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Effect of Variable Suction on Magneto Hydrodynamic Couette Flow through a Porous Medium in the Slip Flow Regime with Heat and Mass Transfer

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Abstract
The effect of variable suction on unsteady magneto hydrodynamic free convective couette flow of a viscous incompressible electrically conducting fluid in the slip flow regime in presence of variable suction with heat and mass transfer was analyzed. The governing equations of the flow field were solved employing perturbation technique which gave the expressions for the velocity, temperature, concentration and the rate of heat and mass transfer. The effects of the flow parameters such as magnetic parameter M, permeability parameter K_p, Grashof number for heat transfer G_r, suction parameters α_1, α_2; slip flow parameter h_1, h_2; Prandtl number P_r etc. on the flow field have been studied and the results are presented graphically and discussed quantitatively.

Keyword: MHD, Couette Flow, Heat and Mass Transfer, Slip Condition, Variable Suction

1. Introduction
The phenomenon of magneto hydrodynamic couette flow with heat transfer has been a subject of interest of many researchers because of its possible applications in many branches of science and technology. Channel flows through porous media have several engineering and geophysical applications such as in the field of chemical engineering for filtration and purification process, in the study of the underground resources, in the petroleum industry to study the movement of natural gas, oil and water through the oil channels and reservoirs.
In recent years, flow through porous media has been a subject of general interest of many researchers. A series of investigations have been made by different scholars where the porous medium are either bounded by horizontal or vertical surfaces.
Tao [1] studied the magneto hydrodynamic effects on the formation of couette flow. Sattar [2] reported the free and forced convection boundary layer flow through a porous medium with large suction. Sattar and Alam [3] analyzed the effect of thermal diffusion and transpiration on MHD free convective and mass transfer flow past an accelerated vertical porous plate. Attia and Kotb [4] have investigated the magneto hydrodynamic flow between two parallel plates with heat transfer. Das *et al*. [5] investigated the hydro magnetic flow and heat transfer

between two stretched/squeezed horizontal porous plates. Nagraju *et al*. [6] estimated the effect of simultaneous radiative and convective heat transfer in a variable porosity medium. Taneja and Jain [7] explained the hydro magnetic flow in the slip flow regime with time dependent
suction. Das et al [8] discussed the hydro magnetic flow and heat transfer of an elastico-viscous fluid between two horizontal parallel porous plates employing finite difference scheme.

The problem of oscillatory MHD slip flow along a porous vertical wall in a medium with variable suction in the presence of radiation was analyzed numerically by Ogulu and Prakash [9]. Makinde [10] investigated the free convection flow with thermal radiation and mass transfer past a moving vertical porous plate. Das et al [11] discussed the laminar flow of an elastico-viscous Rivlin-Ericksen fluid through porous parallel plates with suction and injection, the lower plate being stretched. Ogulu and Motsa [12] investigated the problem of radiative heat transfer to magneto hydrodynamic couette flow with variable wall temperature. Cortell [13] studied the flow and heat transfer of a fluid through a porous medium over a
stretching surface with internal heat generation/absorption and suction/ blowing. Das et al [14] analyzed the effect of heat source and variable magnetic field on unsteady
hydro magnetic flow of a viscous stratified field past a porous flat moving plate in the slip flow regime. In a separate paper Das *et al*. [15] studied the hydro magnetic three dimensional couette
flow and heat transfer. Recently, Das et al [16] estimated the effect of mass transfer on free convective MHD flow of a viscous fluid bounded by an oscillating porous plate in
the slip flow regime in presence of heat source. Sharma and Singh [17] investigated the unsteady MHD free convective flow and heat transfer along a vertical porous plate with variable
suction and internal heat generation. Das et al [18] studied the effect of variable suction and radiative heat transfer on MHD couette flow through a porous medium in the slip flow regime. Manna et al [19] investigated the effects of radiation on unsteady MHD free convective flow past an oscillating vertical porous plate embedded in a porous medium with oscillatory heat flux. K. sarada and B. shanker [20] analyzed the effect of chemical reaction on an unsteady MHD free convection flow past an infinite vertical porous plate with variable suction.

This present paper analyzes the effect of variable suction and radiative heat transfer on unsteady hydromagnetic free convective couette flow of a viscous incompressible electrically conducting fluid in the slip flow regime.

2. Problem/Model Formulation

Considering a two dimensional unsteady free convective magneto hydrodynamic flow of a viscous incompressible electrically conducting fluid between two vertical parallel porous plates placed at a distance *d* apart in the slip flow regime with heat and mass transfer in presence of variable suction. Let the medium between the plates be filled with a porous material of permeability

$K'(t') = K_O\left(1 + \varepsilon B e^{-\omega' t'}\right)$ and time dependent suction

$v'(t') = \; -v_0'\left(1 + \varepsilon A e^{-\omega' t'}\right)$ be applied at the plate y=0 and the same injection velocity be applied at the pate y=1.

