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A Numerical Method for Inverting Bordered k-Tridiagonal Matrices

B. TALIBI[1], A.AIAT HADJ [2], D.SARSRI [1]

[1]Department of Industrial Engineering and Logistics, National School of Applied Sciences of TANGIER; Abdelmalek Essaâdi University, Morocco.

[2] Regional Center of the Trades of Education and Training (CRMEF)-Tangier, Avenue My Abdelaziz, Souani, BP: 3117, Tangier. Morocco.

1 Abstract

We develop a new algorithm to compute the inverse of an interesting class of bordered k-tridiagonal matrices. This algorithm relieson a novel strategy of partitioning and decomposition. We also give examples illustrating the efficiency of the proposed method.
Keywords : Bordered k-Tridiagonal Matrices, Inverse, Decomposition.

2 Introduction

Various areas of numerical analysis or numerically-based computation generate large nonsingular sparse matrices. Such matrix have special structure, so the inverse is harder to calculate. One of the most well-known types of sparse matrices is banded matrices, such as tridiagonal matrices [1], pentadiagonal matrices [2], and heptadiagonal matrices [3]. Matrices of this form are commonly associated with numerical simulations, discretizations of partial differential equations and various other computational tasks.

For many problems in Mathematics and Applied Science, one need to solve linear systems where whose coefficient matrices are bordered $k - tridiagonal$ matrices. These are a subclass of the class of banded matrices, where we have more border elements that increase the multiplicity and make the direct calculation of their inverses even more complex.

Thus, the purpose of this paper is to treat this computing problem by designing efficient algorithms for the inverse of bordered $k - tridiagonal$ matrix. These methods exploit the special structural features of such matrices to make the computation simpler, cheaper, and numerically stable.

In addition, we present comprehensive illustrations to demonstrate the practical and efficient applicability of the proposed methods over a range of fields in science and engineering.

To deal with the bordered matrix of the following type $B_n^{(k)}$ with:

$$B_n^{(k)} = \begin{pmatrix} d_1 & 0 & \cdots & 0 & a_1 & 0 & \cdots & & 0 & p_1 \\ 0 & d_2 & 0 & \cdots & 0 & a_2 & \ddots & & \vdots & p_2 \\ \vdots & \ddots & \ddots & \ddots & \cdots & \ddots & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & \ddots & \cdots & \ddots & & \ddots & 0 & \vdots \\ 0 & \cdots & \ddots & \ddots & \ddots & \cdots & & \ddots & a_{n-k-1} & p_{n-k-1} \\ b_1 & \ddots & \cdots & \ddots & \ddots & \ddots & & \ddots & \ddots & a_{n-k} \\ 0 & b_2 & \ddots & \cdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \cdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & b_{n-k-1} & \ddots & \cdots & & \ddots & d_{n-1} & 0 \\ q_1 & q_2 & \cdots & q_{n-k-1} & b_{n-k} & 0 & \cdots & \cdots & 0 & d_n \end{pmatrix} \tag{1}$$

3 Inverse of bordered $k-$tridiagonal matrices

Herein, we introduce a new symbolic algorithm making use of the Sherman-Morrison-Woodbury formula to calculate the inverse of a bordered $k - tridiagonal$ matrix, (1), in a symbolic way.

Then the matrix $B_n^{(k)}$ can be expressed as:

$$B_n^{(k)} = T_n^{(k)} + VW^T \tag{2}$$

Where:

$$T_n^{(k)} = \begin{pmatrix} d_1 & 0 & \cdots & 0 & a_1 & 0 & \cdots & & 0 & 0 \\ 0 & d_2 & 0 & \cdots & 0 & a_2 & \ddots & & \vdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \cdots & \ddots & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & \ddots & \cdots & \ddots & & \ddots & 0 & \vdots \\ 0 & \cdots & \ddots & \ddots & \ddots & \cdots & & \ddots & a_{n-k-1} & 0 \\ b_1 & \ddots & \cdots & \ddots & \ddots & \ddots & & \ddots & \ddots & a_{n-k} \\ 0 & b_2 & \ddots & \cdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \cdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & b_{n-k-1} & \ddots & \cdots & & \ddots & d_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & b_{n-k} & 0 & \cdots & \cdots & 0 & d_n \end{pmatrix} \tag{3}$$

And

$$V = \begin{pmatrix} p_1 & 0 \\ \vdots & \vdots \\ p_{n-k-1} & \vdots \\ 0 & \vdots \\ \vdots & 0 \\ 0 & 1 \end{pmatrix} \tag{4}$$

