to those results by Kay, (2007) which seem to be reasonable because even though we make the unphysical continuation from it to the rising plume the initial downward motion would still not appear in the plot.

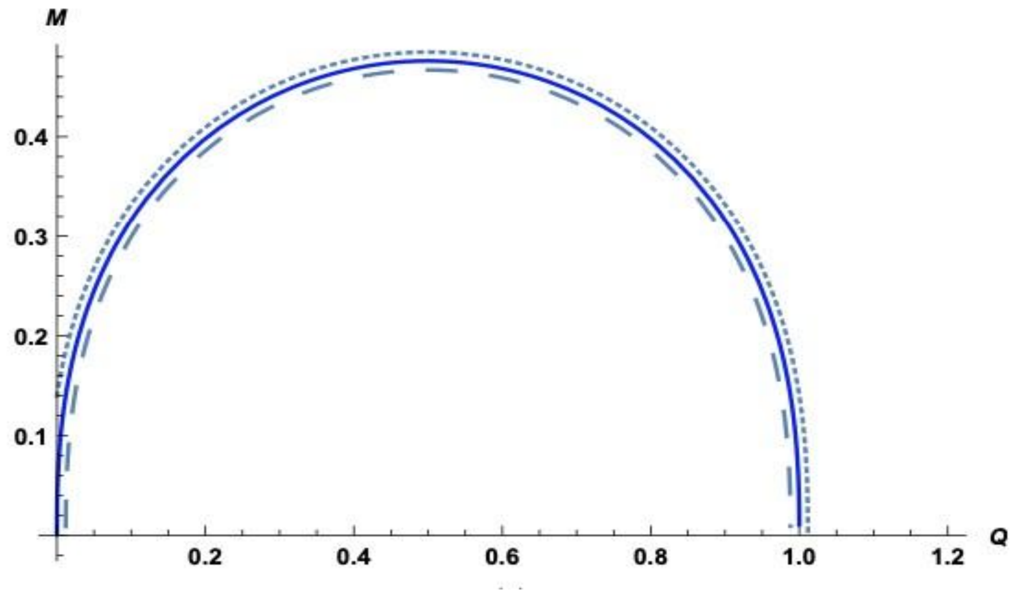

Fig. 2: The courses pattern followed in Q-M space from solution (38) for upward motion with $M_0 = -0.14$,0 and 0.14.