We choose x-axis along the plate and y-axis normal to it. Under the above conditions the equations governing the flow are:

Momentum equation

$$\frac{\partial u'}{\partial t'} - v_0'\left(1 + \varepsilon A e^{-\omega' t'}\right)\frac{\partial u'}{\partial y'} = g\beta(T' - T_n') + g\beta^*(C' - C_n') +$$

$$v\frac{\partial^2 u'}{\partial y'^2} - \frac{\sigma B_0'}{\rho}u' - \frac{v}{K'(t')}u' \tag{1}$$

Energy equation

$$\frac{\partial T'}{\partial t'} - v_0'\left(1 + \varepsilon A e^{-\omega' t'}\right)\frac{\partial T'}{\partial y'} = \frac{k}{\rho C_P}\frac{\partial T'^2}{\partial y'^2} - \frac{1}{\rho C_P}\frac{\partial q_r}{\partial y'} \tag{2}$$

Concentration equation

$$\frac{\partial C'}{\partial t'} - v_0'\left(1 + \varepsilon A e^{-\omega' t'}\right) = D\frac{\partial C'}{\partial y'}\frac{\partial^2 C'}{\partial'^2} \tag{3}$$

The boundary conditions of the problem are:

$$u' - U_1 = L_1\frac{\partial u'}{\partial y'}, \frac{\partial T'}{\partial y'} = -\frac{q}{k}, \frac{\partial C'}{\partial y'} = -\frac{m}{D} \;\; at\; y' = 0$$

$$u' - U_2 = L_2\frac{\partial u'}{\partial y'}, T' = T_h', C' = C_h' \;\; at\; y' = d \tag{4}$$

The radiative heat flux q_r is given by $\frac{\partial q_r}{\partial y'} = 4\left(T' - T_h'\right)I.$ (5)

Where u' is the velocity, T' is the temperature, β is the volumetric coefficient of expansion for heat transfer, β^* is the volumetric coefficient of expansion for mass transfer, k is the thermal conductivity, β_0 is the uniform transverse magnetic field, v is the kinematic viscosity, ρ is the density, t is the time, F is the radiation parameter, C_p is the specific heat at constant pressure, σ is the electical conductivity, g is the acceleration due to gravity, ε is the small positive number, ω is the frequency oscillation, $A\; and\; B$ real positive constants such that

$\varepsilon A \ll 1$ and $\varepsilon B \ll 1$.

Introducing the following non-dimensional variables and parameters:

$y = \frac{y' v_0'}{\nu}$, $t = \frac{t' v_0'^2}{\nu}$, $u = \frac{u'}{v_0'}$, $M = \left(\frac{\sigma B_0^2}{\rho}\right)\frac{\nu}{v_0'^2}$, $K'(t') = K_O\left(1 + \varepsilon B e^{-\omega' t'}\right)$, $K_P = \frac{v_0'^2 K_0}{\nu^2}$,
$P_r = \frac{\rho \nu C_P}{k}$, $G_r = \frac{\nu^2 g \beta q}{k v_0'^4}$, $\theta = \frac{k v'(T' - T_h')}{\nu q}$, $F = \frac{4\nu l}{\rho C_P v_0'^2}$, $C = \frac{D v_0'(C' - C_0')}{\nu m}$, $G_m = \frac{\nu^2 g \beta^* m}{D v_0'^4}$, $S_C = \frac{\nu}{D}$ (6)

In equation (1)-(3) to obtain the following non-dimensional equations.

Momentum equation: $\frac{\partial u}{\partial t} - \left(1 + \varepsilon A e^{\omega' t'}\right)\frac{\partial u}{\partial y} = G_r \theta + G_m C + \frac{\partial^2 u}{\partial y^2} - \left[M + \frac{1}{K_P\left(1 + \varepsilon B e^{\omega' t'}\right)}\right] u$ (7)

Energy equation: $\frac{\partial \theta}{\partial t} - \left(1 + \varepsilon A e^{-\omega' t'}\right)\frac{\partial \theta}{\partial y} = \frac{1}{P_r}\frac{\partial^2 \theta}{\partial y^2} - F\theta$ (8)

Concentration equation: $\frac{\partial c}{\partial t} - \left(1 + \varepsilon A e^{-\omega' t'}\right)\frac{\partial c}{\partial y} = \frac{1}{S_C}\frac{\partial^2 c}{\partial y^2}$ (9)

The corresponding boundary conditions are:

$$u = \alpha_1 + h_1 \frac{\partial u}{\partial y}, \ \frac{\partial \theta}{\partial y} = -1, \frac{\partial c}{\partial y} = -1 \quad at\ y' = 0$$

$$u = \alpha_2 + h_2 \frac{\partial u}{\partial y} \ \theta = 0, C = 0 \quad at\ y' = d \qquad (10)$$

3. Method of Solution

Perturbation techniques and expressions is applied to solve the non-dimensional governing partial differential equations (7) through (9) respectively into dimensionless form with a system of ordinary differential equation, under the boundary conditions in equation (10) for particular case $R = 1$, which is valid for an incompressible fluid. In solving equations (7) through (9) we assume

$$u(y,t) = u_0(y) + \varepsilon u_1(y) e^{-\omega t} + 0(\varepsilon^2) \qquad (11)$$

$$\theta(y,t) = \theta_0(y) + \varepsilon \theta_1(y) e^{-\omega t} + 0(\varepsilon^2) \qquad (12)$$

$$C(y,t) = C_0(y) + \varepsilon C_1(y) e^{\frac{-i}{\omega t}} + 0(\varepsilon^2) \qquad (13)$$

Using equations (11)-(12) in equations (7)-(9), we obtained the following zeroth order and first order equations respectively:

$$u_0''(y) + u_0'(y) - \left(M + \frac{1}{K_P}\right) u_0(y) = G_r \theta_0(y) + G_m C_0(y) \qquad (14)$$

$$u_1''(y) - u_1'(y) - \left(\omega - M + \frac{B}{K_P \varepsilon e^{-i\omega t}}\right) u_1(y) = G_r \theta_1(y) + G_m C_1(y) - u_0'(y) A \qquad (15)$$

$$\theta_1''(y) + P_r \theta_0'(y) - P_r F \theta_0(y) = 0 \qquad (16)$$

$$\theta_1''(y) + P_r \theta_1'(y) + (P_r \omega - P_r F)\theta_1(y) = -P_r A \theta_0'(y) \qquad (17)$$