Also

$$W = \begin{pmatrix} 0 & q_1 \\ \vdots & \vdots \\ \vdots & q_{n-k-1} \\ \vdots & 0 \\ 0 & \vdots \\ 1 & 0 \end{pmatrix} \tag{5}$$

Applying the Sherman-Morrison-Woodbury formula to (2), we obtain:

$$(B_n^{(k)})^{-1} = (T_n^{(k)})^{-1} - (T_n^{(k)})^{-1} V W^T (T_n^{(k)})^{-1} (1 + V W^T (T_n^{(k)})^{-1})^{-1} \tag{6}$$

And

$$det(B_n^{(k)}) = det(T_n^{(k)}) \cdot det(I + V^T (T_n^{(k)})^{-1} V) \tag{7}$$

As we see the matrix $T_n^{(k)}$ is a banded matrix, assume that the matrix $T_n^{(k)}$ is invertible and let us denote by D_j the $j-$th column vector of the inverse matrix $(T_n^{(k)})^{-1}$. Then we obtain:

$$D_{n-j-p} = \frac{1}{a_{n-j-p,n-j}} (E_{n-j} - \sum_{i=n-j-p+1}^{n} a_{i,n-j} D_i) \tag{8}$$

For $j = 0, ..., n-p-1$, where $E_j = [(\delta_{i,j})_{1 \le i \le n}]^T \in \mathbb{C}^n$ is the vector of order j of the canonical basis of $\mathbb{C}^n$.

For the $n \times n$ banded matrix $T_n^{(k)}$, we associate the sequence numbers $(A_{i,k})_{1 \le i \le n+p, 1 \le k \le p}$ defined by the following relations for $k = 1, ..., p$:

$$A_{i,k} = \delta_{i,k} \quad for\ i = 1, ..., p \tag{9}$$

$$-a_{i,p+i} A_{p+i,k} = \sum_{s=1}^{p+i-1} a_{i,s} A_{s,k} \quad for\ i = 1, ..., n-p \tag{10}$$

$$_{p+i,k}= \sum_{s=1}^{n} a_{i,s} A_{s,k} \quad for\ i = n-p+1, ..., n \tag{11}$$

We can write the relation :

$$T_n^k A_k = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ A_{n+1,k} \\ A_{n+p,k} \end{pmatrix} \tag{12}$$

With: $A_k = [A_{1,k}, ..., A_{n,k}]$.
We shall denote Q the $p \times p$ matrix $(A_{n+i,j})_{1 \leq i,j \leq p}$
And $X_j = [x_{1,j}, ..., x_{p,j}]^T$ the $j-$th column vector of the matrix Q^{-1}.

Theorem 1 *The $j-th$ vector column of the inverse matrix $T_n^{(k)}$, $1 \leq j \leq p$, is given by*

$$D_{n-p+j} = \sum_{k=1}^{p} x_{k,j} A_k$$

Proof We get from the relation Equation (9):

$$T_n \cdot \sum_{k=1}^{p} x_{k,j} A_k = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \sum_{k=1}^{p} x_{k,j} A_{n+1,k} \\ \vdots \\ \sum_{k=1}^{p} x_{k,j} A_{n+p,k} \end{pmatrix}$$

The result follows from the fact that:

$$\begin{pmatrix} \sum_{k=1}^{p} x_{k,j} A_{n+1,k} \\ \vdots \\ \sum_{k=1}^{p} x_{k,j} A_{n+p,k} \end{pmatrix} = Q \begin{pmatrix} x_{1,j} \\ \vdots \\ x_{p,j} \end{pmatrix} = [(\delta_{i,j})_{1 \leq i \leq p}]^T$$

Then the proof completed.

Algorithm: New efficient computational algorithm for computing the Inverse of the bordered $k-$tridiagonal matrices INPUT:
The dimension n, .

OUTPUT:

The partition of the matrix B.

Step 1:

$$T_n^{(k)} = \begin{pmatrix} d_1 & 0 & \cdots & 0 & a_1 & 0 & \cdots & & 0 & 0 \\ 0 & d_2 & 0 & \cdots & 0 & a_2 & \ddots & & \vdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \cdots & \ddots & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & \ddots & \cdots & \ddots & & \ddots & 0 & \vdots \\ 0 & \cdots & \ddots & \ddots & \ddots & \cdots & & \ddots & a_{n-k-1} & 0 \\ b_1 & \ddots & \cdots & \ddots & \ddots & \ddots & & \ddots & \ddots & a_{n-k} \\ 0 & b_2 & \ddots & \cdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \cdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & b_{n-k-1} & \ddots & \cdots & & \ddots & d_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & b_{n-k} & 0 & \cdots & \cdots & 0 & d_n \end{pmatrix} \tag{13}$$