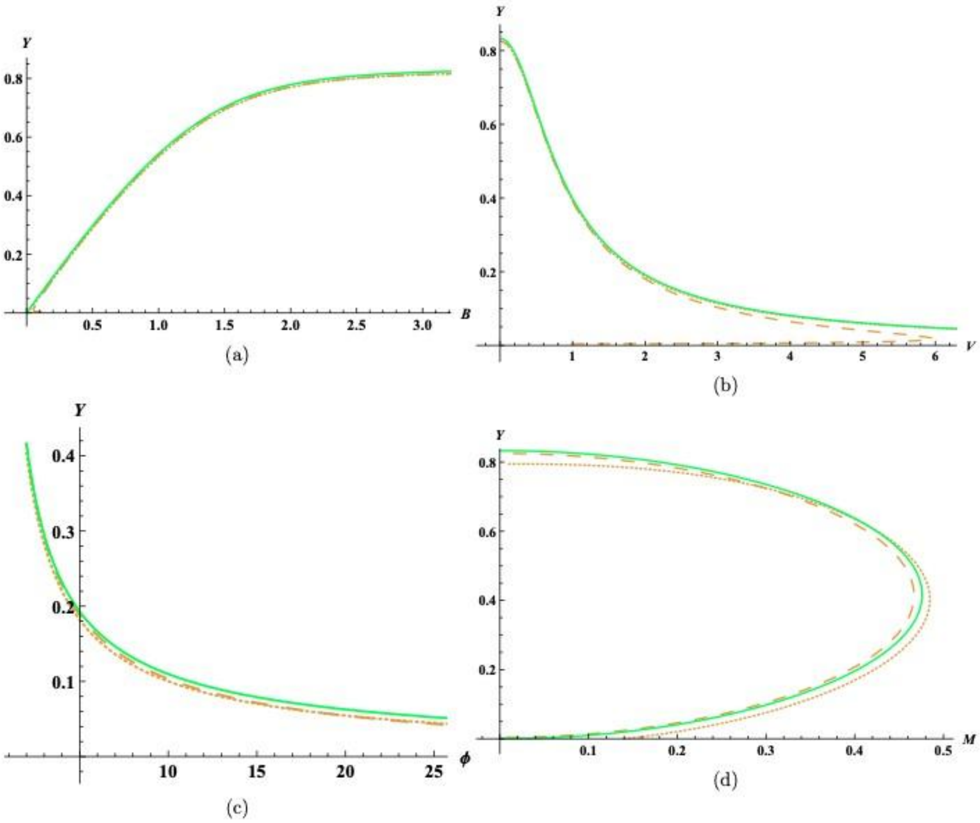

Fig. 3: Dimensionless plume properties against height above virtual source for a pure plume (solid curves), a forced plume with M_0 =0.14 (dashed curves) and a lazy plume with M_0 = -0.14 (dotted curves): (a) Half-width, (b) Vertical velocity, (c) Temperature, (d) Momentum flux.

The plumes as shown in Fig. 2 appears slightly different from those by Kay, (2007) and this is evident near their respective virtual sources. However, our results also showed that theÂ upward momentum flux attains its pick at $Q = \frac{1}{2}$; and we experience a change in sign in the buoyancy force at this point while the momentum flux tends to zero from its maximum rise height. In fact, it is expected that the three plumes will definitely stop progressing and will come to a stop because, the momentum flux had decreased to zero with a finite final volume flux Q_f. Meanwhile, as M_0 approaches the value $-\left(\frac{5}{2^5}\right)^{\frac{2}{5}} \approx 0.475913$, so the Q_f (final volume flux) and the Q_0 (initial volume flux) tends to the value $\frac{1}{2}$. The value $\left(\frac{5}{2^5}\right)^{\frac{2}{5}}$ can be seen as a very critical value as this may also be useful when considering descending plumes. (Note: the said value of M_0 can be obtained from (38) the point at when momentum flux M = 0 and volume flux $Q = \frac{1}{2}$).

Figure 3. shows the results that describes the special behaviours of dimensionless plume properties against height near the source (i.e., the half-width, velocity, temperature and momentum flux are plotted as functions of the pure, forced and lazy plume’s height). The forced plume appears larger as compared to the pure plume this might be as a result of the greater velocity of the forced plume in its starting stages which in turn have resulted to greater entrainment (see Fig. 3b). In the other way round, the lazy plume possesses some sort of infinite width at its source and its width does not speedily shrink or reduce to a minimum even as the velocity increases to its maximum. The lazy plume appears similar to both the forced plume and the pure plume (figure 3(a)) outside of the point of maximum velocity. It is expected that after the forced, pure and the lazy plumes might have come to rest, they are supposed to spread outwards to an infinite width. Though, this we were unable to capture in our results, possibly because of the scale length used in figure 3(a)). Results in figure 3(c) shows the temperature distribution in the various plumes and that, rate of depletion of temperature was high which in turn led to more entrainment in the forced plume. And as a result of this, the forced plume could come to rest much faster as compared to others. This again, makes the forced plume to attain the temperature of zero buoyancy at a slightly reduced height, as also captured in the positions of the maxima in momentum flux (see Fig. 3(d)).On the other hand, the lazy plume is expected to come to rest slowly as a result of its reduced velocity and slow entrainment rate as compared to that of the force plume. But rather, the behaviours in these two plumes as shown here appears similar as compared to those by Kay (2007). However, the lazy plume could penetrate further upwards before the temperature of zero buoyancy (see Fig. 3(d)) as compared to

the other plumes. This is certain because, with its reduced velocity that have led to low entrainment will not enhance quick production of mixed fluid, enabling the lighter fluid in the plume to penetrate further upwards.

