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Common Fixed Point Theorems in Compact Fuzzy 3-Metric Space Using Implicit Relation

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Abstract:
An extensive study in fuzzy metric space, fuzzy 2-metric space,compact fuzzy metric space and compact fuzzy 2-metric space had been done by Sharma [3]and shrivastava[4], using this concept we have generalizing the result of Aliouche [1] in the context of compact fuzzy3-metric space.

Keywords: lass of Implicit Relation, Compact Fuzzy 3- Metric Space. Weak-compatible

Introduction:
Gahlar [2] who furnished the concept of 2-metric space, and this concept has prospered very fast in various directions. It is to be remarked that Sharma, Sharma and Iseki [3] studied for first time contraction type mapping in 2-metric space. Wenzhi [5] and many others preceded the study of probabilisstic 2-metric space. We know that a 2-metric is a real valued function with domain $X\times X\times X$, whose abstract properties were suggested by the area function in Euclidean space. Now it is natural to expect 3-metric space which is suggested by the volume function. Now we generalized the work of shrivastava[4] in the context of fuzzy 3-metric space.

COMMON FIXED POINT THEOREM IN COMPACT FUZZY 3-METRIC SPACE SATISFYING AN IMPLICIT RELATION

Definition 1. The mapping $* : [0, 1] \to [0, 1]$ is called **t-norm** if

(1) $*(a, 1, 1, 1) = a$ for all $a \in [0,1]$

(2) $*(a, b, c, d) = *(a, c, d, b) = *(c, d, a, b)$

(3) $*(a, b, c, d) \leq *(d, c, f, g)$; whenever $a \leq d, b \leq c, c \leq f$ and $d \leq g$

(4) $(*(a, b, c), d, c, f) = *(a, *(b, c, d), e, f)$

$= *(a, b, *(c, d, e), f) = *(a, b, c, *(d, e, f))$

Definition 2. The 3- tuple $(X, M, *)$ is called a **fuzzy 3-metric space** if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set in X^4 x $[0, \infty)$ satisfying the following conditions, for all x, y, z, w, u $\in$ X and $t_1, t_2, t_3, t_4 > 0$

(1) $M(x, y, z, w, 0) = 0$,

(2) $M(x, y, z, w, t) = 1$; for all $t > 0$, [only when at least two of the four points are equal]

(3) $M(x, y, z, w, t) = M(x, w, z, y) = M(y, z, w, x, t) = M(z, w, x, y, t) = \ldots\ldots\ldots,$

(4) $M(x, y, z, w, t_1 + t_2 + t_3 + t_4) \geq M(x, y, z, u, t) * M(x, y, u, w_1t_2) *$

$M(x, u, z, w, t_3) * M(u, y, z, w, t_4)$,

(3) $M(x, y, z, w, .) : [0,\infty) \to [0, 1]$ is left continuous,

(4) $\lim_{t \to \infty} M(x, y, z, w, t) = 1.$

The function value M(x, y, z, w, t) may be interpreted as the probability that the volume of tertrahedron is less then t.

Definition 3. Let (X, M, *) be a fuzzy 3-metric space. Then we define an **open ball** with centre $x_0 \in X$ and radius r, $0 < r < 1$, $t > 0$ as

$$B(x_0, r, t) = \{y, \in X : S(x_0, y, z, w, t) > 1 - r$$

Definition 4.. Let (X, M, *) be a fuzzy 3-metric space. Define

$\tau = \{ A \subset X : x \in A$ if and only if there exist $t > 0$ and $0 < r < 1$

such that $B(x_0, r, t) \subset A\}$.

Then τ is a topology on X.

Definition 5. Let (X, M, *) be fuzzy 3-metric space. Then a collection $C = \{G_\alpha : \alpha \in \Lambda$; where G_α is open sets of X} is said to be a **open cover** of X if $\bigcup_{\alpha \in \Lambda} G_\alpha = X$.

Definition 6. A fuzzy 3-metric space (X, M, *) is said to be **compact** if every open covering of X has a finite subcovering.

Definition 7. Let (X, M, *) is fuzzy 3-metric space

(1) A sequence $\{x_n\}$ in X is said to be **convergent** to a point $x \in X$, if

$$\lim_{n \to \infty} M(x_n\ x, a, b, t) = 1, \text{ for all } a, b \in X \text{ and } t > 0.$$

(2) A sequence $\{x_n\}$ in X is called a **cauchy sequence**, if

$$\lim_{n \to \infty} M(x_{n+p}\, x_{n,}\, a, b, t) = 1, \text{ for all } a, b \in X \text{ and } t > 0, p > 0.$$

(3) A fuzzy 3-metric space in which every cauchy sequence is convergent is said to be **complete**.