$$C_0''(y) + S_C C_0'(y) = 0 \qquad (18)$$

$$C_1''(y) + S_C C_1'(y) + S_C \omega C_1(y) = -A S_C C_0'(y) \qquad (19)$$

The corresponding boundary conditions to be used are

$$u_0 = \alpha_1 + h_1 \frac{\partial u_0}{\partial y}, u_1 = h_1 \frac{\partial u_1}{\partial y}, \frac{\partial \theta_0}{\partial y} = -1 \quad \frac{\partial \theta_1}{\partial y} = 0 \quad at\ y = 0$$

$$u_0 = \alpha_2 + h_2 \frac{\partial u_0}{\partial y}, u_1 = h_2 \frac{\partial u_1}{\partial y} \quad \theta_0 = 0,\ \theta_1 = 0 \quad at\ y = 1$$

$$u_0 = \alpha_3 + h_3 \frac{\partial u_0}{\partial y}, u_1 = h_3 \frac{\partial u_1}{\partial y} \quad C_0 = 0, C_1 = 0 \quad at\ y = 1$$

(20)

The solutions of equations (14)-(19) under boundary condition (20) are given by

$$u_0(y) = C_9 e^{m_9 y} + C_{10} e^{m_{10} y} + K_5 e^{m_5 y} + K_6 e^{m_6 y} K_7 e^{m_1 y} + K_8 e^{m_2 y} \quad (21)$$

$$u_1(y) = C_{11} e^{m_{11} y} + C_{12} e^{m_{12} y} + K_{15} e^{m_7 y} + K_{16} e^{m_8 y} + K_{17} e^{m_3 y} + K_{18} e^{m_4 y} + K_{19} e^{m_9 y} + K_{20} e^{m_{10} y} \quad (22)$$

$$\theta_0(y) = C_5 e^{m_5 y} + C_6 e^{m_6 y} \quad (23)$$

$$\theta_1(y) = C_7 e^{m_7 y} + C_8 e^{m_8 y} + K_3 e^{m_3 y} + K_4 e^{m_4 y} \quad (24)$$

$$C_0(y) = C_1 e^{m_1 y} + C_2 e^{m_2 y} \quad (25)$$

$$C_1(y) = C_3 e^{m_3 y} + C_4 e^{m_4 y} + K_1 e^{m_1 y} + K_2 e^{m_2 y} \quad (26)$$

4. Results and Discussion

The effect of variable suction on unsteady hydromagnetic free convective couette flow of a viscous incompressible electrically conducting fluid in the slip flow regime with heat and mass transfer has been studied. The governing equations of the flow fluid are solved for velocity, temperature, concentration and the rate of heat and mass transfer. The effects of the pertinent parameters on the flow fluid have been discussed with the aid of velocity profiles figure 1-8, temperature profiles figures 9-10 and concentration profiles figures 11-12 are presented graphically and discussed numerically for magnetic parameter M, prandtl number P_r, permeability parameter K_P, suction parameters α_1, α_2, slip flow parameters h_1, h_2, grashof number G_r, radiation parameter F, prandtl number P_r, schmidt number S_c, and frequency oscillation ω respectively.

4.1 Velocity field

In Figure 1 a magnetic parameter M is found increasing and velocity retards on the flow field.

Figure 2 shows an increase in permeability parameter K_P with velocity retarding on the flow fluid.

The velocity in the flow field increases at all points of the flow field due to an increase in Grashof number G_r in figure 3.

It is observed clearly that an increase in radiation parameter F decreases the velocity of the flow field at all points in figure 4.

Figure 5 and 6 shows an increase in both velocity and suction parameters α_1, α_2 in the flow filed. But thereafter the velocity decreases on figure 5.

The behaviour of velocity slip parameters h_1 and h_2 on Figures 7 and 8 respectively. Shows both the parameters tends to increase causes velocity retardation in the flow field at all points.

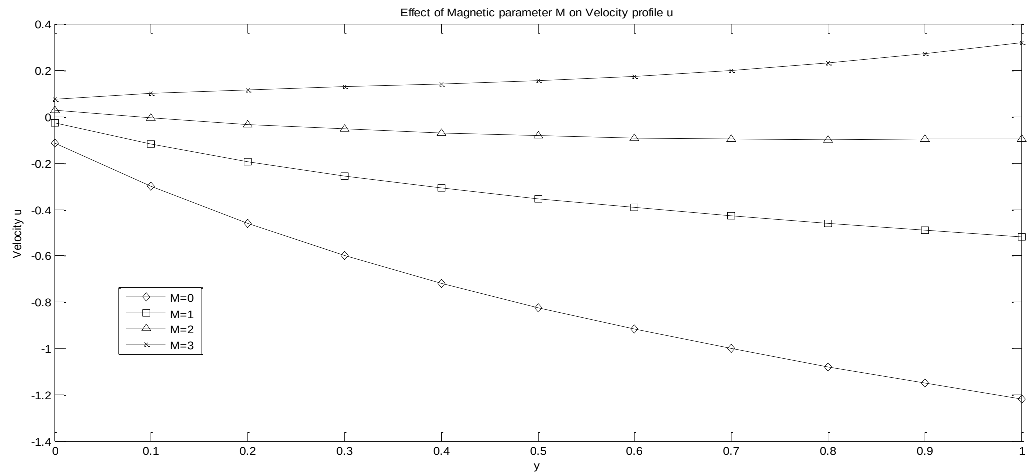

Figure 1. The effect of magnetic parameter $\boldsymbol{M}$ on the velocity field u. $\boldsymbol{M}$=0, 1, 2, 3.

