International Journal of Mathematics and Computer Research

ISSN: 2320-7167

Volume 13 Issue 01 January 2025, Page no. – 4711-4717

Index Copernicus ICV: 57.55, Impact Factor: 8.316

DOI: 10.47191/ijmcr/v13i1.02

Analytical Solution of Fractional Order Mathematical Model in the Time of COVID-19 by Fractional Differential Transform Method

A. D. Nagargoje[1], **R. S. Teppawar**[2]

[1,2] P. G. Department of Mathematics, N.E.S. Science College, Nanded - 431602, (MH), India

ARTICLE INFO	ABSTRACT
Published Online: 04 January 2025 **Corresponding Author:** Dr. ARUN NAGARGOJE	In this paper, we will discuss an analytical solution and numerical simulation of fractional order mathematical model on COVID-19 under Caputo sense with the help of fractional differential transform method for different values of *q*, where $q \in (0,1]$. The underlying mathematical model on COVID-19 is consist of two compartments, like, healthy individual and infected individual. We show the reliability and simplicity of the method by comparing our solution of given model with the solution obtained by Laplace Adomian decomposition method via graphically and numerically. Further, we analyse the stability of model using Lyapunov direct method under Caputo sense. We conclude that use of fractional epidemic model provides better understanding and biologically more insights about the disease dynamics. **Mathematics Subject Classification:** 2010 MSC: 26A33, 37M05

KEYWORDS: Caputo fractional model, COVID-19, Numerical simulation, Stability analysis, Analytical solution, fractional differential transform method

1 INTRODUCTION

The idea of models on the CORONA virus disease have been discussed by some authors using the various method Adams-Moulton type, Laplace transform coupled with Adomain decomposition method, generalized - BashforthMoulton method, q-homotopy analysis transform method which can be found in [1, 2, 3, 4, 5].

The differential transform method was applied in the different field by many researcher which can be found in [6, 7]. In this paper, we will study a new application of the fractional differential transform method to obtain an approximate solutions for the system of fractional order mathematical model on COVID-19 (1.1).

For further use we will recall the two most commonly used definitions of fractional derivative like Riemann–Liouville and Caputo which are given below.

Definition 1.1 Riemann–Liouville definition *[10, 11]: For* $\alpha \in [n-1,n)$ *the* α *- derivative of f is*

$$D_a^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n x}{dt^n} \int_a^\alpha \frac{f(x)}{(t-x)^{\alpha-n+1}} dx$$

Definition 1.2 Caputo definition*[10, 11]:For* $\alpha \in (n-1,n)$ *the* α *- derivative of f is*

$${}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(\alpha-n)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau$$

Recently, some researchers have constructed the COVID-19 Models under ordinary derivatives, as a modification to some previously studied preypredator models which can be found in [9]. The study of the mathematical models under fractional derivatives instead of classical ordinary derivatives produces more significant results which are more helpful for understanding. Now in this paper, we will investigate the COVID-19 model (1.1), under Caputo fractional derivative sense which is constructed as follows:

$$\begin{cases} {}^cD^{\mu_1} H = \alpha H(t) - \beta H(t) I(t) + \rho I(t), \\ {}^cD^{\mu_2} I = \beta H(t) I(t) + \gamma I(t) - \delta I(t) - \rho I(t), \\ H(0) = H_0, I(0) = I_0, \end{cases} \tag{1.1}$$

where $0 < \mu_1, \mu_2 \leq 1$.

In the above model,

$H \longrightarrow$ stands for healthy individual, $I \longrightarrow$ for infected individuals, and $\beta \longrightarrow$ for the infection rate, $\alpha \longrightarrow$ the rate of immigration of healthy individuals from one place to another place,

$\gamma \longrightarrow$ the rate at which infection take immigration,

$\delta \longrightarrow$ death rate, $\rho \longrightarrow$ cure rate.

Since immigration of people is also a big cause of spreading of this disease, it is evident to check the impact of the immigration of individuals on the transmission dynamics of the current disease. On the other hand, such a study may help in forming some precautionary measures to protect more people from this infection.

