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On the Minimum Modified Sombor Index of the Unicyclic Graphs by the Restrictions on the Girth

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ARTICLE INFO	ABSTRACT
Published Online: 07 February 2025	The Modified Sombor Index of a graph **G** is defined as the reciprocal of the well known Sombor Index. The girth of **G,** by short **g(G),** is the length of the smallest cycle in **G**. A graph with exactly one cycle is a unicyclic graph. If it is further, connected, it is a connected unicyclic graph. In this article, we achieved the minimum and second minimum modified Sombor Index of unicyclic graphs and also found the graphs achieving the minimum and second minimum modified Sombor Index.
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KEYWORDS: Modified Sombor Index, Topological Index, Graph invariant, Unicyclic graph, girth, extremal problem, characterization.

I. INTRODUCTION

By a graph **G,** throughout the paper, we mean only the connected and finite graphs. In this paper, we concentrate only on the connected, finite graphs. A graph **G** is an ordered pair **(V_G,E_G),** and the vertices and edges of a graphs are the members of the sets **V_G** and **E_G** respectively and the set of vertices that are adjacent to a vertex ***u*** in **G** is denoted as ***$N_G(u)$*** and is called the open-neighbourhood of ***u*** in **G**. The closed neighbourhood is $N_G[v] = N_G(v) \cup \{v\}$. A (x, y) – Path $xux_1x_2 \ldots y$ is a sequence of distinct members of the set V_G and the vertices x, y are respectively known as the origin and terminus of the path. The concept of distance between two random vertices $x, y \in V_G$ is defined as the length of the shortest (x, y) – Path in G. If $d_G(x) = 1$, then the vertex x is a pendent vertex and its only adjacent vertex in G, say y is a support vertex in G. The graph denoted by $G - w$ is the graph resulted after removal of a vetex $w \in V_G$ and all its incident edges. A connected graph with exactly one cycle is a unicyclic graph. For more on graphs and related works, the reader is referred to [1,6,7].

The Sombor (SO) Index for a graph G is defined by Gutman [11] as

$$SO(G) = \sum_{uv \in E(G)} \sqrt{d_G(u)^2 + d_G(v)^2}$$

.A lot of work and modifications on the Sombor (SO) Index has been done and received a numerous attention from researchers in the past in the form of surveys [2,11] and research articles [3,4,8,9,12,13,16,17]. For the applications in chemical domain of this index one may refer the papers [5,14].

The modified Sombor Index is defined as

$$SO^m(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_G(u)^2 + d_G(v)^2}}.$$

The Modified Sombor Index is a recently introduced term and some of the works can be found in [10,15,18,19].

The minimum values of modified Sombor index is studied in this article and provided the unicyclic graphs with minimum modified Sombor index. As the minimum modified Sombor index is not characterized or proposed for the class of unicyclic graphs in the literature so far, we characterized the unicyclic graphs with the minimum $SO^m(G)$.

II. RESULTS

In this section, the minimum and second minimum of the modified Sombor index for unicyclic graphs are found. If the order n of G is one. There is no edge in the graph, hence by a graph G. We mean a graph with minimum two vertices throughout this article. Also, by a graph we mean a connected graph. First, let us prove a main theorem on the minimum modified Sombor index of unicyclic graphs with a given girth. The following lemmas are useful in proving our main theorem. In the following lemmas, we assume x to be non-negative.

Lemma 1 The function $f(x) = \frac{x}{\sqrt{(x+2)^2+1}} - \frac{x-1}{\sqrt{(x+1)^2+1}}$ is decreasing for any real values $x \geq 0$.

Proof Consider the function $f(x) = g(x) - h(x)$ where $g(x) = \frac{x}{\sqrt{(x+2)^2+1}}$ and $h(x) = \frac{x-1}{\sqrt{(x+1)^2+1}}$. Then, $g'(x) = \frac{2x+5}{(x^2+4x+5)^{\frac{3}{2}}}$ and $h'(x) = \frac{2x+3}{(x^2+2x+3)^{\frac{3}{2}}}$.

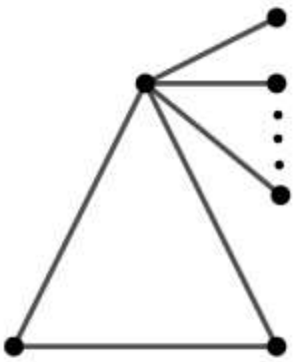
Now consider,

$$f'(x) = \frac{2x+5}{(x^2+4x+5)^{\frac{3}{2}}} - \frac{2x+3}{(x^2+2x+3)^{\frac{3}{2}}}$$

$$< \frac{2x+5}{(x^2+4x+5)^{\frac{3}{2}}} - \frac{2x+3}{(x^2+2x+3)^{\frac{3}{2}}}$$

$$< 0.$$

The following theorem provides a characterization of all unicyclic graphs with minimum modified Sombor index of graphs. The notation $\mathcal{G}$ denotes the unicyclic graphs on n vertices with girth k. The structure of a random graph from the collection $\mathcal{G}$ is shown in figure:



$C_{k,n-k}$

Theorem 1 If G is unicyclic graph with n vertices and of girth g, then $SO^m(G) \geq \frac{(g-2)}{\sqrt{8}} + \frac{2}{\sqrt{(n-g+2)^2+4}} + \frac{(n-g)}{\sqrt{(n-g+2)^2+1}}$ with equality iff $G \in \mathcal{G}$.