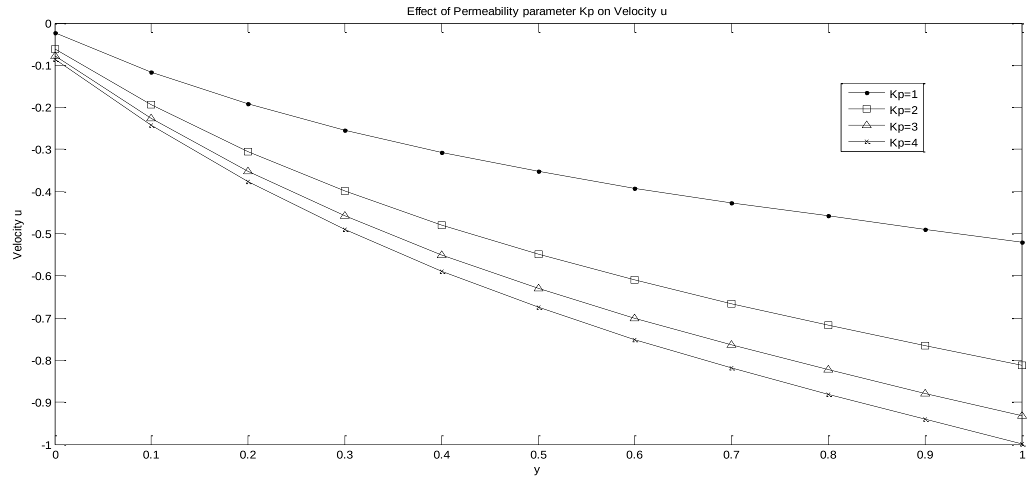

Figure 2. The effect of permearbility parameter $\boldsymbol{K_p}$on the velocity field u. $\boldsymbol{K_p}$= 1, 2, 3, 4.

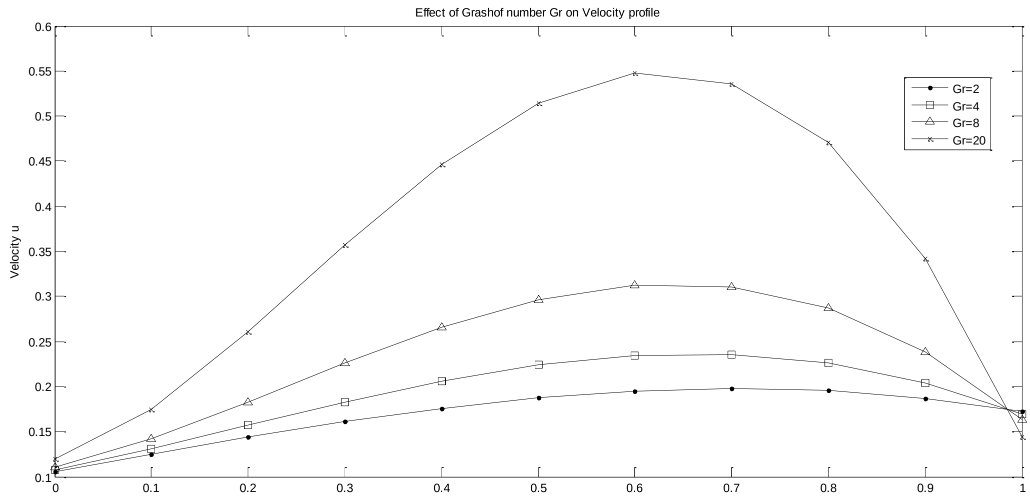

Figure 3. The effect of Grashof number $\boldsymbol{G_r}$on the velocity field u. $\boldsymbol{G_r}$= 2, 4, 8, 20.

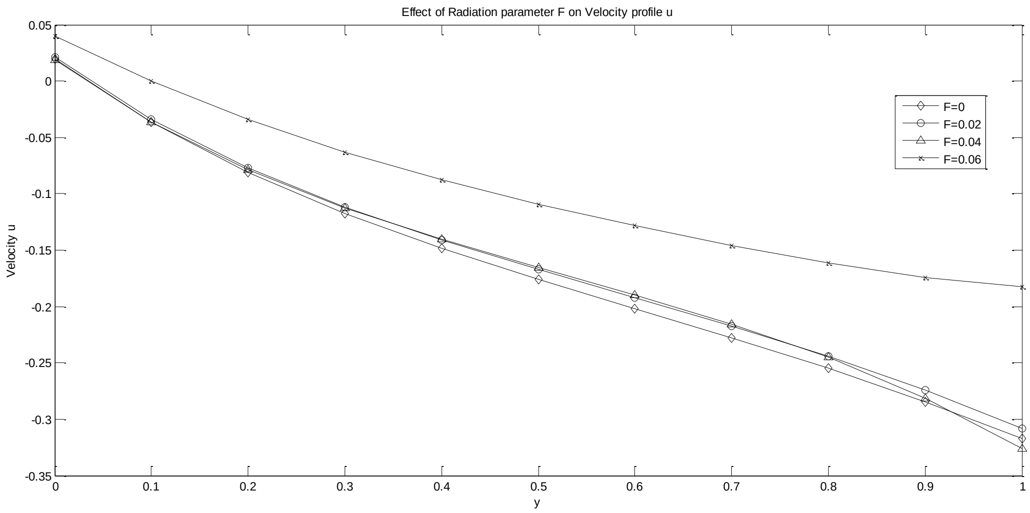

Figure 4. The effect of radiation parameter $\boldsymbol{F}$ on the velocity field u. $\boldsymbol{F}$= 0, 0.2, 0.4, 0.6.