2 STABILITY ANALYSIS

We will discuss the Mittag-Leffler and asymptotical stability analysis of model (1.1). Now consider the following fractional system as a particular case of model (1.1)

where 0

$$ {}_{t_0}^{C}D_t^q h(t) = Ah(t) + \phi(t, h(t)), \tag{2.6}$$

where $0 < q \leq 1$, $h(t) = (h_1(t), h_2(t))^T, \phi(t,h(t)) = (sinh_1(t), sinh_2(t))^T$,

$$A = \begin{pmatrix} -3 & 1 \\ 1 & -2 \end{pmatrix}.$$

Here ϕ is Lipschitz continuous with Lipschitz constant $L = 1$. If we select $V(t) = h^T h$, where $P = I_2$, then we have,

$$\begin{aligned} {}_{t_0}^{C}D_t^q V(t) &\leq 2h^T(t) {}_{t_0}^{C}D_t^q h(t) \\ &= [Ah + \phi(t,h)]^T h + h^T [Ah + \phi(t,h)] \\ &= -6h_1^2 + 4h_1h_2 - 4h_2^2 + 2h_1 sinh_1 + 2h_2 sinh_2 \\ &\leq -4(h_1 - \frac{1}{2}h_2)^2 - h_2^2. \end{aligned}$$

Hence, ${}_{t_0}^{C}D_t^q V(t)$ is negative define, which implies the trivial solution of system (2.6) is Mittag- Leffler stability according to Theorem 3.1[12]. Furthermore, since $A^T P + PA + \eta I = \begin{pmatrix} -3 & 1 \\ 1 & -2 \end{pmatrix} < 0$, and hence by Theorem 3.1[12], the trivial solution of system (2.6) is also asymptotically stable . Hence by Theorem 3.1[12] and Theorem 3.3[12], we conclude that our model (1.1) is Mittag-Leffler and asymptotically stable.

3 FRACTIONAL DIFFERENTIAL TRANSFORM METHOD

In this section, we will recall some results of fractional differential transform method [7]. Since the initial conditions are implemented to the integer order derivatives, the transformation of the initial conditions are defined by using FDTM as follows.

$$F(k) = \begin{cases} If\ \frac{k}{\xi} \in Z^+, \frac{1}{\frac{k}{\xi}!}\left[\frac{d^{\frac{k}{\xi}} f(x)}{dx^{\frac{k}{\xi}}}\right]_{x=x_0} & for\quad k = 0,1,2,3,.....,(q\xi - 1) \\ If\ \frac{k}{\xi} \notin Z^+, 0 \end{cases} \tag{3.5}$$

where, q is the order of fractional differential equation considered. The following theorems that can be used to deduced solution of given fractional model.

Theorem 3.1 *[7] If $f(x) = g(x) \pm h(x)$, then $F(k) = G(k) \pm H(k)$.*

Theorem 3.2 *[7] If $f(x) = g(x)h(x)$, then* $F(k) = \sum_{l=0}^{k} G(l)H(k-l)$.

Theorem 3.3 *[7] If $f(x) = (x - x_0)^p$, then $F(k) = \delta(k - \xi p)$ where,*

$$\delta(k) = \begin{cases} 1\ if\ k = 0 \\ 0\ if\ k \neq 0. \end{cases},$$

Theorem 3.4 *[7] If $f(x) = D_x^q{}_0[g(x)]$, then* $F(k) = \frac{\Gamma(q+1+\frac{k}{\xi})}{\Gamma(1+\frac{k}{\xi})} G(k+\xi q)$.