Proof Let G and G' represent the graphs of order n and $(n-1)$ respectively. Consider,

$SO^m(G) - SO^m(G)'$

$$= \frac{(g-2)}{\sqrt{8}} + \frac{2}{\sqrt{(n-g+2)^2+4}} + \frac{(n-g)}{\sqrt{(n-g+2)^2+1}} - \left(\frac{(g-2)}{\sqrt{8}} + \frac{2}{\sqrt{(n-g+1)^2+4}} + \frac{(n-g-1)}{\sqrt{(n-g+1)^2+1}}\right)$$

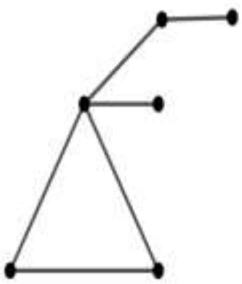
$$= \frac{2}{\sqrt{(n-g+2)^2+4}} + \frac{(n-g)}{\sqrt{(n-g+2)^2+1}} - \left(\frac{2}{\sqrt{(n-g+1)^2+4}} + \frac{(n-g-1)}{\sqrt{(n-g+1)^2+1}}\right)$$

$$= 2\left(\frac{1}{\sqrt{(n-g+2)^2+4}} - \frac{1}{\sqrt{(n-g+1)^2+4}}\right) + \left(\frac{(n-g)}{\sqrt{(n-g+2)^2+1}} - \frac{(n-g-1)}{\sqrt{(n-g+1)^2+1}}\right)$$

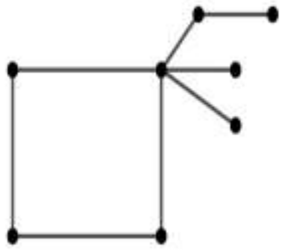
$$< \frac{(n-g)}{\sqrt{(n-g+2)^2+1}} - \frac{(n-g-1)}{\sqrt{(n-g+1)^2+1}}$$

$< 0.$ (by lemma 1.

Examples of unicyclic graphs on different number of vertices and girth with minimum modified sombor index is shown in the following figure:



$C^{*}_{3,2}$



$C^{*}_{4,3}$

Let G be a unicyclic graph such that a vertex v in the cycle with $d_G(v) \geq 3$ with a pendant vertex w adjacent to v, and u be the vertex in the cycle of G such that $d_G(u) = 2$. Let H be the graph constructed from G using the following operations:

- **Operation - $\mathcal{H}_1$:** delete the vertex u and its incident edges.

- **Operation - $\mathcal{H}_2$:** join the vertices u' and u'' by an edge.
- **Operation - $\mathcal{H}_3$:** attach u to the vertex v.

The transformation of the unicyclic graph $G = C^*_{n,k}$ into the unicyclic graph H is shown in the following figure:

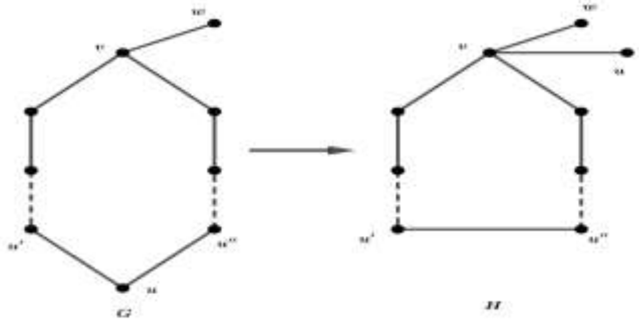


Fig. 1 Transformation of G into H.

Lemma 2 Let G be a unicyclic graph such that a vertex v in the cycle with $d_G(v) \geq 3$ with a pendant vertex w and u be the vertex in the cycle of G such that $d_G(u) = 2$. Let H be the graph constructed from G using the above operations. Then, $SO^m(H) < SO^m(G)$.

Proof Let u, v be the vertices in the cycle of G such that $d_G(u) = 2$ and $d_G(v) \geq 3$. Let w be a pendant vertex adjacent to v in G. Let uu' and uu'' be the edges removed from G to form H. Let the edge $u'u''wu$ be the newly added edge in H. Now,

$$SO^m(H) = SO^m(G) - \frac{1}{\sqrt{d_G(u)^2 + d_G(u')^2}} - \frac{1}{\sqrt{d_G(u)^2 + d_G(u'')^2}} + \frac{1}{\sqrt{d_H(u')^2 + d_H(u'')^2}} + \frac{1}{\sqrt{d_H(u)^2 + d_H(v)^2}}$$

$$= SO^m(G) - \frac{2}{\sqrt{2^2+2^2}} + \frac{1}{\sqrt{d_H(u')^2+d_H(u'')^2}} + \frac{1}{\sqrt{d_H(u)^2+d_H(v)^2}}$$