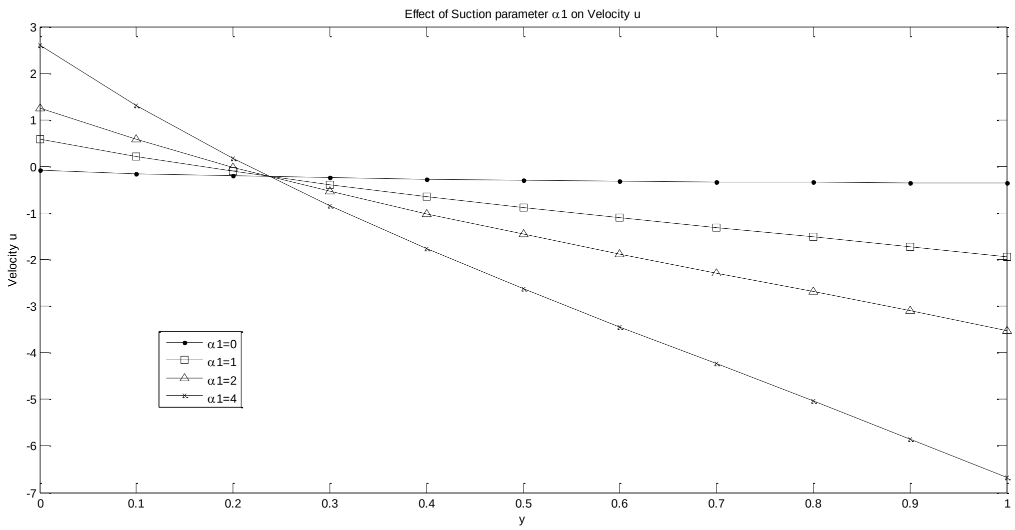

Figure 5. The effect of Suction parameter $\boldsymbol{\alpha_1}$on the velocity field u. $\boldsymbol{\alpha_1}$= 0, 1, 2, 4.

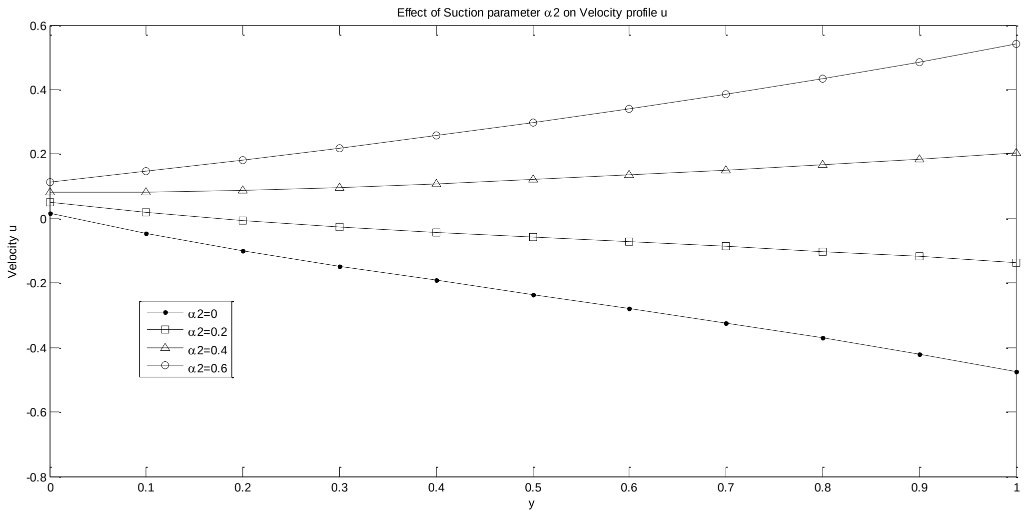

Figure 6. The effect of Suction parameter $\boldsymbol{\alpha_2}$on the velocity field u. $\boldsymbol{\alpha_2}$= 0, 0.2, 0.4, 0.6.

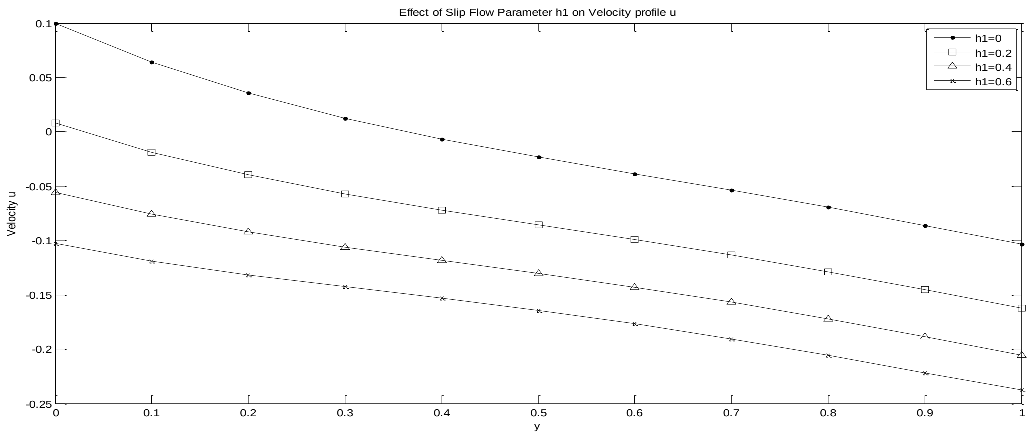

Figure 7. The effect of Slip flow parameter $\boldsymbol{h_1}$ on the velocity field u. $\boldsymbol{h_1}$= 0, 0.2, 0.4, 0.6.