4 SOLUTION OF MODEL IN TIME OF COVID-19 USING FDTM

Now apply FDTM on the first equation of model system (1.1), and then by using Theorem 3.1 to Theorem 3.4, we have,

$$\frac{\Gamma(\mu_1 + 1 + \frac{k}{\xi_1})}{\Gamma(1 + \frac{k}{\xi_1})} H(k + \xi_1\mu_1) = \alpha H(k) - \beta \sum_{l=0}^{k} H(l)I(k-l) + \rho I(k)$$

this implies

$$H(k+\xi_1\mu_1)=\frac{\Gamma(1+\frac{k}{\xi_1})\left[\alpha H(k) \not\beta \quad \sum_{l=0}^{k} H(l)I(k-l)+\rho I(k)\right]}{\Gamma(\mu_1+1+\frac{k}{\xi_1})}$$

(4.1) Similarly, the second equation of model (1.1) is transformed into

$$\frac{\Gamma(\mu_2+1+\frac{k}{\xi_2})}{\Gamma(1+\frac{k}{\xi_2})}I(k+\xi_2\mu_2) \not= \beta \quad \sum_{l=0}^{k} H(l)I(k-l)+\gamma I(k)-\delta I(k)-\rho I(k)$$

this implies

$$I(k+\xi_2\mu_2)=\frac{\Gamma(1+\frac{k}{\xi_2})\beta\left[\quad \sum_{l=0}^{k} H(l)I(k-l)+\gamma I(k)-\delta I(k)-\rho I(k)\right]}{\Gamma(\mu_2+1+\frac{k}{\xi_2})} \quad (4.2)$$

Subject to initial conditions and using (3.5) which reduces to

$H(0) = H_0 = 0.7, I(0) = I_0 = 0.3$

Now, we take values for parameters as follows,

$\alpha = 0.0, \beta = 0.03, \gamma = 0.05, \delta = 0.05,$ *and* $\rho = 0.05$.

Therefore, the series solution of the transformed expressions when $k = 6$

and $\mu_1=\frac{1}{2}, \mu_2=\frac{1}{2}, \xi_1=2, \xi_2=2$ for $H(t$) and $I(t)$ are obtained as

$h(t) = 0.7 + 0.009816898753730959t^2 - 0.0003306t^4 + 1.1625273084042954e - 05t^6$

$- 2.936249556503726e - 07t^8 + 1.2808810766608499e - 08t^{10} - 4.931204308682898e - 10t^{12} + 1.407481129690695e - 11t^{14} + ...$

$i(t) = 0.3 - 0.009816898753730959t^2 + 0.0003306t^4 - 1.1625273084042954e - 05t^6$

$+ 2.936249556503726e - 07t^8 - 1.2808810766608499e - 08t^{10} + 4.931204308682898e - 10t^{12} - 1.407481129690695e - 11t^{14} + ...$

For $\mu_1 = 0.3, \mu_2 = 0.4, \xi_1 = 10, \xi_2 = 10$ and using (3.5)initial conditions are reduces to

$H(0) = 0.7, H(1) = H(2) = 0,$

$I(0) = 0.3, I(1) = I(2) = I(3) = 0.$

Hence $H(t)$ and $I(t)$ are obtained as

$$h(t)=0.7+0.009693909824361522t^{10}-0.00039844335699694406t^{\frac{40}{3}}+1.865493365962163e-05t^{\frac{50}{3}}$$
$$-7.062526448001621e-07t^{20}+4.114235320785562e-08t^{\frac{70}{3}}-2.272616649585302e-09t^{\frac{80}{3}}$$
$$+8.816579301827117e-11t^{\frac{90}{3}}+.....$$
$$i(t)=0.3-0.009805426332478438t^{10}+0.00039890953866837 17t^{\frac{50}{4}}-1.8502764966344836e-05t^{15}$$
$$+6.944977753945928e-07t^{\frac{70}{4}}-4.01376283871082e-08t^{20}+2.200824977204085e-09t^{\frac{90}{4}}$$
$$-8.816579301827117e-11t^{25}+......$$

For $\mu_1 = 0.7, \mu_2 = 0.7, \xi_1 = 10, \xi_2 = 10$ and using (3.5)initial conditions are reduces to