And

$$V = \begin{pmatrix} p_1 & 0 \\ \vdots & \vdots \\ p_{n-k-1} & \vdots \\ 0 & \vdots \\ \vdots & 0 \\ 0 & 1 \end{pmatrix} \tag{14}$$

Also

$$W = \begin{pmatrix} 0 & q_1 \\ \vdots & \vdots \\ \vdots & q_{n-k-1} \\ \vdots & 0 \\ 0 & \vdots \\ 1 & 0 \end{pmatrix} \tag{15}$$

Step 2: Inverse of $T_n^{(k)}$

$$D_{n-p+j} = \sum_{k=1}^{p} x_{k,j} A_k$$

$$D_{n-j-p} = \frac{1}{a_{n-j-p,n-j}} (E_{n-j} - \sum_{i=n-j-p+1}^{n} a_{i,n-j} D_i) \tag{16}$$

Step 3:

$$(B_n^{(k)})^{-1} = (T_n^{(k)})^{-1} - (T_n^{(k)})^{-1}V(I + W^T(T_n^{(k)})^{-1}V)^{-1}W^T(T_n^{(k)})^{-1} \quad (17)$$

4 The inverse of bodered $1-$tridiagonal matrices

In this section we consider a particular case of bodered $k-$tridiagonal is bodered $1-$tridiagonal or bodered tridiagonal matrices, then the matrix T_n have the form:

$$T_n = \begin{pmatrix} d_1 & a_1 & 0 & \cdots & 0 \\ b_1 & d_2 & a_2 & 0 & \vdots \\ 0 & b_2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ & & & & a_{n-1} \\ 0 & \cdots & 0 & b_{n-1} & d_n \end{pmatrix} \quad (18)$$

We assume that T_n is non-singular and from the relation $T_n^{-1}T_n$ we get the column vector for the inverse T_n^{-1} as:

$$Dn - 1 = \frac{1}{a_{n-1}}(E_n - d_n D_n)$$

$$Dj - 1 = \frac{1}{a_{j-1}}(E_n - d_j D_j - b_{j+j}D_{j+1}) \quad j = n-1, ..., 2$$

Consider the sequence of numbers $(A_i)_{0\le i\le n}$ and $(B_i)_{0\le i\le n}$ characterized by a term recurrence relation:

$$A_0 = 1$$

$$d_1 A_0 + a_1 A_1 = 0$$

And

$$b_{i+1}A_{i-1} + d_{i+1}A_i + c_{i+1}A_{i+1} = 0 \quad for \;\; 1 \le i \le n-1,$$

Theorem 2 *Suppose that $A_n \neq 0$, then T_n is invetible and the column D_n will be*

$$D_n = [\frac{-A_0}{A_n}, ..., \frac{-A_0}{A_n}]$$

Algorithm: New efficient computational algorithm for computing the Inverse of the bordered $k-$tridiagonal matrices INPUT:
The dimension n, .

OUTPUT:
The partition of the matrix B.
Step 1:

$$T_n^{(k)} = \begin{pmatrix} d_1 & 0 & \cdots & 0 & a_1 & 0 & \cdots & & 0 & 0 \\ 0 & d_2 & 0 & \cdots & 0 & a_2 & \ddots & & \vdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \cdots & \ddots & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & \ddots & \cdots & \ddots & & \ddots & 0 & \vdots \\ 0 & \cdots & \ddots & \ddots & \ddots & \cdots & & \ddots & a_{n-k-1} & 0 \\ b_1 & \ddots & \cdots & \ddots & \ddots & \ddots & & \ddots & \ddots & a_{n-k} \\ 0 & b_2 & \ddots & \cdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \cdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & b_{n-k-1} & \ddots & \cdots & & \ddots & d_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & b_{n-k} & 0 & \cdots & \cdots & 0 & d_n \end{pmatrix} \tag{19}$$

And

$$V = \begin{pmatrix} p_1 & 0 \\ \vdots & \vdots \\ p_{n-k-1} & \vdots \\ 0 & \vdots \\ \vdots & 0 \\ 0 & 1 \end{pmatrix} \tag{20}$$

Also

$$W = \begin{pmatrix} 0 & q_1 \\ \vdots & \vdots \\ \vdots & q_{n-k-1} \\ \vdots & 0 \\ 0 & \vdots \\ 1 & 0 \end{pmatrix} \tag{21}$$