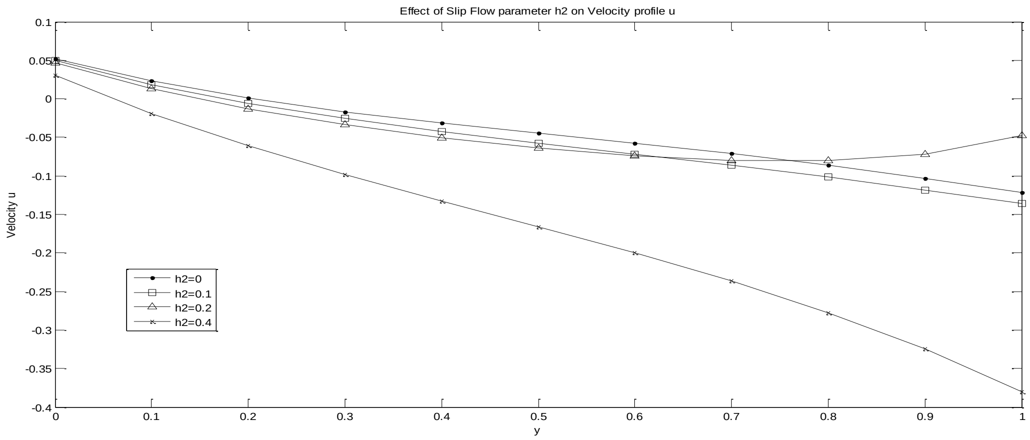

Figure 8. The effect of slip flow parameter $\boldsymbol{h_2}$ on the velocity field u. $\boldsymbol{h_2}$= 0, 0.1, 0.2, 0.4.

4.2 Temperature field

An increase in both radiation parameter F and Prandtl number P_r. And temperature θ decreases in the flow field at all points is shown on figure 9 and 10. The curves of both the figures is observed that both the parameters shows a decrease in temperature θ of the flow field at all points.

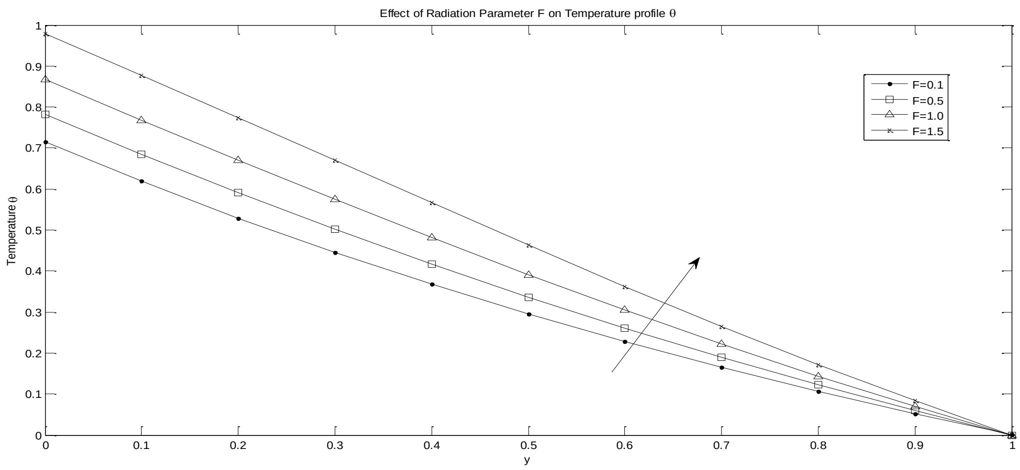

Figure 9. The effect of radiation parameter $\boldsymbol{F}$ on the temprature field $\boldsymbol{\theta}$. $\boldsymbol{F}$= 0.1, 0.5, 1.0, 1.5.

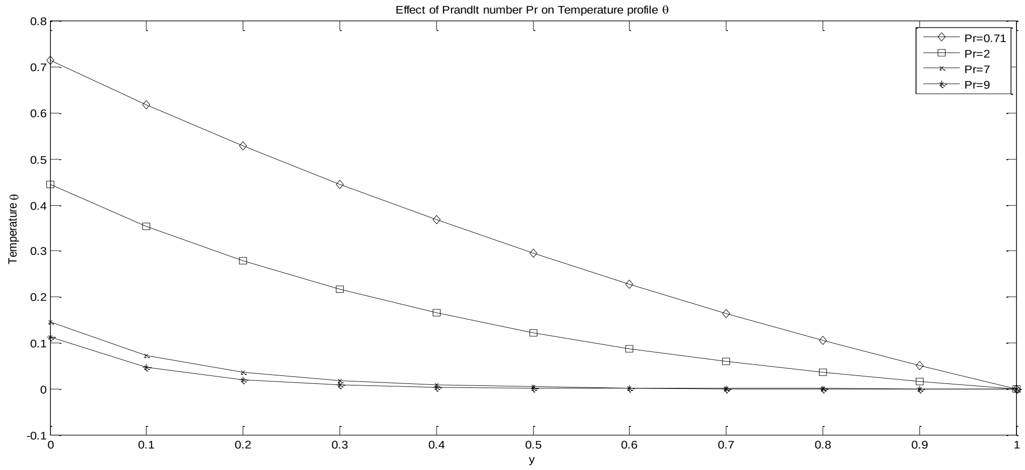

Figure 10. The effect of Prandtl number $\boldsymbol{P_r}$ on the temprature field $\boldsymbol{\theta}$. $\boldsymbol{P_r}$= 0.71, 2, 7, 9.

4.3 Concentration feild

Shmidt number S_c and frequency oscillation ω increases while concentration C is found decreasing in the flow field in figures 11 and 12 respectively.

