International Journal of Mathematics and Computer Research

ISSN: 2320-7167

Volume 13 Issue 04 April 2025, Page no. – 5033-5043

Index Copernicus ICV: 57.55, Impact Factor: 8.615

DOI: 10.47191/ijmcr/v13i4.02

Stability and Boundedness of Solutions to a Kind of Third Order Neutral Stochastic Differential Equations with Delay

Ademola, A. T.[1], Appah, F. Z.[2], Omoniyi, G. E.[3] Adesina, O. A.[4]

[1,4]Department of Mathematics, Obafemi Awolowo University, Post Code 220005 Ile-Ife, Nigeria

[2,3]Chevron Nigeria Limited Lekki, Lagos, Nigeria

ARTICLE INFO	ABSTRACT
Published Online: 03 April 2025 **Corresponding Author:** Ademola, A. T.	This study investigates the stability and boundedness of solutions to a specific type of third order neutral stochastic differential equations with delay. The analysis focuses on the stability, asymptotic stability and boundedness of solutions of the nonlinear neutral stochastic system. By employing appropriate Lyapunov-Krasovskii functional techniques, the study provides insights into the behavior of the solutions under various conditions. The results contribute to and extend the understanding of the dynamics of third order neutral stochastic differential equations with delay which have existed in the literature and have implications for applications in various fields. Examples are provided to show the effectiveness of the technique used and the reliability of the obtained theoretical results.

KEYWORDS: Third order; Nonlinear neutral stochastic differential equation; stability boundedness of solution

I. INTRODUCTION

In this paper, we investigate the qualitative properties of solutions to a specific type of third order neutral stochastic differential equations with delay. The study focuses on understanding the behavior of solutions to these equations and analyzing criteria for stability, asymptotic stability and boundedness of solution. By examining the qualitative properties of the solutions, we aim to provide insights into the dynamics of the system and its long-term behavior (i.e., as $t \to \infty$). This analysis contributes to the broader understanding of neutral stochastic differential equations with delay and their applications in various fields.

Functional differential equations are generally applicable differential equations that include classical ordinary and partial differential equations. Their applications can be found in technical problems, mechanical systems under the action of dissipative and gyroscopic forces, and hydraulic engineering applications (see [17, 18, 19, 20, 22, 23, 24, 31, 33, 39,43]). Due to the unending applications of differential equations to real life problems, authors have developed standard techniques to study qualitative properties of solutions to various branches of differential equations such as ordinary differential equations ([10, 11, 13, 14, 15]), delay differential equations ([2, 3, 5, 8, 9, 12, 16, 34, 35, 36, 38, 40, 41, 42]), stochastic differential equations with or without delay ([4, 25, 26, 28, 29, 30, 37]), neutral delay differential equations ([1, 6, 7, 8, 21, 32]), nonlinear third order neutral stochastic differential equations with delay (see [27]) and the references cited therein.

You see that recent work on nonlinear neutral stochastic differential equations is scare in the literature, this scarcity in this direction is not unconnected to difficulty in obtaining the standard tool such as the Lyapunov functional, to obtain qualitative properties of solutions. Neutral differential equations with delay have a wide range of applications in various fields. Some major areas where these equations are commonly used include but not limited to: (i) Neutral differential equations with delay are often used in modeling biological systems such as population dynamics, disease spread, and ecological interactions. The delay in these equations can represent the time it takes for a biological process to occur or for a population to respond to changes in its environment; (ii) In engineering, neutral differential equations with delay are used to model systems with time delays, such as control systems, communication networks, and mechanical systems. These equations help engineers analyze the stability and performance of these systems under different conditions; (iii) Economic models often involve time delays, such as the lag between changes in economic policies and their effects on the economy. Neutral differential equations with delay can be used to study the dynamics of economic systems and predict future trends; (iv) In physics, neutral differential equations with delay are used to model systems with memory effects, such as viscoelastic materials or systems with delayed feedback. These equations help physicists understand the behavior of complex systems and predict their future evolution; and (iv) In neuroscience, neutral differential equations with delay are used to model neural networks and brain dynamics. The delay in these equations can represent the time it takes for signals to propagate through the brain or for neurons to respond to stimuli.

Overall, neutral differential equations with delay are a powerful tool for modeling dynamic systems with memory effects and time delays, making them valuable in a wide range of applications across different disciplines. Therefore, research in this direction cannot be over-emphasized. In 2021 the authors [21] employed Lyapounv functional to establish some new sufficient conditions under which all solutions of nonlinear neutral delay differential equations

$$\left[x''(t)+\Omega\big(x''(t-r)\big)\right]'+\Psi\big(x(t)\big)x'' \\ +\Phi\big(x(t)\big)x'(t)+h\big(x(t-\sigma)\big)=e(t)$$

are stable, bounded, and square integrable. Authors in [6] discussed the stability, boundedness and existence of periodic solutions of the nonlinear third order neutral differential equations

$$\left[r(t)\left(x''(t)+q(t)\Phi\big(x''(t-\tau_0)\big)\right)\right]' \\ +\boldsymbol{\varphi}(\boldsymbol{t})\boldsymbol{f}\big(\boldsymbol{x}(\boldsymbol{t})\big)\boldsymbol{x}''(\boldsymbol{t})+\boldsymbol{\psi}(\boldsymbol{t})\boldsymbol{g}\big(\boldsymbol{x}(\boldsymbol{t}-\boldsymbol{\tau_1}),\boldsymbol{x}'(\boldsymbol{t}-\boldsymbol{\tau_1})\big) \\ +\mu(t)h\big(x(t-\tau_1)\big)=p(t).$$

In [7] the asymptotic stability of a neutral differential system with variable delay and nonlinear perturbation is considered, namely,

$$\dot{x}(t)=Ax(t)+Bx\big(t-\tau(t)\big)+C\dot{x}\big(t-\tau(t)\big) \\ +Q\left(x(t),x\big(t-\tau(t)\big),\dot{x}\big(t-\tau(t)\big)\right),$$

where A,B,C are square matrices, $Q\in C(\mathbb{R}^{3n},\mathbb{R}^n)$, and $x\in\mathbb{R}^n$. Recent work on third order nonlinear stochastic differential equations with delay are as follows. Authors in [26] studied asymptotic stability of certain nonlinear differential equation

$$x'''(t)+ax''(t)+\phi\left(x'\big(t-r(t)\big)\right) \\ +\psi\left(x\big(t-r(t)\big)\right)+\sigma x(t-h)\omega'(t)=0.$$

In [29], properties of solutions for non autonomous third order stochastic differential equation with a constant delay is discussed, namely,

$$x'''(t)+a(t)f\big(x(t),x'(t)\big)x''(t)+b(t)\varphi\big(x(t)\big)x'(t) \\ +c(t)\psi\big(x(t-r)\big)+g(t,x)\omega'(t)= \\ p\big(t,x(t),x'(t),x''(t)\big).$$

Recently in [28] the behaviour of solution to a kind of third order stochastic integro-differential equation with time delay is investigated, namely

$$x'''(t)+a(t)f\big(t,x'(t)\big)x''(t)+b(t)g_1\left(x'\big(t-r(t)\big)\right) \\ +c(t)g_2\left(x\big(t-r(t)\big)\right)+\sigma x\big(t-h(t)\big)\omega'(t)= \\ P\big(t,x(t)\big)\int_0^t G\big(s,x'(s)\big)x'(s)ds.$$

Finally, according to our observation from relevant literature the only paper on third order nonlinear neutral stochastic differential equation is discussed in [27], the equation discussed is

$$[x''(t)+\phi x''(t-r(t))]'+ax''+bx'(t-r(t)) \\ +\psi h(x(t-r(t)))+\sigma x(t)\omega'(t)=p(\cdot),$$

where $p(\cdot)=p\left(t,x(t),x\big(t-r(t)\big),x'(t)\right)$, ϕ is a constant satisfying $0\le\phi\le\frac{1}{2}$, the continuous functions $\psi(t),h(x)$ and $p(\cdot)$ depending only on the arguments shown and $h'(x)$ exist and are continuous for all x; the constants σ,a,b and β are positive with $0\le r(t)\le\beta$, which will be determined later, $\omega(t)\in\mathbb{R}$ is the standard Brownian motion.