$H(0) = 0.7, H(1) = H(2) = H(3) = H(4) = H(5) = H(6) = 0,$

$I(0) = 0.3, I(1) = I(2) = I(3) = I(4) = I(5) = I(6) = 0.$

Now by using above values we obtained, $H(t)$ and $I(t)$ are as

$h(t) = 0.7 + 0.009574762428055888t^{10/7} - 0.00037164099253113466t^{20/7} + 1.6107798925693248\ e - 05t^{30/7} - 5.493385456765303e - 07t^{40/7} + 2.905760594937488e - 08t^{50/7} - 1.4308899215846184e - 09t^{60/7} + 5.004325516403819e - 11t^{10} + ...$

$i(t) = 0.3 + -0.009574762428055888t^{10/7} + 0.00037164099253113466t^{20/7} - 1.6107798925693248\ e - 05t^{30/7} + 5.493385456765302e - 07t^{40/7} - 2.905760594937488e - 08t^{50/7} + 1.4308899215846184e - 09t^{60/7} - 5.004325516403819e - 11t^{10} + ...$

For $\mu_1 = 1, \mu_2 = 1, \xi_1 = 1, \xi_2 = 1$ and hence $H(t)$ and $I(t)$ are obtained as

$h(t) = 0.7 + 0.0087t - 0.0001653t^2 + 2.8507e - 06t^3 - 2.708165e - 08t^4$

$+ 6.673781600000001e - 10t^5 - 1.1295038996666667e - 11t^6 + 1.5523752840761906e - 13t^7 + ...$

$i(t) = 0.3 - 0.0087t + 0.0001653t^2 - 2.8507e - 06t^3 + 2.7081650000000005e - 08t^4$

$- 6.673781600000001e - 10t^5 + 1.1295038996666667e - 11t^6 - 1.5523752840761906e - 13t^7 + ...$

5 SOLUTION BY LADM

Solutions by LADM which can be adoted from[9], series solution up to four terms as follows:

$$h(t) = 0.7 + 0.087\left[1-\mu+\frac{t^{\mu}}{\Gamma(\mu)}\right] + 0.02412\left[(1-\mu)^2+\frac{2(1-\mu)t^{\theta}}{\Gamma(\mu)}+\frac{\mu t^{2\mu}}{2\Gamma(2\mu)}\right] + 0.008436$$
$$\left[(1-\mu)^3+\frac{[2(1-\mu)+(1-\mu)^2t^{\mu}]}{\Gamma(\mu)}+\frac{[5\mu(1-\mu)]t^{2\mu}}{2\Gamma(2\mu)}+\frac{[\mu^2]t^{3\mu}}{\Gamma(3\mu)}\right] - 0.0011745$$
$$\left[(1-\mu)^3+2(1-\mu)^2\frac{t^{\mu}}{\Gamma(\mu)}+(1-\mu)\frac{t^{2\mu}}{\Gamma(2\mu)}+\frac{\mu(1-\theta)t^{2\mu}}{\Gamma(2\mu)}+\frac{\mu\Gamma(2\mu+1)t^{3\mu}}{\Gamma(2\mu)\Gamma(3\mu+1)}\right],$$

$$I(t) = 0.3 - 0.045\left[1-\mu+\frac{t^{\mu}}{\Gamma(\mu)}\right] - 0.02412\left[(1-\mu)^2+\frac{2(1-\mu)t^{\theta}}{\Gamma(\mu)}+\frac{\mu t^{2\mu}}{2\Gamma(2\mu)}\right] - 0.0190044$$
$$\left[(1-\mu)^3+\frac{[2(1-\mu)+(1-\mu)^2t^{\mu}]}{\Gamma(\mu)}+\frac{[5\mu(1-\mu)]t^{2\mu}}{2\Gamma(2\mu)}+\frac{[\mu^2]t^{3\mu}}{\Gamma(3\mu)}\right] + 0.0011745$$
$$\left[(1-\mu)^3+2(1-\mu)^2\frac{t^{\mu}}{\Gamma(\mu)}+(1-\mu)\frac{t^{2\mu}}{\Gamma(2\mu)}+\frac{\mu(1-\theta)t^{2\mu}}{\Gamma(2\mu)}+\frac{\mu\Gamma(2\mu+1)t^{3\mu}}{\Gamma(2\mu)\Gamma(3\mu+1)}\right],$$

6 NUMERICAL SIMULATION AND DISCUSSION

In this section, we provide numerical simulation by using table and graphs. The mathematical analysis of epidemic model with non linear system of fractional differential equation has been presented. We observe that fractional order COVID -19 model has more degree of freedom as compared to ordinary derivatives. The compression for some different values of q has been obtained and numerical simulation are shown using FDTM and LADM.