$$Y_n = \int_{Q_0}^{\frac{1}{2}} \frac{1}{2M^{\frac{1}{2}}} dQ \tag{46}$$

Now the height of zero buoyancy can be computed using the following from (46). The notation "Y_n" represent the height of maximum depth of water wherein which the plumes could reach the surface and if still positively buoyant, then spread outwards forming surface current. And as widely use, Q_0 is taken to be zero for forced and pure plumes (Kay, 2007). This height of maximum depth here is plotted as a function of M_0 (see Fig. 4). The result in figure 4 also shows that the maximum value of "Y_n" turns out to be 0.487 which decreases with increase in M_0. This figure (Fig. 4) also agrees to the fact that an increase in the forcing at the source for the forced plumes will eventually result to a decrease in height before getting to the point of zero-buoyancy. Meanwhile the maximum rise height of the plume (47) and (48) the dimensionless temperature at this height are all plotted as functions of M_0 and shown in Fig.5(a and b). These terms in (47) and (48) can also be computed from (35) and (34). Based on our model here, the rising plumes are expected to comes to rest at the height Y_f. Whereas, this is supposed to be maximum rise height of the plumes. This can also be seen as the maximum depth of water where the most buoyant fluid or plume will spread outwards or impinge on the surface as the case may be. The overall results for a rising plume as presented here are very good even as they look very similar though, with little variations to those by Kay (2007) as we have highlighted above.

$$Y_f = \int_{Q_0}^{Q_f} \frac{1}{2M^{\frac{1}{2}}} dQ \tag{47}$$

$$\phi_f = \frac{1}{Q_f} \tag{48}$$

Wiiest et al. (1992) in the past have also investigated on turbulent plumes using some advanced model which enable the authors to take into consideration nonlinear equation of state and some other factors. Be that as it may, there is the need for numerical solution for each specific application. It is ideal to state here that some level of carefulness is needed whenever we are intending to make use of the present results to foretell the behaviour of real plumes. In particular, if laboratory experiments are required either to examine the theory or to model larger-scale flows in the environment.

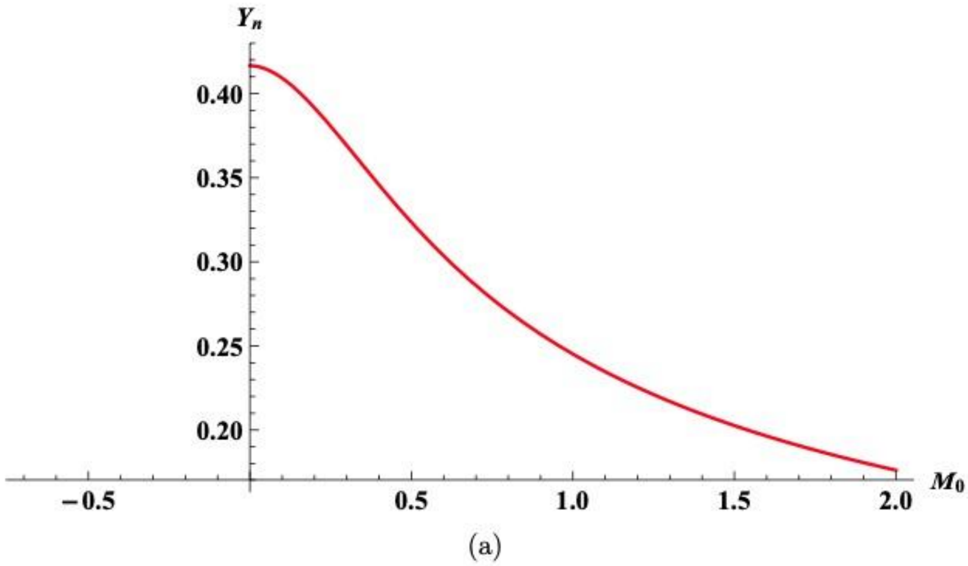

Fig. 4: Dimensionless height of zero buoyancy, calculated from (46), as a function of M_0.