The main focus of this expository paper is to obtain sufficient conditions for the stability and boundedness of solutions of the following third order nonlinear neutral stochastic differential equation with delay, namely,

$$[x''(t)+\varepsilon x''(t-\tau)]'+f\big(x(t),x'(t)\big)x''+g\big(x'(t-\tau)\big)+ \\ h\big(x(t-\tau)\big)+\sigma x(t)\omega'(t)=p(\cdot) \quad (1.1)$$

where $p(\cdot)=p\left(t,x(t),x\big(t-r(t)\big),x'(t)\right)$, $p:\mathbb{R}^+\times\mathbb{R}^3\to\mathbb{R}$, $f:\mathbb{R}^2\to\mathbb{R}$, $g,h:\mathbb{R}\to\mathbb{R}$, the derivative $g'(x')$ and $h'(x)$ exist and continuous for all x,x', and $\sigma>0$ is a constant. The condition on ϕ in [27], which is equivalent to constant ε in (1.1) is not necessary in this investigation. Clearly, equation (1.1) includes and extends the discussed equation in [27], since equation (1.1) contains more nonlinear functions except that $\tau>0$ is a constant delay term define in $\mathbb{R}$ while $r(t)$, defined in the closed interval $[0,\beta]$, is a variable delay. Suppose that $x'(t)=y(t)$, $x''(t)=z(t)$, and $Z(t)=z(t)+\varepsilon z(t-\tau)$ equation (1.1) becomes

$$x'=y,\qquad y'=z$$

$$Z'=p(\cdot)-f\big(x(t),y(t)\big)z(t)-g\big(y(t)\big)-h\big(x(t)\big) \\ -\sigma x(t)\omega'(t)+\int_{t-\tau}^{t}\left[h'\big(x(s)\big)y(s)+g'\big(y(s)\big)\right]ds. \quad (1.2)$$

Motivation for this work comes from the works in [7, 27, 28], where Lyapunov functionals are exploited to acquire asymptotic stability, boundedness, existence and uniqueness of solutions of the equations considered. Our notation shall be $x(t)=x$, $y(t)=y$, $z(t)=z$, and $f\big(x(t),y(t)\big)=f(x,y)$. Section 2 presents definitions of terms and basic results used in this paper. Stability of the trivial solution is stated and proved in Section 3, boundedness result is communicated in Section 4, and special cases of the theoretical results obtained in Sections 3 and 4 is presented in Section 5 and conclusion is finally presented in Section 6.

II. PRELIMINARY RESULTS

Let $(\Omega,\mathfrak{F},\{\mathfrak{F}_t\}_{t>0},\mathbb{P})$ be a complete probability space with a filtration $\{\mathfrak{F}_t\}_{t>0}$ satisfying the usual conditions (i.e., it is right continuous and $\{\mathfrak{F}_0\}$ contains all $\mathbb{P}-$null sets). Let $B(t)=(B_1(t),\cdots,B_m(t))^T$ be an $m-$ dimensional Brownian motion defined on the probability space. Let $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^n$. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by

$$\| A \| = \sqrt{trace\ (A^T A)}.$$

Consider a non autonomous $n-$dimensional stochastic delay differential equation

$$dx(t) = F\big(t, x(t), x(t-\tau)\big)dt$$
$$+G\big(t, x(t), x(t-\tau)\big)dB(t) \qquad (2.1)$$

on $t > 0$ with initial data $\{x(\theta): -\tau \leq \theta \leq 0\}, x_0 \in C([-\tau, 0], \mathbb{R}^n)$. Here $F: \mathbb{R}^+ \times \mathbb{R}^{2n} \to \mathbb{R}^n$ and $G: \mathbb{R}^+ \times \mathbb{R}^{2n} \to \mathbb{R}^{n\times m}$ are measurable functions. Suppose that the functions F, G satisfy the local Lipschitz condition, given any $b > 0$, $p \geq 2$, $F(t, 0,0) \in C^1([0, b], \mathbb{R}^n)$, and $G(t, 0,0) \in C^p([0, b], \mathbb{R}^{m\times n})$. Then there must be a stopping time $\beta = \beta(\omega) > 0$ such that equation (2.1) with $x_0 \in C^p_{\mathfrak{F}_{t_0}}$ [class of $\mathfrak{F}_t -$ measurable $C([-\tau, 0], \mathbb{R}^n) -$ valued random variables ξ_t and $E \| \xi_t \|^p < \infty$] has a unique maximal solution on $t \in [t_0, \beta]$ which is denoted by $x(t, x_0)$. Assume further that

$$F(t, 0,0) = G(t, 0,0) = 0$$

for all $t \geq 0$. Hence, the stochastic delay differential equation admits zero solution $x(t, 0) \equiv 0$ for any given initial value $x_0 \in C([-\tau, 0], \mathbb{R}^n)$.

Let $\mathbb{K}$ denote the family of all continuous non-decreasing functions $\rho: \mathbb{R}^+ \to \mathbb{R}^+$ such that $\rho(0) = 0$ and $\rho(r) > 0$ if $r \neq 0$. In addition, $\mathbb{K}_\infty$ denotes the family of all functions $\rho \in \mathbb{K}$ with

$$\lim_{r\to\infty} \rho(r) = \infty.$$

Suppose that $C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^+)$, denotes the family of all non negative functions $V = V(t, x_t)$ (Lyapunov functional) defined on $\mathbb{R}^+ \times \mathbb{R}^n$ which are twice continuously differentiable in x and once in t. By Itô's formula we have

$$dV(t, x_t) = LV(t, x_t)dt + V_x(t, x_t)G(t, x_t)dB(t),$$

where

$$LV(t, x_t) = \frac{\partial V(t, x_t)}{\partial t} + \frac{\partial V(t, x_t)}{\partial x_i} F\big(t, x(t)\big)$$
$$+\frac{1}{2} trace\ [G^T(t, x_t)V_{xx}(t, x_t)G(t, x_t)] \qquad (2.2)$$

with

$$V_{xx}(t, x_t) = (\frac{\partial^2 V(t, x_t)}{\partial x_i \partial x_j})_{n\times n}, i, j = 1, \cdots, n$$

In this study we will use the diffusion operator $LV(t, x_t)$ defined in (2.2) to replace $V'\big(t, x(t)\big) = \frac{d}{dt} V\big(t, x(t)\big)$. We now present the basic results that will be used in the proofs of the main results.

Lemma 2.1 *(See [17]) Assume that there exist* $V \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^+)$, *and* $\eta \in \mathbb{K}$ *such that*

(I) $V(t, 0) = 0$, for all $t \geq 0$;

(ii) $V(t, x_t) \geq \eta(\| x(t) \|)$, $\eta(r) \to \infty$ as $r \to \infty$; and

(iii) $LV(t, x_t) \leq 0$ for all $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^n$.

Then the zero solution of stochastic delay differential equation (2.1) is *stochastically stable.* If conditions (ii) and (iii) hold then (2.1) with $x_0 \in C^p_{\mathfrak{F}_{t_0}}$ has a unique global solution for $t > 0$ denoted by $x(t; x_0)$.