Through a novel methods FDTM, we have derived approximate solution for the corresponding model under investigation. Further, some numerical result have been presented for different fractional orders by using Python software for different immigration states. We observed that as the immigration infected class is increasing, it will cause the decrease in health population and hence the population of infected people increases. Therefore the increase in the infection of current and break is free immigration. We plot the solutions of given model (1.1) for different fractional order by using Python software, which is shown in Figures.

Table 1: Table of healthy individual for different fractional orders using FDTM.

S.N.	μ = 0.5	μ = 0.3	μ = 0.7	μ = 1
1	7.00000000e-01	7.00000000e-01	7.00000000e-01	7.00000000e-01
2	8.47143191e+12	3.87176726e+29	4.40156704e+06	1.06217720e+00
3	1.40256270e+17	4.09227088e+36	4.81097260e+09	1.09620304e+01
4	4.10245857e+19	5.25729328e+40	2.82250566e+11	1.81456123e+02
5	2.30399857e+21	4.32513610e+43	5.04957996e+12	1.45302184e+03
6	5.24033779e+22	7.89284438e+45	4.72167157e+13	7.30040365e+03
7	6.72931872e+23	5.55643195e+47	2.93070846e+14	2.71841335e+04

Table 2: Table of infected individuals, for different fractional orders using FDTM.

S.N.	μ = 0.5	μ = 0.4	μ = 0.7	μ = 1
1	3.00000000e-01	3.00000000e-01	3.00000000e-01	3.00000000e-01
2	-8.47143191e+12	-4.74965575e+19	-4.40156604e+06	-6.21771972e-02
3	-1.40256270e+17	-8.81437888e+24	-4.81097260e+09	-9.96203044e+00
4	-4.10245857e+19	-1.06379467e+28	-2.82250566e+11	-1.80456123e+02
5	-2.30399857e+21	-1.63420253e+30	-5.04957996e+12	-1.45202184e+03
6	-5.24033779e+22	-8.11408027e+31	-4.72167157e+13	-7.29940365e+03

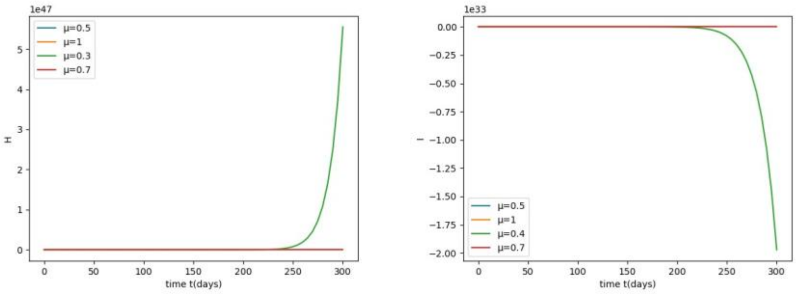
7	-6.72931872e+23	-1.97203728e+33	-2.93070846e+14	-2.71831335e+04

Table 3: Table of healthy individual for different fractional orders using LADM.

S.N.	$\mu = 0.5$	$\mu = 0.3$	$\mu = 0.7$	$\mu = 1$
1	7.50437687e-01	7.75209494e-01	7.28466860e-01	7.000000e-01
2	1.14204032e+02	2.83892641e+01	1.45104382e+02	-1.199300e+03
3	8.53089931e+02	1.86898766e+02	1.07239002e+03	-9.746000e+03
4	2.81730573e+03	5.90515426e+02	3.52872979e+03	-3.304640e+04
5	6.60653508e+03	1.35335440e+03	8.25995606e+03	-7.850750e+04
6	1.28204479e+04	2.58952725e+03	1.60118720e+04	-1.535363e+05
7	2.20587098e+04	4.41314454e+03	2.75302711e+04	-2.655398e+05

Table 4: Table of infected individuals, for different fractional orders using LADM.