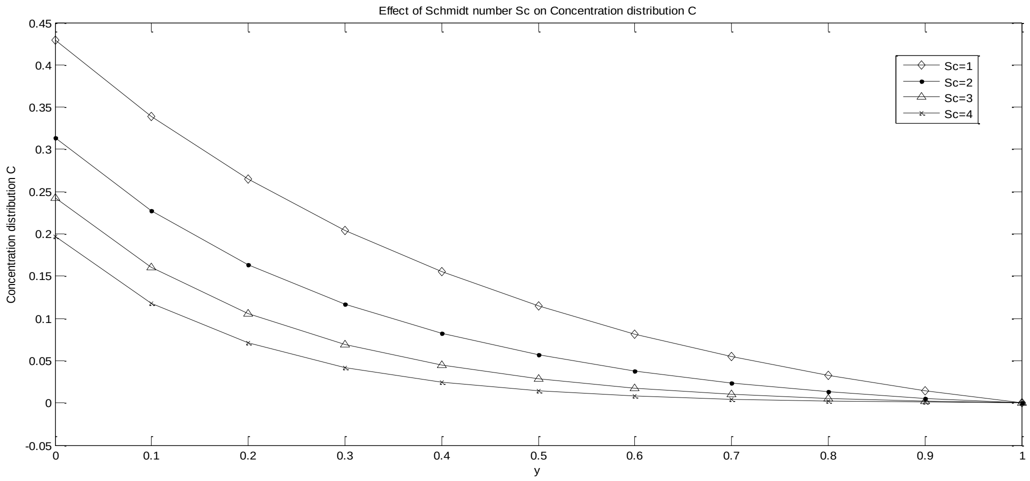

Figure 11. The effect of Schmidt number $\boldsymbol{S_c}$ on the concentration field $\boldsymbol{C}$. $\boldsymbol{S_c}$ = 1, 2, 3, 4.

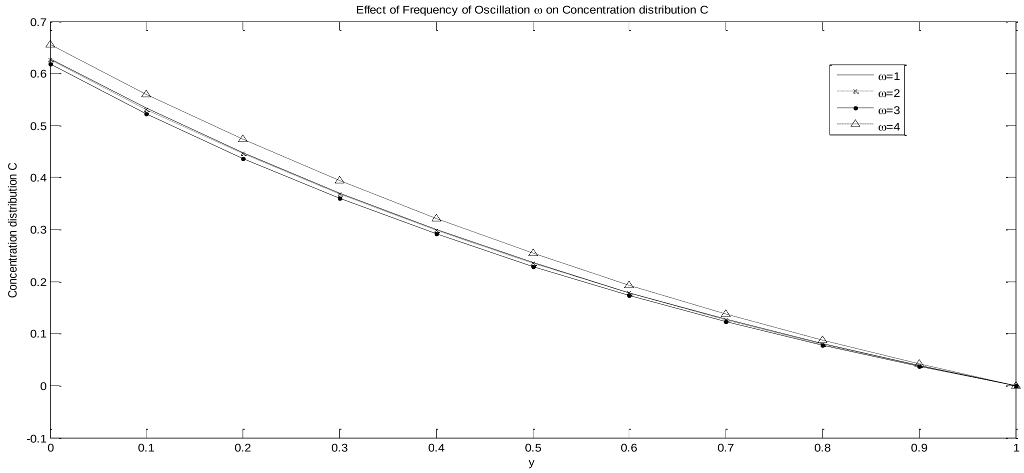

Figure 12. The effect of Frequency of oscillation $\boldsymbol{\omega}$ on the concentration field $\boldsymbol{C}$. $\boldsymbol{\omega}$ = 1, 2, 3, 4.

5. Conclusion

In this paper, we have considered the effect of variable suction on unsteady hydromagnetic free convective couette flow of a viscous incompressible electrically conducting fluid through a porous medium in the slip flow regime with heat and mass transfer. The unsteady free convective MHD flow is between two vertical parallel porous plates at a distance apart in the slip flow regime in presence of variable suction with heat and mass source. The medium between the plates filled with a porous material of permearbility and a time dependent suction applied at the plate y=0 and the same injuction velocity applied at the other plate y=1. We choosed x-axis along the plate and y-axis normal to it.

The problem is solved by employing perturbation techniques and expressions for concentration, energy and momentum equations. The analysis of the fluid flow are presented for velocity, temperature, concentration and the rate of heat and mass transfer at the wall in the flow field and the effect of the pertinent parameters on the fluid flow have been discussed with the aid of velocity profile, temperature profile and concentration profile respectively.

In conclusion, it was discovered that the effect of certain pertinent parameter such as magnetic parameter M, grashof number G_r, permeability parameter K_P are found increasing in the velocity flow field up to a certain distance and thereafter the effect reverses, both radiation parameter F and Prandtl number P_r shows an increase in temperature θ of the flow field, Shmidt number S_c and frequency oscillation ω increases in concentration C of the flow field.

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CONSTANTS

$a_1 = S_C$ $\quad a_2 = S_C$

$a_3 = S_C\omega$ $\quad a_4 = m_1K_1 + m_2K_2$

$a_5 = K_1e^{m_1} + K_2e^{m_2}$ $\quad a_6 = P_r$ $\quad a_7 = P_rF$

$a_8 = P_r$ $\quad a_9 = (P_r\omega - P_rF)$

$a_{10} = m_3K_3 + m_4K_4$ $\quad a_{11} = K_3e^{m_3} + K_4e^{m_4}$

$$a_{12} = \left(M + \frac{1}{K_P}\right) \qquad a_{13} = \left(\omega - M + \frac{B}{K_P\varepsilon e^{-i\omega t}}\right)$$

$a_{14} = 1 - h_1m_9$ $\quad a_{15} = 1 - h_1m_{10}$

$a_{16} = e^{m_9} - h_2m_9e^{m_9}$ $\quad a_{17} = e^{m_{10}} - h_2m_{10}e^{m_{10}}$