Lemma 2.2 *(See [17]) Suppose that there exist* $V \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^+)$, *and* $\eta_0, \eta_1, \eta_2 \in \mathbb{K}$ *such that*

(i) $V(t, 0) = 0$, for all $t \geq 0$;

(ii) $\eta_0(\| x(t) \|) \leq V(t, x_t)$, $\eta_0(r) \to \infty$ as $r \to \infty$; and

(iii) $LV(t, x_t) \leq -\eta_2(\| x(t) \|)$ for all $(t, x_t) \in \mathbb{R}^+ \times \mathbb{R}^n$.

Then the zero solution of stochastic delay differential equation (2.1) is *stochastically asymptotically stable in the large*

Assumption 2.3 *(See [25, 33]) Let* $V \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^+)$, *suppose that for any solutions* $x(t_0, x_0)$ *of stochastic delay differential equation (2.1) and for any fixed* $0 \leq t_0 \leq T < \infty$, *we have*

$$E^{x_0}\left\{\int_{t_0}^{T} V^2_{x_i}(t, x_t)G^2_{ik}(t, x_t)dt\right\} < \infty, 1 \leq i \leq n, 1 \leq k \leq m. \qquad (2.3)$$

Assumption 2.4 *(See [25, 33]) A special case of the general condition (2.3) is the following condition. Assume that there exists a function* $\rho(t)$ *such that*

$$\left|V_{x_i}(t, x_t)G_{ik}(t, x_t)\right| < \rho(t), x \in \mathbb{R}^n\ 1 \leq i \leq n, 1 \leq k \leq m, \qquad (2.4)$$

for any fixed $0 \leq t_0 \leq T < \infty$,

$$\int_{t_0}^{T} \rho^2(t)dt < \infty. \qquad (2.5)$$

$$E^{x_0} \| x(t, x_0) \| \leq \{V(t_0, x_0)e^{-\int_{t_0}^{t} \alpha(s)ds} + A\}^{\frac{1}{p}}, \qquad (2.6)$$

for all $t \geq t_0$, where

$$A := \int_{t_0}^{t} \big(\mu\alpha(u) + \psi(u)\big)e^{-\int_{u}^{t} \alpha(s)ds} du.$$

Lemma 2.5 *(See [25, 37]) Assume there exists a Lyapunov function* $V \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^+)$, *satisfying Assumption 2.3, such that for all* $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^n$,

(i) $\| x(t) \|^p \leq V(t, x_t)$,

(ii) $LV(t, x_t) \leq -\alpha(t)V^q(t, x_t) + \psi(t)$,

(iii) $V(t, x_t) - V^q(t, x_t) \leq \mu$,

where $\alpha, \psi \in C(\mathbb{R}^+, \mathbb{R}^+)$, p, q are positive constants, $p \geq 1$, and μ is a non negative constant. Then all solutions of stochastic delay differential equation (2.1) satisfy (2.6) for all $t \geq t_0$.

Corollary 2.6 *Suppose that*

$$\int_{t_0}^{t} \big(\mu\alpha(u) + \psi(u)\big)e^{-\int_{u}^{t} \alpha(s)ds} du \leq M, \qquad (2.7)$$

for all $t \geq t_0 \geq 0$ for some positive constant M.

Assume the hypotheses (i) to (iii) of Lemma 2.5 hold. If condition (2.7) is satisfied, then all solutions of stochastic delay differential equation (2.1) are *stochastically bounded.*

III. STABILITY OF SOLUTION

In this section we shall state and prove stability theorems. The proofs of the theorems depend on properties of the continuously differentiable functional $V = V(x_t, y_t, Z_t)$ defined by

$$V = a\int_0^x h(s)ds + yh(x) + \frac{1}{2}(y^2 + Z^2) + \frac{1}{2}ax^2 + \frac{1}{2}af(x,y)y^2 + yg(y) + a(x+y)Z + \int_{-\tau}^0\int_{t+s}^t [\lambda_1 y^2(\theta) + \lambda_2 z^2(\theta)]d\theta ds + \int_{t-\tau}^t \lambda_3 z^2(s)ds. \quad (3.1)$$

Next, we shall present our assumptions as follows.

Assumption 3.1 *Suppose that* $H_i > 0$ $i = (1,2,3,4,5)$, $0 < c < 1$, $c < a$, $a < a_1$ *and for all* $t \ge 0$

(I) $0 < d \le \frac{h(x)}{x} \le H_1$ $x \ne 0$, $h'(x) \le c$ and $|h'(x)| \le H_2$ for all x

(ii) $0 < b \le \frac{g(y)}{y} \le H_3$ $y \ne 0$, $|g'(y)| \le H_4$ for all y

(iii) $0 < a_1 \le f(x,y) \le H_5$ for all x, y;

(iv) $|y|f_x(x,y) \le 0$, $|z|f_y(x,y) \le 0$ for all x, y, z; and

(v) $|p(\cdot)| \le A < \infty$.

Theorem 3.2 *If* $p(\cdot) \equiv 0$, *and hypotheses (i) to (iv) are satisfied and the inequality*

$$\tau < \min\left\{\frac{B_1}{B_2}, \frac{B_3}{B_4}, \frac{B_{51}}{B_6}\right\}, \quad (3.2)$$

where

$$B_1 = ad - \sigma^2 - \varepsilon H_1; \quad B_2 := a(H_2 + H_4);$$
$$B_3 = ab - c - \varepsilon(a + H_3) - (1 + 2a + H_4 - b);$$
$$B_4 := (2a+1)H_2 + a(H_2 + H_4);$$
$$B_{51} = a_1 - a - \varepsilon(2a + H_1 + H_3 + H_5) - (1 + 2a + H_4 - b) - \varepsilon(a + H_5);$$
$$B_6 = (2a + \varepsilon)H_4 + (\varepsilon + 1)^2(H_2 + H_4),$$

holds. Then the trivial solution $X_t \equiv 0$ of system (1.2) is not only stable but asymptotically stable.

Proof. Let $X_t = (x_t, y_t, Z_t)$ be any solution of system (1.2), first we shall show that the continuously differentiable functional V satisfy $V(t,0) = 0$, is positive semi-definite, and that V is radially unbounded that is

$$V \to \infty \; as \; x^2 + y^2 + Z^2 \to \infty.$$

To see this the functional V defined by (3.1) obviously satisfies the equation

$$V(t,0) = 0 \quad (3.3)$$

for all $t \ge 0$ and can be recast as

$$V = \frac{1}{2}(h(x) + y)^2 + \int_0^x [a - h'(s)]h(s)ds + \frac{1}{8}(2ax + Z)^2 + \frac{a}{2}[f(x.y) - a]y^2 + \frac{a}{2}(1-a)x^2 + \frac{g(y)}{y}y^2 + \frac{1}{4}Z^2 + \frac{1}{8}(2ay + Z)^2 + \int_{t-\tau}^t \lambda_3 z^2(s)ds + \int_{-\tau}^0\int_{t+s}^t [\lambda_1 y^2(\theta) + \lambda_2 z^2(\theta)]d\theta ds.$$

Applying the lower inequalities in hypotheses (I) to (iii), $h'(x) \le c$ and the fact that $(dx + y)^2 + \frac{1}{8}(2ax + Z)^2 + \frac{1}{8}(2ay + Z)^2 \ge 0 \; \forall \, x, y, Z$,

there exist a positive constant

$$M_1 = \min\left\{(a - c)d + a(1 - a), \frac{a}{2}(a_1 - a) + b, \frac{1}{4}\right\}$$

such that

$$V \ge M_1(x^2 + y^2 + Z^2), \quad (3.4)$$

for all x, y and Z. From inequality (3.4) we find that

$$V = 0 \;\Leftrightarrow\; x^2 + y^2 + Z^2 = 0$$

it follows that

$$V \to \infty \; as \; x^2 + y^2 + Z^2 \to \infty. \quad (3.5)$$

Inequality (3.4) and estimate (3.5) clearly established that the functional V defined by (3.1) is positive semi-definite.