S.N.	$\mu = 0.5$	$\mu = 0.4$	$\mu = 0.7$	$\mu = 1$
1	2.69241262e-01	2.60465542e-01	2.83847793e-01	3.0000000e-01
2	-3.07656724e+02	-1.62885329e+02	-5.85285852e+02	5.4187500e+02
3	-2.37557937e+03	-1.23124971e+03	-4.56910354e+03	4.4670000e+03
4	-7.92132406e+03	-4.07424516e+03	-1.52951490e+04	1.5219525e+04
5	-1.86625531e+04	-9.56116251e+03	-3.61071912e+04	3.6243300e+04
6	-3.63169182e+04	-1.85612864e+04	-7.03489791e+04	7.0982175e+04
7	-6.26020681e+04	-3.19438998e+04	-1.21364255e+05	1.2288000e+05

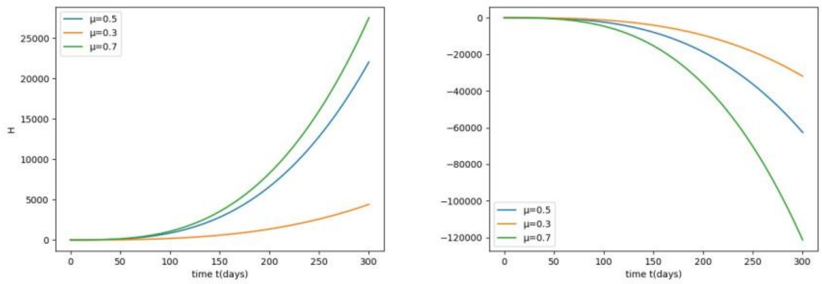


(a) Plots of healthy individual (b) Plots of infected individuals

Figure 1: Plots of healthy individual and infected individuals at different fractional orders using FDTM.

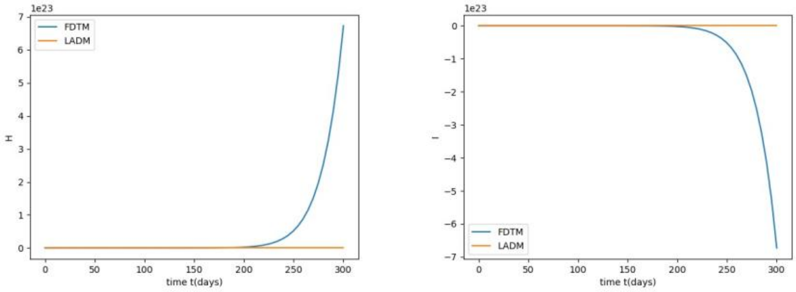
7 CONCLUSION

In this work, a fractional order corona virus model under Caputo sense has been investigated and solved by using fractional differential transform method.



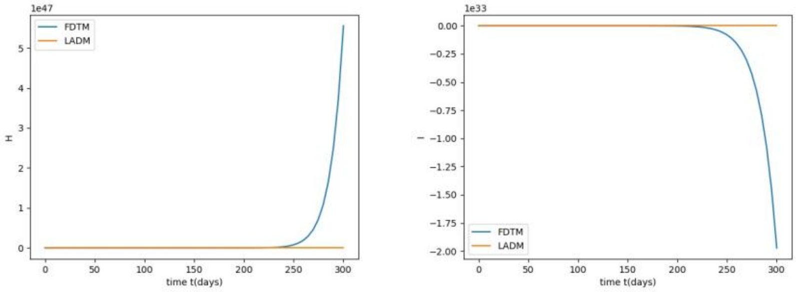
(a) Plots of healthy individual (b) Plots of infected individuals

Figure 2: Plots of healthy individual and infected individuals at different fractional orders using LADM.