Definition 8. A function M is **continuous** in fuzzy 3-metric space iff whenever $x_n \to x$, $y_n \to y$, then

$$\lim_{n \to \infty} M(x_{n\,,}\, y_n, a, b, t) = M(x, y, a, b, t) ;$$

for all a, b, $\in X$ and $t > 0$.

Definition 9. Two mappings A and S on fuzzy 3-metric space (X, M, *) are **compatible** if

$$\lim_{n \to \infty} M(ASx_n, SAx_n, a, b, t) = 1 \ ; \text{ for all } a, b \in X, \ t > 0.$$

Whenever, $\{x_n\}$ is a sequence such that

$$\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = x \in X.$$

Definition 10. Self mapping A and S of a fuzzy 3-metric space (X, M, *) are said to be **weak-compatible** if they commute of their coincidence point i.e. if $Ax = Sx$ for some $x \in X$ then $SAx = ASx$.

Theorem 1. Let A, B, S and T be self mappings of a compact fuzzy 3-metric space (X, M, *) satisfying

(a) $S(X) \subset B(X)$ and $T(X) \subset A(X)$,

(b) the pair (A, S) and (T, B) are weak-compatible,

(c) S and A are continuous,

(d) inequality

$$F[\,M(Sx, Ty, a, b, t), M(Ax, By, a, b, t), M(Ax, Sx, a, b, t),$$

$$M(By, Ty, a, b, t), M(Ax, Ty, a, b, t), M(Sx, By, a, b, t)\,] > 0$$

$\forall\, x, y, a, b \in X$ and $F \in F_6$ satisfying (C_1), (C_2) and (C_3) for which one of $M(Ax, By, a, b, t)$, $M(Ax, Sx, a, b, t)$ and $M(By, Ty, a, b, t)$ is positive.

Then A, B, S and T have a unique common fixed point in X.

Proof. Let $K = \operatorname{Sup}\{\, M(Ax, Sx, a, b, t) ; b, x, a \in X, t > 0 \,\}$.

Since X is compact, then there is a convergent sequence $\{x_n\}$ with limit $x_0 \in X$ such that

$$\lim_{n\to\infty} M(Ax_n, Sx_n, a, b, t) = K, \quad \forall\, t > 0.$$

Since
$$M(Ax_0, Sx_0, a, b, t) \geq M(Ax_0, Sx_0, a, Sx_n, 1/16) * M(Ax_0, Sx_0, Sx_n, Ax_n, 1/16) *$$

$$M(Ax_0, Sx_n, a, Ax_n, 1/16) * M(Sx_n, Sx_0, a, Ax_n, 1/16) *$$

$$M(Ax_0, Sx_0, Sx_n, b, 1/16) * M(Ax_0, Sx_n, Ax_n, b, 1/16) *$$

$$M(Sx_n, Sx_0, Ax_n, b, 1/16) * M(Ax_0, Sx_n, a, b, 1/16) *$$

$$M(Sx_n, Ax_n, a, b, 1/16) * M(Sx_n, Sx_0, a, b, 1/16).$$

By the continuity of A and S and $\lim_{n\to\infty} x_n = x_0$, we get

$$M(Ax_0, Sx_0, a, b, t) \geq K, \quad \forall\, t > 0$$

Hence $M(Ax_0, Sx_0, a, b, t) = K$. [by the definition of K]

Since $S(x) \subset B(x)$, then there exists $v \in X$ such that $Sx_0 = Bv$ and

$$M(Ax_0, Bv, a, b, t) = K$$

We have to prove that $K = 1$, suppose on the contrary that $K \neq 1$ then $K < 1$. Putting $x = x_0$, $y = v$ in (d), we get,

$$F[\,M(Sx_0, Tv, a, b, t), M(Ax_0, Bv, a, b, t), M(Ax_0, Sx_0, a, b, t),$$

$$M(Bv, Tv, a, b, t), M(Ax_0, Tv, a, b, t), M(Sx_0, Bv, a, b, t)] > 0$$

$$\Rightarrow \quad F[\,M(Bv, Tv, a, b, t), K, K, M(Bv, Tv, a, b, t), M(Ax_0, Tv, a, b, t), 1] > 0.$$