Next, we shall prove that the functional V is negative semi-definite noting that $p(\cdot) \equiv 0$. To see this the Itô's formula defined by (2.2) with respect to the functional V is calculated and simplified to be

$$\begin{aligned} LV = &-\left[a\frac{h(x)}{x} - \frac{1}{2}\sigma^2\right]x^2 - \left[a\frac{g(y)}{y} - h'(x) - \tau\lambda_1\right]y^2 \\ &-[2(f(x,y) - a) - \tau\lambda_2 - \lambda_3]z^2 - \left[a\frac{g(y)}{y} - a\right]xy \\ &-\left[af(x,y) + \frac{h(x)}{x}\right]xz \\ &+\left[g'(y) + a + 1 - \left(\frac{g(y)}{y} - a\right)\right]yz \\ &+\frac{1}{2}af_x(x,y)y^3 + \frac{1}{2}azf_y(x,y)y^2 - \lambda_3 z^2(t-\tau) \\ &+[a(x+y) + Z]\int_{t-\tau}^t [h'(x(s))y(s) + g'(y(s))z(s)]ds \\ &-\varepsilon\left[\frac{h(x)}{x}x + \left(\frac{g(y)}{y} - a\right)y + (f(x,y) - a)z\right]z(t-\tau) \\ &-\int_{t-\tau}^t (\lambda_1 y^2(\mu) + \lambda_2 z^2(\mu))d\mu. \end{aligned} \quad (3.6)$$

Employing hypotheses (i) to (iv), equation (3.6) becomes

$$\begin{aligned} LV \le &-\frac{1}{2}\{a_1 - a - 2\lambda_3 - (1 + 2a + H_4 - b) \\ &-\varepsilon(a + H_5) - [2\lambda_2 + (1+\varepsilon)(H_2 + H_4)]\tau\}z^2 \\ &-\frac{1}{2}\{ab - c - \varepsilon(a + H_3) - (1 + 2a + H_4 - b) \\ &-[2\lambda_1 + a(H_2 + H_4)]\tau\}y^2 \\ &-\frac{1}{2}\{ad - \sigma^2 - bH_1 - a(H_2 + H_4)\tau\}x^2 \\ &-\left[\lambda_1 - \frac{1}{2}(2a+1)H_2\right]\int_{t-\tau}^t y^2(\mu)d\mu \\ &-[\lambda_3 - \frac{1}{2}\varepsilon(2a + H_1 + H_3 + H_5) \\ &-\frac{1}{2}\varepsilon(1+\varepsilon)(H_2 + H_4)\tau]\, z^2(t-\tau) \\ &-\left[\lambda_2 - \frac{1}{2}(2a+\varepsilon)H_4\right]\int_{t-\tau}^t z^2(\mu)d\mu - N_1 - N_2 - N_3, \end{aligned} \quad (3.7)$$

$$N_1 := \frac{1}{4}adx^2 + \left(\frac{ag(y)}{y} - a\right)xy + \frac{1}{4}(ab-c)y^2;$$
where $$N_2 := \frac{1}{4}adx^2 + \left[af(x,y) + \frac{h(x)}{x}\right]xz + \frac{1}{2}(a_1 - a)z^2;$$
$$N_3 := \frac{1}{4}(ab-c)y^2 + \left(\frac{g(y)}{y} - a\right)yz + \frac{1}{2}(a_1 - a)z^2.$$

Let
$$\lambda_1 := \frac{1}{2}(2a+1)H_2, \lambda_2 := \frac{1}{2}(2a+\varepsilon)H_4 \text{ and}$$
$$\lambda_3 := \frac{1}{2}[\varepsilon(2a + H_1 + H_3 + H_5) + \varepsilon(1+\varepsilon)(H_2+H_4)\tau].$$

Now since N_1, N_2 and N_3 are quadratic functions, applying Hessian matrix, it can be shown that $N_1 \geq 0$, $N_2 \geq 0$ and $N_3 \geq 0$ for all $x, y,$ and z. There exists positive constant
$$M_2 := \frac{1}{2}\min\{B_1 - B_2\tau,\ B_3 - B_4\tau,\ B_{51} - B_6\tau,\}$$
such that
$$LV \leq -M_2(x^2+y^2+z^2), \qquad (3.8)$$
for all x, y, z and
$$LV \leq 0, \qquad (3.9)$$
for all x, y, z. From equation (3.3), inequality (3.4), estimate (3.5) and (3.8), hypotheses of Lemmas 2.1 and 2.2 hold, hence by Lemmas 2.1 and 2.2 the trivial solution X_t of the neutral stochastic system (1.2) for $p(\cdot) \equiv 0$ is stochastically asymptotically stable and stochastically stable respectively. This completes the proof of Theorem 3.2.

IV. BOUNDEDNESS OF SOLUTION

In this section, we shall state and prove a boundedness theorem using (3.1), some inequalities and estimates (3.5) already obtained in Section 3.

Theorem 4.1 *If in addition to Theorem 3.2 hypothesis (v) hold and the inequality*
$$\tau < \min\left\{\frac{B_1}{B_2}, \frac{B_3}{B_4}, \frac{B_{52}}{B_6},\right\} \qquad (4.1)$$
where
$$B_1 := ad - \sigma^2 - \varepsilon H_1; \quad B_2 := a(H_2 + H_4);$$
$$B_3 := ab - c - \varepsilon(a + H_3) - (1 + 2a + H_4 - b);$$
$$B_4 := (2a+1)H_2 + a(H_2 + H_4);$$
$$B_{52} := a_1 - a - \varepsilon(2a + H_1 + H_3 + H_5) - 2A\varepsilon$$
$$-(1 + 2a + H_4 - b) - \varepsilon(a + H_5);$$
$$B_6 := (2a+b)H_4 + (\varepsilon+1)^2(H_2+H_4)$$
holds. Then the solutions (X_t) of system (1.2) is stochastically bounded.

Proof. Let (X_t) be any solution of system (1.2). Equation (3.6) can be modified to accommodate $p(\cdot) \neq 0$ as
$$\text{LV} = -\left[a\frac{h(x)}{x} - \frac{1}{2}\sigma^2\right]x^2 - \left[a\frac{g(y)}{y} - h'(x) - \tau\lambda_1\right]y^2$$
$$-[2(f(x,y) - a) - \tau\lambda_2 - \lambda_3]z^2 - \left[a\frac{g(y)}{y} - a\right]xy$$
$$-\left[af(x,y) + \frac{h(x)}{x}\right]xz + \left[g'(y) + 2a + 1 - \frac{g(y)}{y}\right]yz$$
$$+\frac{1}{2}af_x(x,y)y^3 + \frac{1}{2}azf_y(x,y)y^2 + [a(x+y) + Z] \times$$
$$\left\{\int_{t-\tau}^{t}[h'(x(s))y(s) + g'(y(s))z(s)]ds + p(\cdot)\right\}$$
$$-\varepsilon\left[\frac{h(x)}{x}x + \left(\frac{g(y)}{y} - a\right)y + (f(x,y) - a)z\right]z(t-\tau)$$
$$-\lambda_3 z^2(t-\tau) - \int_{t-\tau}^{t}\left(\lambda_1 y^2(\mu) + \lambda_2 z^2(\mu)\right)d\mu. \qquad (4.2)$$