(a) Plots of healthy individual (b) Plots of infected individuals

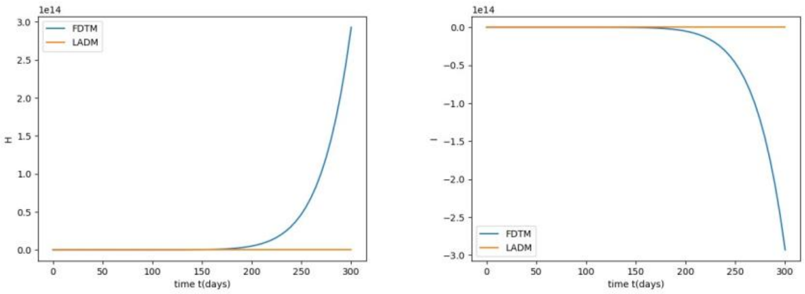
Figure 3: Plots of healthy individual and infected individuals at μ = 0.5



(a) Plots of healthy individual (b) Plots of infected individuals

Figure 4: Plots of healthy individual and infected individuals at μ = 0.3,0.4

The analytical solution have been obtained in terms of converges series with easily computable components in a direct way without using linearisation or perturbation or restrictive assumptions. We obtained the approximate solution of fractional order corona virus model by using FDTM, which are in good



(a) Plots of healthy individual (b) Plots of infected individuals

Figure 5: Plots of healthy individual and infected individuals at μ = 0.7

agreement with those obtained by using the Laplace Adomian decomposition method. As an advantage of FDTM method over the Laplace Adomian decomposition method, in this method we do not need to do the difficult computation for finding the Adomian polynomial. The compression for some different values of q has been obtained and numerical simulation are shown. Further, we analysed the stability of non-linear model by using Lyapunov direct method under Caputo sense. This analysis place an important roll to describe the physical behaviour of the systems when the model is constructed in fractional differential equations.

REFERENCES

1. Musibau A. Omoloye, Sunday O. Adewale, Aliyu M. Umar, Asimiyu O. Oladapo; analysis of Coronavirus disease model by Differential Transformation Method (DTM), (2021).
2. Abdullah, Saeed Ahmad, Saud Owyed, Abdel-Haleem Abdel-Aty, Emad E. Mahmoud, Kamal Shah, Hussam Alrabaiah; Mathematical analysis of COVID-19 via new mathematical model, (2021).
3. V. Padmavathi, A. Prakash, K. Alagesan, N. Magesh; Analysis and numerical simulation of novel coronavirus (COVID-19) model with MittagLeffler Kernel, (2020).
4. Mohammed A. Aba Oud, Aatif Ali, Hussam Alrabaiah, Saif Ullah, Muhammad Altaf Khan, and Saeed Islam; A fractional order mathematical model for COVID-19 dynamics with quarantine, isolation, and environmental viral load, (2021).

5. Nguyen Huy Tuan, Hakimeh Mohammadi, Shahram Rezapour; A mathematical model for COVID-19 transmission by using the Caputo fractional derivative, (2020).
6. J.K. Zhou, Differential Transformation and its Applications for Electrical Circuits, (1986).
7. Aytac Arikoglu, Ibrahim Ozkol; Solution of fractional differential equations by using differential transform method, (2007).
8. M. Caputo; Linear models of dissipation whose Q is almost frequency independent II, (1967).
9. Kamal Shah, Thabet Abdeljawad, Ibrahim Mahariq, and Fahd Jarad; Qualitative Analysis of a Mathematical Model in the Time of COVID-19, (2020).
10. Shantanu Das, Functional Fractional Calculus, Springer, (2011).
11. I. Podlubny, Fractional Differential equations, Academic Press USA, (1999).
12. S. Liu, W. Jiang, X. Li, X.-F. Zhou; Lyapunov stability analysis of fractional nonlinear systems, Appl. Math. Lett. (2015).