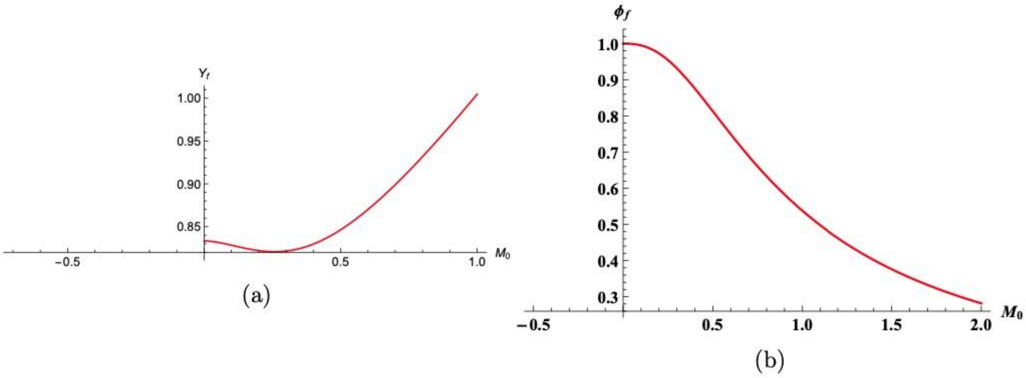

Fig. 5: (a) Maximum rise height and (b) temperature at this height, as a function of M_0.

Having that the governing equations here also assume self-similarity as those by Kay (2007) and the entrainment model of Morton et al. (1956), Thus, these assumptions will be valid at a distance of several outfall widths from the plume source. Lastly, with the entrainment model, it is required that the plume will be fully turbulent a condition usually obtained with a high Reynolds number as also stated in the literature by Fischer et al. (1979). In our case, this can also be likened to the power station warm discharge which will definitely be turbulent with its large volume flux. These are basic conditions on the validity of these assumptions. In overall, the numerical results here are very good as they present us with more insight into rising plume with the quadratic dependence relation assumption.

4 CONCLUSION

We have just concluded an investigation on some possible behaviours of rising plumes axisymmetrically. A quadratic dependence relation assumption was made which enables us to obtain some numerical solutions with the consideration that the rising plumes arises from a virtual sources. Our results showed that both the forced and pure plumes provides buoyancy flux at the source. Meanwhile, the lazy plume rises with a significant volume flux and finite temperature while zero volume flux for both the forced plume and pure at the source. The forced plume also possesses some significant upward momentum flux and zero momentum flux for the pure plume. The determination of zero-buoyancy height and the fountain-top height are also key when it comes to considering rising plumes. If the ambient water depth is less than the zero-buoyancy height then the warm water will definitely spread outwards as surface gravity current; otherwise, it will form a fountain, falling back to the floor of the domain as also recorded by George and Kay, 2017. Our results also showed that the upward momentum flux attains its pick at $Q = \frac{1}{2}$; and changes sign in the buoyancy force at this point while the momentum flux tends to zero from its maximum rise height. All the plumes are expected to stop their upward motion and come to a stop because, the momentum flux had decreased to zero with a finite final volume flux

Q_f. Meanwhile, as M_0 approaches the value $-(\frac{5}{2^5})^{\frac{2}{5}} \approx 0.475913$, so the Q_f and the Q_0 tends to the value $\frac{1}{2}$. It is worth stating also that the fountain-top height will be of great significant if our main focus is on erosion of an ice cover. We therefore suggest maintaining a low source Froude number which will result to minimising impact on the lake floor or on the surface. It is expected that after the forced, pure and the lazy plumes might have rested, they are supposed to spread outwards to an infinite width. Though, this we were unable to capture in our results, possibly because of the scale length used in figure 3(a)). Furthermore, the lazy plume is expected to come to rest slowly as a result of its reduced velocity and slow entrainment rate as compared to that of the force plume. But rather, the behaviours in these two plumes as shown here appears similar when compared to those by Kay (2007). All these can be considered as limitations. But then, the lazy plume could penetrate further upwards before the temperature of zero buoyancy (see Fig. 3(d)) as compared to the other plumes. This is certain because, with its reduced velocity that have led to low entrainment, the process will not enhance quick production of mixed fluid, enabling the lighter fluid in the plume to penetrate further upwards. It is also ideal to state clearly that some level of carefulness is needed whenever we are intending to make use of the present results to foretell the behaviour of real plumes. In particular, if laboratory experiments are required either to examine the theory or to model larger-scale flows in the environment. Lastly, with the entrainment model, it is required that the plume will be fully turbulent, a condition usually obtained with a Reynolds number Re > 2000 according to Fischer et al. (1979). Whereas, our case here can be likened to the power station warm discharge which will definitely be fully turbulent with it large volume flux. These are basic conditions on the validity of these assumptions. In overall, the numerical results here are very good as they present us with more insight into rising plume with the quadratic dependence relation assumption.

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