Employing hypotheses (I) to (v), equation (4.2) becomes
$$LV \leq -\frac{1}{2}\{a_1 - a - 2\lambda_3 - (1 + 2a + H_4 - b)$$
$$-\varepsilon(a + H_5) - [2\lambda_2 + (1+b)(H_2+H_4)]\tau\}z^2$$
$$-\frac{1}{2}\{ab - c - \varepsilon(a + H_3) - (1 + 2a + H_4 - b)$$
$$-[2\lambda_1 + a(H_2+H_4)]\tau\}y^2$$
$$-\frac{1}{2}\{ad - \sigma^2 - bH_1 - a(H_2+H_4)\tau\}x^2$$
$$-\left[\lambda_1 - \frac{1}{2}(2a+1)H_2\right]\int_{t-\tau}^{t} y^2(\mu)d\mu$$
$$-[\lambda_3 - \frac{1}{2}\varepsilon(2a + H_1 + H_3 + H_5) - A\varepsilon$$
$$-\frac{1}{2}\varepsilon(1+\varepsilon)(H_2+H_4)\tau - A\varepsilon - N_1$$
$$-N_2 - N_3]z^2(t-\tau)$$
$$-\left[\lambda_2 - \frac{1}{2}(2a+\varepsilon)H_4\right]\int_{t-\tau}^{t} z^2(\mu)d\mu$$
$$+A\max\{1,a\}(|x| + |y| + |z|) + Ab, \qquad (4.3)$$
Where N_1, N_2 and N_3 are defined above. There exists positive constant
$$M_3 := \frac{1}{2}\min\{B_1 - B_2\tau, \quad B_3 - B_4\tau, \quad B_{52} - B_6\tau,\}$$
such that
$$LV \leq -M_3(x^2+y^2+z^2)$$
$$+A\max\{1,a\}(|x| + |y| + |z|) + A\varepsilon, \qquad (4.4)$$
for all x, y, z. Since $|q| \leq q^2 + 1$ if follows that
$$A\max\{1,a\}(|x| + |y| + |z|) \leq A\max\{1,a\}(x^2 + y^2 + z^2) + 3A\max\{1,a\} \qquad (4.5)$$
for all x, y, z. From inequalities (4.4) and (4.5) there exist positive constants M_4 and M_5 such that
$$LV \leq -M_4(x^2+y^2+z^2) + M_5 \qquad (4.6)$$
for all x, y, z provided that $A\max\{1,a\} < M_3$ where $M_4 := M_3 - A\max\{1,a\}$ and $M_5 = 3A\max\{1,a\} + A\varepsilon$. Finally, from inequalities (3.4) and (4.6), estimates (iii) of Lemma 2.5 holds so that $\mu \geq 0$. Also, From Lemma 2.5 inequalities 2.6 and 2.7 hold so that by Corollary 2.6, all solutions of the neutral stochastic system (1.2) are stochastically bounded. This completes the proof of Theorem 4.1

V. EXAMPLES

Consider the following special cases of equation (1.1). We shall test the effectiveness and reliability of our tool and the results obtained in Sections 3 and 4.

Example 1. *Consider a nonlinear third order neutral stochastic differential equation*
$$[x'' + \varepsilon x''(t-\tau)]' + \left[\frac{20(\sin x + \sin x') + 21}{\sin x + \sin x' + 3}\right]x''$$

$$+\frac{5x'(t-\tau)+3(x'(t-\tau))^3}{1+(x'(t-\tau))^2}$$
$$+\frac{21x(t-\tau)[x^2(t-\tau)+6]+40x(t-\tau)\text{sin}x(t-\tau)}{40[x^2(t-\tau)+6]}$$
$$+1.264x\omega'=0, \qquad (5.1)$$

where $\varepsilon = 0.001 > 0$. Equation (5.1) converted to system of first order neutral stochastic is given by

$x' = y, \quad y' = z,$

$$Z' = -\left[\frac{20(\text{sin}x+\text{sin}y)+21}{\text{sin}x+\text{sin}y+3}\right]z - \frac{y(3y^2+5)}{y^2+1}$$
$$-\frac{21x(x^2+6)+40x\text{sin}x}{40(x^2+6)} - 1.264x\omega'$$
$$+\int_{t-\tau}^{t}\left[\frac{(y^2(s)+1)(3y^2(s)+5)-4y^2(s)}{(1+y^2(s))^2}\right]z(s)ds$$
$$+\int_{t-\tau}^{t}[21(6+x^2(s))^2+40(6+x^2(s))(\text{sin}x(s)$$
$$+x(s)\text{cos}x(s))-80x^2(s)\text{sin}x(s)]\times$$
$$[40(6+x^2(s))^2]^{-1}y(s)ds. \qquad (5.2)$$

Comparing (1.2) and (5.2) we find that:

(i) The function

$$h(x) = \frac{21x(6+x^2)+40x\text{sin}x}{40(6+x^2)} = \frac{21x}{40} + xH(x)$$

where

$$H(x) := \frac{\text{sin}x}{x^2+6}.$$

The graph of $H(x)$ is shown in Figure 1 where we found that

$$-0.125 \le H(x) \le 0.125$$

and that

$$0.4 = d \le \frac{h(x)}{x} \le H_1 = 0.65, \qquad x \neq 0,$$

as depicted in Figure 2. This fulfills the first inequalities of Assumption 3.1 (i).

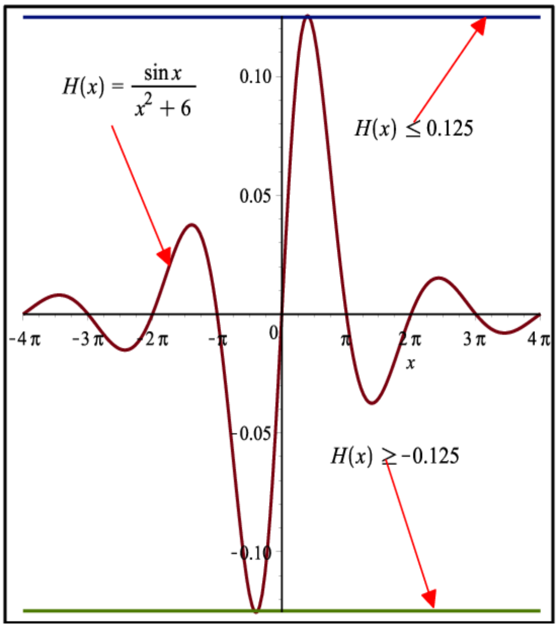

Figure 1: Upper and lower bounds on the functions $H(x)$ on the inteval $-4\pi \le x \le 4\pi$]

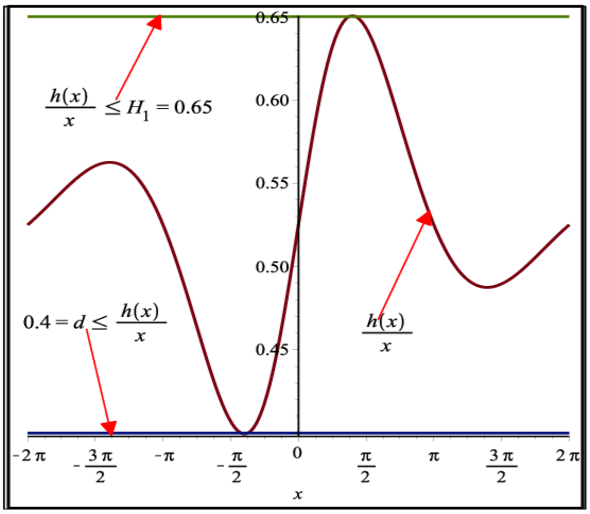

Figure 2: Upper bound on the functions $\frac{h(x)}{x}$ for all $x \neq 0$ on $-2\pi \le x \le 2\pi$.

(ii) Next the derivative of the function h is

$$h'(x) = \frac{21}{40} + \frac{\text{sin}x + x\text{cos}x}{6+x^2} - \frac{2x^2\text{sin}x}{(6+x^2)^2} \le c = 0.724$$

and $|h'(x)| \le H_2 = 0.724$, the coincide paths of $h'(x)$ and $|h'(x)|$ are shown in Figure 3 and Figure 4 at different intervals of real $-4\pi \le x \le 4\pi$ and $-50\pi \le x \le 50\pi$ respectively, thus 2nd and 3rd inequalities of Assumption 3.1 (i) are satisfied.