$a_{18} = 1 - h_1m_{11}$ $\quad a_{19} = 1 - h_1m_{11}$

$a_{20} = e^{m_{11}} - h_2m_{11}e^{m_{11}}$ $\quad a_{21} = e^{m_{11}} - h_2m_{11}e^{m_{11}}$

$m_1 = 0$ $\quad m_2 = -a_1$

$$m_3 = \frac{-a_2 + \sqrt{a_2^2 - 4a_3}}{2} \qquad m_4 = \frac{-a_2 - \sqrt{a_2^2 - 4a_3}}{2}$$

$$m_5 = \frac{-a_6 + \sqrt{a_6^2 - 4a_7}}{2} \qquad m_6 = \frac{-a_6 - \sqrt{a_6^2 - 4a_7}}{2}$$

$$m_7 \frac{-a_8 + \sqrt{a_8^2 - 4a_9}}{2} \qquad m_8 \frac{-a_8 - \sqrt{a_8^2 - 4a_9}}{2}$$

$$m_9 = \frac{-1 + \sqrt{1 - 4a_{12}}}{2} \qquad m_{10} = \frac{-1 - \sqrt{1 - 4a_{12}}}{2}$$

$$m_{11} = \frac{-1 + \sqrt{1 - 4a_{13}}}{2} \qquad m_{12} = \frac{-1 - \sqrt{1 - 4a_{13}}}{2}$$

$$C_1 = \frac{-C_2e^{m_2}}{e^{m_1}} \qquad C_2 = \frac{-e^{m_1}}{m_2e^{m_1} + m_1e^{m_2}}$$

$$C_3 = \frac{a_4 - m_4C_4}{m_3} \qquad C_4 = \frac{a_5m_3 - a_4e^{m_3}}{m_4e^{m_3} - m_3e^{m_4}}$$

$$C_5 = -\frac{-C_6e^{m_6}}{e^{m_5}} \qquad C_6 = \frac{-e^{m_5}}{(m_6e^{m_5} - m_5e^{m_6})}$$

$$C_8 = \frac{a_{11}m_7 - a_{10}\, e^{m_7}}{m_8e^{m_7} - m_7e^{m_8}} \qquad C_7 = \frac{-C_8e^{m_7} - a_{11}}{e^{m_7}}$$

$$C_9 = \frac{K_{11} - K_{14}}{a_{14} - a_{16}} \qquad C_{10} = \frac{K_{11} - C_9a_{14}}{a_{15}}$$

$$C_{11} = \frac{K_{23} - K_{26}}{a_{18} - a_{20}} \qquad C_{12} = \frac{K_{23} - C_{11}a_{18}}{a_{19}}$$

$$K_1 = \frac{-AS_Cm_1C_1}{m_1^2 + a_2m_1 + a_3} \qquad K_2 = \frac{-AS_Cm_2C_2}{m_2^2K_2 + a_2m_2K_2 + a_3K_2}$$

$$K_3 = \frac{-P_rAm_5C_5}{m_5^2 + a_8m_5 + a_9} \qquad K_4 = \frac{-P_rAm_6C_6}{m_6^2 + a_8m_6 + a_9}$$

$$K_5 = \frac{G_rC_5}{m_5^2 - m_5 - a_{12}} \qquad K_6 = \frac{G_rC_6}{m_6^2 - m_6 - a_{12}}$$

$$K_7 = \frac{G_m C_1}{m_1^2 - m_1 - a_{12}} \qquad K_8 = \frac{G_m C_2}{m_2^2 - m_2 - a_{12}}$$

$$K_9 = K_5 + K_6 + K_7 + K_8 \qquad K_{10} = \alpha_1 + h_1(m_5 K_5 + m_6 K_6 + m_1 K_7 + m_2 K_8)$$

$$K_{11} = K_{10} - K_9 \qquad K_{12} = K_5 e^{m_5} + K_6 e^{m_6} + K_7 e^{m_1} + K_8 e^{m_2}$$

$$K_{13} = \alpha_2 + h_2(m_5 K_5 e^{m_5 y} + m_6 K_6 e^{m_6 y} + m_1 K_7 e^{m_1 y} + m_2 K_8 e^{m_2 y})$$

$$K_{14} = K_{13} - K_{12} \qquad K_{15} = \frac{G_r C_7}{m_7^2 + m_7 - a_{13}}$$

$$K_{16} = \frac{G_r C_8}{m_8^2 + m_8 - a_{13}} \qquad K_{17} = \frac{G_m C_3}{m_3^2 + m_3 - a_{13}}$$

$$K_{18} = \frac{G_m C_4}{m_4^2 + m_4 - a_{13}} \qquad K_{19} = \frac{-A m_9 C_9}{m_9^2 + m_9 - a_{13}}$$

$$K_{20} = \frac{-A m_{10} C_{10}}{m_{10}^2 + m_{10} - a_{13}} \qquad K_{21} = K_{15} + K_{16} + K_{17} + K_{18} + K_{19} + K_{20}$$

$$K_{22} = h_1(m_{11} C_{11} + m_{12} C_{12} + m_7 K_{15} + m_8 K_{16} + m_3 K_{17} + m_4 K_{18} e^{m_4 y} + m_9 K_{19} + m_{10} K_{20})$$

$$K_{23} = K_{22} - K_{21} \qquad K_{24} = K_{15} e^{m_7} + K_{16} e^{m_8} + K_{17} e^{m_3} + K_{18} e^{m_4} + K_{19} e^{m_9} + K_{20} e^{m_{10}}$$

$$K_{25} = h_2(m_7 K_{15} e^{m_7} + m_8 K_{16} e^{m_8} + m_3 K_{17} e^{m_3} + m_4 K_{18} e^{m_4} + m_9 K_{19} e^{m_9} + m_{10} K_{20} e^{m_{10}})$$