Step 2: Inverse of T_n

$$Dn-1=\frac{1}{a_{n-1}}(E_n-d_nD_n)$$

$$Dj-1=\frac{1}{a_{j-1}}(E_n-d_jD_j-b_{j+j}D_{j+1})\quad j=n-1,...,2$$

$$D_{n-j-p}=\frac{1}{a_{n-j-p,n-j}}(E_{n-j}-\sum_{i=n-j-p+1}^{n}a_{i,n-j}D_i) \tag{22}$$

$$D_n=[\frac{-A_0}{A_n},...,\frac{-A_0}{A_n}]$$

Step 3:

$$(B_n^{(k)})^{-1}=(T_n)^{-1}-(T_n)^{-1}V(I+W^T(T_n)^{-1}V)^{-1}W^T(T_n)^{-1} \tag{23}$$

5 Exemple:

We presenta specific numerical example in this section to verify the performance of our algorithm. We use the MATLAB R2024b software to implement the algorithm and evaluate its performance. As a concrete application example of our method, we consider a 7-by-7 tridiagonal matrix. Covering a range of problems over multiple domains, tridiagonal matrices play a crucial role in scientific and other engineering applications, being used as test cases for finite difference based methods that tackle differential equations.

With:

$$\mathbf{B}=\begin{pmatrix}-2&3&0&0&5\\1&-2&3&0&5\\0&1&-2&3&5\\0&0&1&-2&3\\2&2&2&1&-2\end{pmatrix} \tag{24}$$

Then the matrices T , V and W will be:

$$\mathbf{T}=\begin{pmatrix}-2&3&0&0&0\\1&-2&3&0&0\\0&1&-2&3&0\\0&0&1&-2&3\\0&0&0&1&-2\end{pmatrix} \tag{25}$$

$$\mathbf{V}=\begin{pmatrix}5&0\\5&0\\5&0\\0&0\\0&1\end{pmatrix} \tag{26}$$

$$\mathbf{W} = \begin{pmatrix} 0 & 2 \\ 0 & 2 \\ 0 & 2 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{27}$$

And the columns inverse of the matrix T are:

$$\mathbf{T_1^{-1}} = \begin{pmatrix} -1.1000 \\ -0.4000 \\ 0.1000 \\ 0.2000 \\ 0.1000 \end{pmatrix}, \quad \mathbf{T_2^{-1}} = \begin{pmatrix} -1.2000 \\ -0.8000 \\ 0.2000 \\ 0.4000 \\ 0.2000 \end{pmatrix}, \quad \mathbf{T_3^{-1}} = \begin{pmatrix} 0.9000 \\ 0.6000 \\ 0.1000 \\ 0.2000 \\ 0.1000 \end{pmatrix} \tag{28}$$

And

$$T_4^{-1} = \begin{pmatrix} 5.4000 \\ 3.6000 \\ 0.6000 \\ -0.8000 \\ -0.4000 \end{pmatrix}, \quad T_5^{-1} = \begin{pmatrix} 8.1000 \\ 5.4000 \\ 0.9000 \\ -1.2000 \\ -1.1000 \end{pmatrix} \tag{29}$$

Also the matrix B^{-1} take the form:

$$B^{-1} = \begin{pmatrix} -0.3426 & -0.1476 & 0.2702 & 0.5209 & 0.2312 \\ 0.1072 & -0.1448 & 0.0292 & 0.1337 & 0.1797 \\ 0.1880 & 0.2396 & -0.2214 & -0.2981 & 0.0682 \\ 0.0919 & 0.1616 & 0.0251 & -0.4568 & 0.0111 \\ -0.0014 & 0.0279 & 0.0905 & 0.1281 & -0.0153 \end{pmatrix} \tag{30}$$

Table 1: The running time

Size of the matrix (n)	The fast algorithm	LU
100	0.082813	6.203181
200	0.394041	7.255524
300	0.531047	8.049020
500	0.964163	14.346657
1000	3.866233	108.681497

6 Conclusions

Here we demonstrate the efficiency and fast computation of our algorithm using the Sherman-Morrison-Woodbury (SMW) formula. We will specifically use our algorithm to compute the inverse of matrix T whichis required to compute the inverse of bordered matrix. Our method based on the SMW framework greatly mitigates the required computational complexity and alleviates the inversion process speed. Such an approach becomes especially useful when the scale of the matrices is particularly large as it minimizes direct inversion of the whole matrix and builds efficient mutations to the current system. We effectively illustrate theusefulness and benefits of our algorithm concerning bordered inverse computations via this method.

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