(iii) Next the function

$$g(y) := 3y + yG(y), \qquad where \;\; G(y) := \frac{2}{y^2+1}.$$

It is shown in Figure 5 that

$$0.01 \le G(y) \le 2$$

for all y on $-10 \le y \le 10$. It follows from the last inequalities and Figure 6 that

$$b = 3.01 \le \frac{g(y)}{y} \le H_3 = 5, \qquad y \neq 0.$$

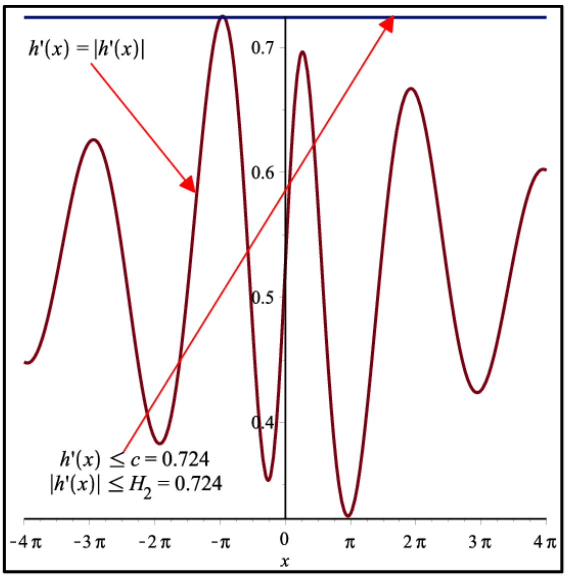

Figure 3: The functions $h'(x)$ and $|h'(x)|$ on the inteval $-4\pi \le x \le 4\pi$.

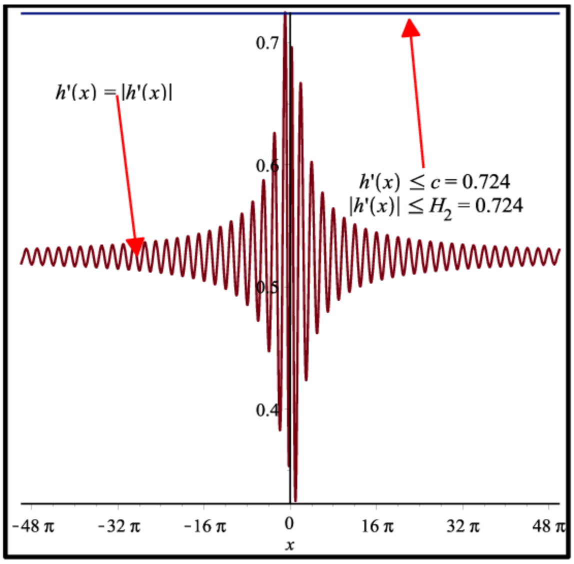

Figure 4: The functions $h'(x)$ and $|h'(x)|$ on the interval $-50\pi \leq x \leq 50\pi$.

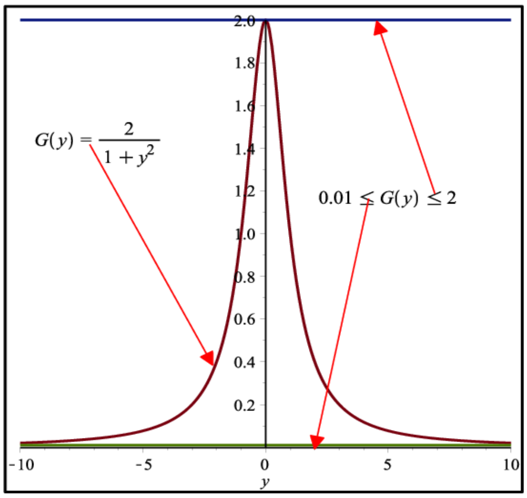

Figure 5: Upper and lower bounds on the functions $G(y)$ on $-10\pi \leq y \leq 10\pi$.

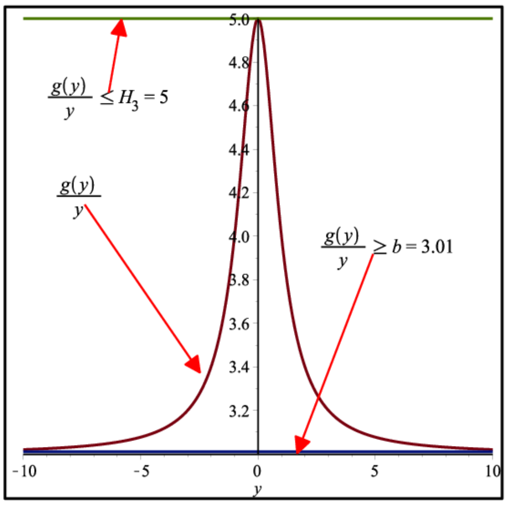

Figure 6: The functions $\frac{g(y)}{y}$ on the interval $-10 \leq y \leq 10$.

(iv) Furthermore, the derivative of the function g is

$$g'(y) = 3 + \frac{2}{1+y^2} - \frac{4y^2}{(1+y^2)^2}$$

and

$$|g'(y)| \leq H_4 = 5$$

for all y and its path is shown in Figure 7 on the real interval $-15 \leq y \leq 15$.

(v) Figure 8 shows the function

$$f(x,y) := 20 + F(x,y)$$

on $-2\pi \leq x, y \leq 2\pi$, where

$$F(x,y) = \frac{1}{\sin x + \sin y + 3}$$

and it can be shown that $0 < F(x,y) \leq \frac{1}{3}$ for all x, y. Therefore,

$$20 = a_1 \leq f(x,y) \leq H_5 = 20.33$$

for all x and y, this satisfies inequality in hypothesis (iii) of Assumption 3.1.

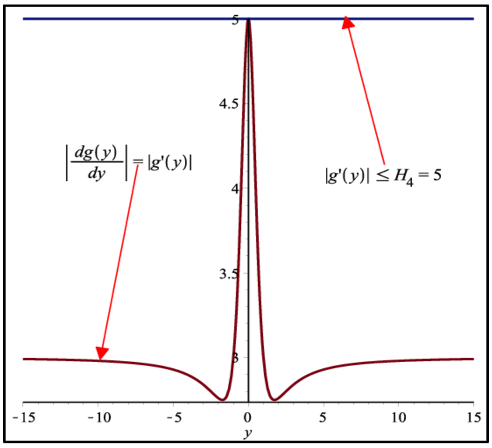

Figure 7: The upper bound on function $|g'(y)|$ on $-15 \leq y \leq 15$

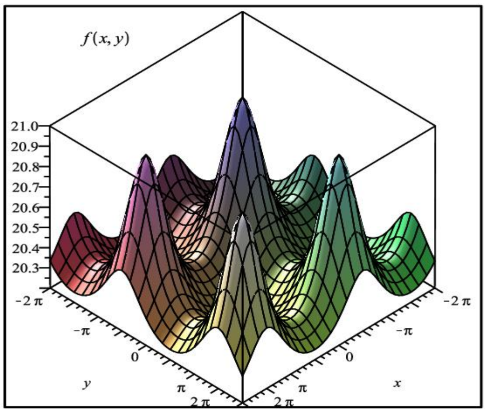

Figure 8: The bounds on function $f(x,y)$ on $-2\pi \leq x, y \leq 2\pi$

(vi) In addition, the derivative of the function f with respect to x is defined as

$$f_x(x,y) = -\frac{\cos x}{(\sin x + \sin y + 3)^2} < 0 \; \forall \; x, y.$$

The solid generate by this rate of change in f with respect to x is shown in Figure 9 on $-2\pi \le x, y \le 2\pi$ and

$$|y| f_x(x,y) = -\frac{|y| \cos x}{(\sin x + \sin y + 3)^2} \le 0 \; \forall \; x, y$$

as shown in Figure 10 on $-2\pi \le x, y \le 2\pi$.