By (C_a), we get

$$M(Bv, Tv, a, b, t) > K.$$

Since $T(X) \subset A(X)$ then there exists $u \in X$ such that $Tv = Au$ and

$M(Au, Bv, a, b, t) > K, \forall t > 0.$

Since by the definition of K,

$M(Au, Su, a, b, t) \leq K > 0.$

Putting x = u, y = v in (d), we obtain

$$F[M(Su, Tv, a, b, t), M(Au, Bv, a, b, t), M(Au, Su, a, b, t), M(Bv, Tv, a, b, t), M(Au, Tv, a, b, t), M(Su, Bv, a, b, t)] > 0$$

$$\Rightarrow F[M(Au, Su, a, b, t), M(Bv, Tv, a, b, t), M(Au, Su, a, b, t), M(Bv, Tv, a, b, t), 1, M(Su, Bv, a, b, t)] > 0, \forall t > 0.$$

By (C_b), we get,

$$K \geq M(Au, Su, a, b, t) > M(Bv, Tv, a, b, t) > K$$

which is a contradiction.

Then K = 1 which implies that $Ax_0 = Sx_0 = Bv$.

If $M(Bv, Tv, a, b, t) < 1$.

Putting $x = x_0$, $y = v$ in (d), we have

$$F[M(Sx_0, Tv, a, b, t), M(Ax_0, Bv, a, b, t), M(Ax_0, Sx_0, a, b, t), M(Bv, Tv, a, b, t), M(Ax_0, Tv, a, b, t), M(Sx_0, Bv, a, b, t)] > 0$$

$$\Rightarrow F[M(Bv, Tv, a, b, t), 1, 1, M(Bv,Tv, a, b, t), M(Bv, Tv, a, b, t), 1] > 0$$

which is a contradiction of (C_3).

Therefore $M(Bv, Tv, a, b, t) = 1$.

We get $Bv = Tv = Sx_0 = Ax_0 = z$.

Since the pair (S, A) is weak-compatible, then Az = Sz

Now, we are going to show that z = Sz i.e. $M(z, Sz, a, b, t) = 1$

Suppose on the contrary that $M(z, Sz, a, b, t) \neq 1$, then $M(z, Sz, a, b, t) < 1$.

Putting x = z, y = v in (d), we get

$$F[M(Sz, Tv, a, b, t), M(Az, Bv, a, b, t), M(Az, Sz, a, b, t), M(Bv, Tv, a, b, t), M(Az, Tv, a, b, t), M(Sz, Bv, a, b, t)] > 0$$

$$\Rightarrow F[M(Sz, z, a, b, t), M(Sz, z, a, b, t), 1, 1, M(Sz, z, a, b, t), M(Sz, z, a, b, t)] > 0$$

which is a contradiction of (C_3)

Therefore $z = Sz = Az$. (1)

Again,since the pair (B,T) is weak-compatible, we get Tz = Bz. (2)

For showing z = Tz, suppose on the contrary that $z \neq Tz$,

then $M(z, Tz, a, t) < 1$. Putting x = z, y = z in (d), we have

$$F[\,M(Sz, Tz, a, t), M(Az, Bz, a, t), M(Az, Sz, a, t),$$

$$M(Bz, Tz, a, t), M(Az, Tz, a, t), M(Sz, Bz, a, t)\,] > 0$$

$$\Rightarrow \quad F[\,M(z, Tz, a, t), M(z, Tz, a, t), 1, 1, M(z, Tz, t), M(z, Tz, t)\,] > 0$$

which is a contradiction of (C_3).

Therefore $z = Tz$. (3)

From (1) (2) and (3)

$$z = Bz = Tz = Az = Sz.$$

Hence z is a common fixed point of A, B, S and T.

Uniqueness

Let z' be another common fixed point of A, B, S and T

i.e. $z' = Az' = Bz' = Sz' = Tz'$ and $M(z, z', a, b) < 1$.

By putting $x = z$, $y = z'$ in (d), we have

$$F[\,M(Sz, Tz', a, b, t), M(Az, Bz', a, b, t), M(Az, Bz, a, b, t),$$

$$M(Bz', Tz', a, b, t), M(Az, Tz', a, b, t), M(Sz, Bz', a, b, t)] > 0$$

$$\Rightarrow \quad F[\,M(z, z', a, b, t), M(z, z', a, b, t), 1, 1, M(z, z', a, b, t),$$

$$M(z, z', a, b, t)\,] > 0$$

which is a contradiction. Hence $z' = z$.

Hence z is the unique common fixed point of A, B, S and T.

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