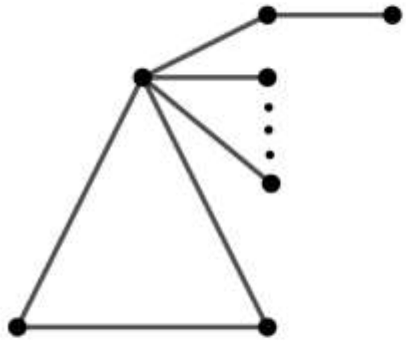
$$= SO^m(G) - \frac{2}{\sqrt{8}} + \frac{1}{\sqrt{8}} + \frac{1}{\sqrt{d_H(u)^2+d_H(v)^2}}$$

$$= SO^m(G) - \frac{1}{\sqrt{8}} + \frac{1}{\sqrt{1+d_H(v)^2}}$$

$$< SO^m(G) - \frac{1}{\sqrt{8}} + \frac{1}{\sqrt{16}}$$

$$< SO^m(G).$$

Now, lwt us prove a theorem on the second minimum modified Sombar index of a given unicyclic graph on n vertices and girth g. Consider $\mathcal{G}^*$ as the collection of unicyclic graphs on n vertices and girth g with the restriction that there are $n-k-1$ pendant vertices adjacent to a vertex in the cycle and one vertex is adjacent to one of these pendant vertices. The structure of the graphs belongs to the collection $\mathcal{G}$ is shown in the following figure:



$C^*_{k,n-k}$

Theorem 2 If G is a unicyclic graph with n vertices and of girth g, then the second minimum is given by,

$$SO^m(G) \geq \frac{(g-2)}{\sqrt{8}} + \frac{2}{\sqrt{(n-g+1)^2+4}} + \frac{(n-g-1)}{\sqrt{(n-g+1)^2+1}} + \frac{1}{\sqrt{5}}$$

With equality iff $G \in \mathcal{G}^*$.

Proof If $G \in \mathcal{G}^*$, then clearly the bound is satisfied. Now assume that G is a graph with the given modified Sombor index. Let G and G' represent the graphs of order n and $(n-1)$ respectively from the collection $\mathcal{G}^*$. Let us prove the result by induction. The result is true for the base case for any $n \geq 4$ and $g = 3$. Assume that the result is true for graphs with less than or equal to $n-1$ vertices. Now consider,

$$SO^m(G) - SO^m(G')$$

$$= \frac{(g-2)}{\sqrt{8}} + \frac{2}{\sqrt{(n-g+2)^2+4}} + \frac{(n-g)}{\sqrt{(n-g+2)^2+1}} - \left(\frac{(g-2)}{\sqrt{8}} + \frac{2}{\sqrt{(n-g+1)^2+4}} + \frac{(n-g-1)}{\sqrt{(n-g+1)^2+1}}\right)$$

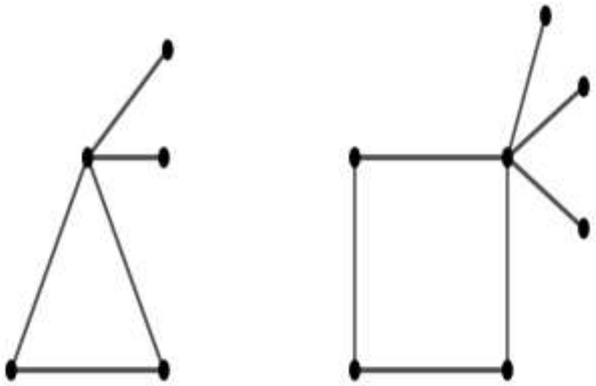
$$= \frac{2}{\sqrt{(n-g+2)^2+4}} + \frac{(n-g)}{\sqrt{(n-g+2)^2+1}} - \left(\frac{2}{\sqrt{(n-g+1)^2+4}} + \frac{(n-g-1)}{\sqrt{(n-g+1)^2+1}}\right)$$

$$= 2\left(\frac{1}{\sqrt{(n-g+2)^2+4}} - \frac{1}{\sqrt{(n-g+1)^2+4}}\right) + \left(\frac{(n-g)}{\sqrt{(n-g+2)^2+1}} - \frac{(n-g-1)}{\sqrt{(n-g+1)^2+1}}\right)$$

$$< \frac{(n-g)}{\sqrt{(n-g+2)^2+1}} - \frac{(n-g-1)}{\sqrt{(n-g+1)^2+1}}$$

$$< 0. \qquad \text{(by lemma 1.}$$

Examples of unicyclic graphs on different number of vertices and girth with second minimum modified sombor index is shown in the following figure:



$C_{3,2}$ $C_{4,3}$

Conclusion

In this article, we found the minimum and second minimum modified sombor index of unicyclic graphs. Also, characterized the graphs attaining the minimum and second minimum modified sombor index and a list of unicyclic graphs with second minimum modified sombor index is provided.

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