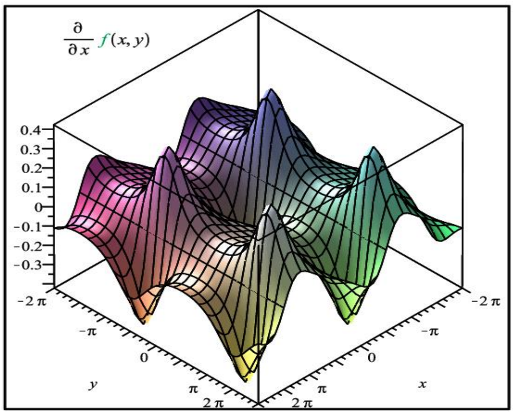

Figure 9: The function $f_x(x,y)$ for $-2\pi \le x, y \le 2\pi$

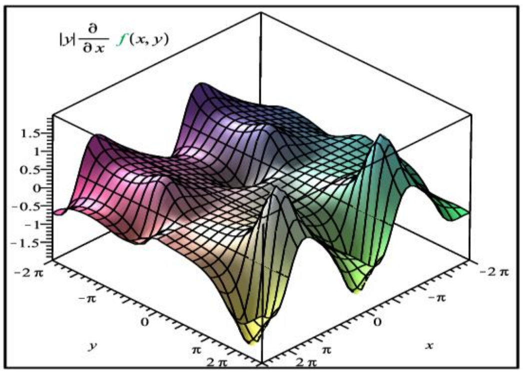

Figure 10: The function $|y| f_x(x,y)$ on $-2\pi \le x, y \le 2\pi$

(vii) The derivative of the function f with respect to variable y is

$$f_y(x,y) = -\frac{\cos y}{(\sin x + \sin y + 3)^2} < 0$$

for all x, y and

$$|z| f_y(x,y) = -|z| \frac{\cos y}{(\sin x + \sin y + 3)^2} \le 0$$

for all x, y, z. Figures 11 and 12 depict the solids represented by $f_y(x,y)$ and $|z| f_y(x,y)$ respectively on $-2\pi \le x, y \le 2\pi$.

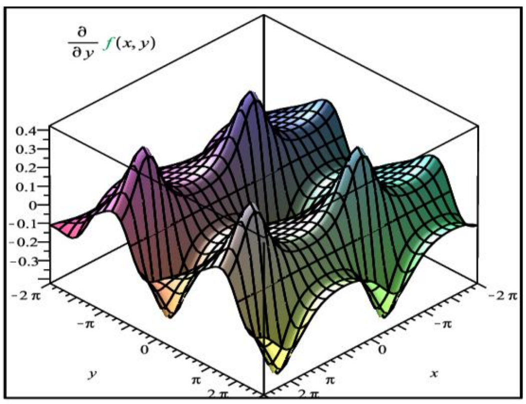

Figure 11: The function $f_y(x,y) < 0$ on $-2\pi \le x, y \le 2\pi$

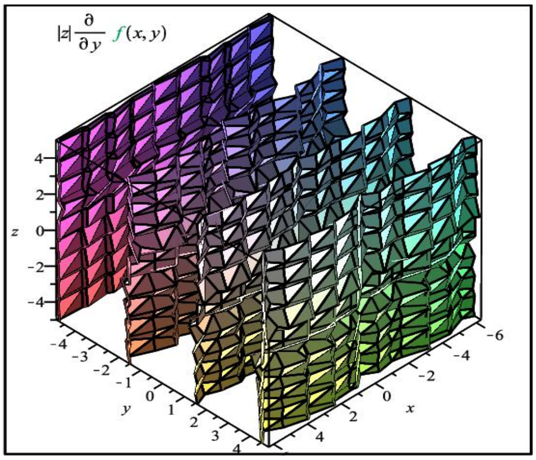

Figure 12: The function $|z| f_y(x,y)$ on $-2\pi \le x, y \le 2\pi$

To estimate the value of the constant τ, we calculate the following relations

$$B_1 := ad - \sigma^2 - \varepsilon H_1 = 0.001654 > 0;$$

$$B_2 := a(H_2 + H_4) = 4 > 0;$$

$$B_3 := ab - c - \varepsilon(a + H_3) - (1 + 2a + H_4 - b) = 2.325 > 0;$$

$$B_4 := (2a+1)H_2 + a(H_2 + H_4) = 10.516 > 0;$$

$$B_{51} := a_1 - a - \varepsilon(2a + H_1 + H_3 + H_5) - (1 + 2a + H_4 - b) - \varepsilon(a + H_5) = 6.95169 > 0;$$

$$B_6 := (2a + \varepsilon)H_4 + (\varepsilon + 1)^2 (H_2 + H_4) = 27.73445172 > 0,$$

holds. It follows that

$$\tau < \min\left\{\frac{B_1}{B_2}, \frac{B_3}{B_4}, \frac{B_{51}}{B_6}\right\}$$
$$= \min\{0.0004135, 0.2210916698, 0.250651791\}$$
$$= 0.0004135,$$

this implies that $\tau < 0.0004135$. The hypotheses of Theorem 3.2 hold and by Theorem 3.2 the trivial solution $X_t \equiv 0$ of system (5.2) is not only stable but asymptotically stable.

Example 2. *Consider a nonlinear third order neutral stochastic differential equation with nonzero forcing term*

$$[x'' + \varepsilon x''(t-\tau)]' + \left[\frac{20(\sin x + \sin x') + 21}{\sin x + \sin x' + 3}\right] x'' + \frac{5x'(t-\tau) + 3(x'(t-\tau))^3}{1 + (x'(t-\tau))^2} + \frac{21x(t-\tau)[x^2(t-\tau) + 6] + 40x(t-\tau)\sin x(t-\tau)}{40[x^2(t-\tau) + 6]}$$
$$+1.264 x\omega' = \mathrm{p}(.), \qquad (5.3)$$

Equation (5.3) converted to system of first order neutral stochastic is given by

$x' = y, \quad y' = z,$

$$Z' = -\left[\frac{20(\sin x + \sin y) + 21}{\sin x + \sin y + 3}\right] z - \frac{y(3y^2 + 5)}{y^2 + 1}$$
$$-\frac{21x(x^2 + 6) + 40x\sin x}{40(x^2 + 6)} - 1.264 x\omega'$$

$$+\int_{t-\tau}^{t}\left[\frac{(y^2(s)+1)(3y^2(s)+5)-4y^2(s)}{(1+y^2(s))^2}\right]z(s)ds$$
$$+\int_{t-\tau}^{t}[21(6+x^2(s))^2+40(6+x^2(s))(\sin x(s)$$
$$+x(s)\cos x(s))-80x^2(s)\sin x(s)]\times$$
$$[40(6+x^2(s))^2]^{-1}y(s)ds \qquad (5.4)$$

Comparing (1.2) and (5.4) we find that items (i) to (vii) in Example 1 hold and in addition let

$$P(\cdot)=\frac{1}{5}+\frac{1}{3+t^2+x^2+x^2(t-\tau)+y^2(t)}$$

Clearly $3+t^2+x^2+x^2(t-\tau)+y^2(t)\geq 3$ so that

$$0<\frac{1}{3+t^2+x^2+x^2(t-\tau)+y^2(t)}\leq\frac{1}{3}$$

from this inequality we conclude that

$$p(\cdot)\leq A=\frac{8}{15}<\infty,$$

so that Assumption 3.1 (v) is satisfied. To estimate the value of the constant τ in this case, we calculate the following relations

$$B_1:=ad-\sigma^2-\varepsilon H_1=0.001654>0;$$
$$B_2:=a(H_2+H_4)=4>0;$$
$$B_3:=ab-c-\varepsilon(a+H_3)-(1+2a+H_4-b)=2.325>0;$$
$$B_4:=(2a+1)H_2+a(H_2+H_4)=10.516>0;$$
$$B_{52}:=a_1-a-\varepsilon(2a+H_1+H_3+H_5)$$
$$-2A\varepsilon-(1+2a+H_4-b)$$
$$-\varepsilon(a+H_5)=6.950623333>0>0;$$
$$B_6:=(2a+\varepsilon)H_4+(\varepsilon+1)^2(H_2+H_4)$$
$$=27.73445172>0,$$

holds. It follows that

$$\tau<\min\left\{\frac{B_1}{B_2},\frac{B_3}{B_4},\frac{B_{52}}{B_6}\right\}$$
$$=\min\{0.0004135,0.2210916698,0.250651791\}$$
$$=0.0004135,$$

this implies that the inclusion of the forcing term in this example is insignificant, i.e., the constant τ is the same as in Example 1 ($\tau<0.0004135$). The hypotheses of Theorem 4.1 hold and by Theorem 4.1 the solutions X_t of system (5.4) is stochastically bounded.

VI. CONCLUSION

In conclusion, this study investigated the stability and boundedness of solutions to a specific type of third order neutral stochastic differential equations with delay. The results provide valuable insights into the behavior of these equations and contribute to the understanding of their dynamics. Further research in this area could focus on exploring additional properties of solutions and their applications in various fields.

REFERENCES

1. Ademola, A.T., Mahmoud, A.M., Arawomo, P.O. 2019. On the behaviour of solutions for a class of third order neutral delay differential equations. Analele Universităţii Oradea Fascicola Matematica ; 26 (2): 85-10
2. Ademola, A.T., Arawomo, P.O., Ogunlaran, O.M., and Oyekan, E.A. 2013. Uniform stability, boundedness and asymptotic behaviour of solutions of some third order nonlinear delay differential equations, Differential Equations and Control Processes, No. 4, 43-66
3. Ademola, A.T., and Arawomo, P.O. 2013. Uniform Stability and boundedness of solutions of nonlinear delay differential equations of the third order, Math. J. Okayama Univ. 55, 157-166 https://www.math.okayama-u.ac.jp/mjou/mjou
4. Ademola, A.T. 2017. Stability and uniqueness of solution to a certain third order stochastic delay differential equations. Differential Equation and Control Process N 2, 24-50
5. Ademola, A.T., Ogundare, B.S. Ogundiran, M.O. and O. A. Adesina, O.A. 2015. Stability, boundedness, and existence of periodic solutions to certain third order delay differential equations with multiple deviating arguments, International Journal of Differential Equations, 2015, Article ID 213935, 12 pages. http://dx.doi.org/10.1155/2015/213935
6. Ademola, A., Arawomo, P., Adesina, A., and Okoya, S. 2022. On the behaviour of solutions to a kind of third order nonlinear neutral differential equation with delay, Turkish Journal of Mathematics, 46,(8) 3139–3158, https://doi.org/10.55730/1300-0098.3324
7. Ademola, A.T. Aderogba, A.A., Ogundipe, O.L., Akinbo, G., and Onasanya, B.O. 2024. Asymptotic stability of neutral differential systems with variable delay and nonlinear perturbations, Math. Sci. Appl. E-Notes, 12(2), 71–80, https://doi.org/10.32323/msaen.1320286
8. Ademola, A. T. 2021. Asymptotic behaviour of solutions to certain nonlinear third order neutral functional differential equation, Heliyon, 7(4), 1-8, https://doi.org/10.1016/j.heliyon.2021.e06615
9. Ademola, A.T. 2013. Existence and uniqueness of a periodic solution to certain third order nonlinear delay differential equation with multiple deviating arguments, Acta Univ. Sapientiae, Mathematica, 5(2), 113-131, https://doi.org/0.2478/ausm-2014-0008
10. Ademola, A.T., and Arawomo, P.O. 2010. Stability and ultimate boundedness of solutions to certain third order differential equations, Applied Mathematics E-Notes, 10 61- 69. http://eudml.org/doc/233275
11. Ademola, A.T., and Arawomo, P.O. 2011. Stability, boundedness and asymptotic behaviour of solution of a certain nonlinear differential equation of the third order, Kragujevac Journal of Mathematics, 35(3), (2011), 431-445 http://elib.mi.sanu.ac.rs/files/journals/kjm

12. Ademola, A.T., and Arawomo, P.O. 2012. Boundedness and asymptotic behaviour of solutions of a nonlinear differential equation of the third order, Afr. Mat., 23, 261–271. https://doi.org/10.1007/s13370-011-0034-x
13. Ademola, T.A., Ogundiran, M.O., Arawomo, P.O., and Adesina, O.A. 2010. Boundedness results for a certain third order nonlinear differential equation, Appl. Math. Comput., 216, no. 10, 044- 3049. https://doi.org/10.1016/j.amc.2010.04.022
14. Adesina, O.A. 2014. Results on the qualitative behaviour of solutions for a class of third order nonlinear differential equations, AIP Conference Proceedings, 1637, no. 1, 5-12. https://doi.org/10.1063/1.4904558
15. Afuwape, A.U., and Omeike, M.O. 2010. Stability and boundedness of solutions of a kind of third order delay differential equations, Comp. Appl. Math., 29(3)329 -342, https://doi.org/10.1590/S1807-03022010000030000
16. Arnold, L. Stochastic differential equations: Theory and applications, John Wiley & Sons, 1974.
17. Arino, O., Hbid M. L., Ait Dads, E. Delay differential equations and application. NATO Science Series II: Mathematics, Physics and Chemistry, 205, 2002.
18. Burton, T.A. Volterra Integral and Differential Equations, (Mathematics in science and engineering 202) Elsevier, 1983.
19. Burton, T.A. Stability and periodic solutions of ordinary and functional differential equations of Mathematics in Science and Engineering, 178. Academic Press Inc., Orlando, Fla, USA, 1985.
20. Graef, J. R., Beldjerd, C.D., and Remili, M. On stability, boundedness, and square integrability of solutions of certain third order neutral differential equations, Mathematica Bohemica, Online first 2021. https://doi.org/10.21136/MB.2021.0081-19
21. Hale, J.K. Functional Differential Equations. Applied Mathematical Sciences, vol. 3. Springer, New York (1971)
22. Kolmanovskii, V., and Shaiket L. 2002. Construction of Lyapunov functionals for stochastic hereditary systems. a survey of some recent results, Mathematical and computer modelling, 36(6), 691-716. https://doi.org/10.1016/S0895-7177(02)00168-1
23. Kolmanovskii, V., and Myshkis, A. Introduction to the Theory and Applications of Functional Differential Equations, Kluwer, Dodrecht, 1999.
24. Liu, R., and Raffoul, Y. 2009. Boundedness and exponential stability of highly nonlinear stochastic differential equations. Electronic Journal of Differential Equations, 2009(143), 1–10. http://eudml.org/doc/223106
25. Mahmoud, A.M., and Tunç, C. 2019. Asymptotic stability of solutions for a kind of third order stochastic differential equation with delays, Miskole Mathematical Note, 20(1), 381-393, https://doi.org/10.18514/MMN.2019.2800
26. Mahmoud, A.M., and Ademola, A.T. 2022. On the behaviour of solutions to a kind of third order neutral stochastic differential equation with delay, Advances in Continuous and Discrete Models (2022) 2022:28 https://doi.org/10.1186/s13662-022-03703-x
27. Mahmoud, A.M., and Bakhit, D. A. M. 2024. On behaviours for the solution to a certain third order stochastic integro-differential equation with time delay, Filomat 38:2 (2024), 487-504. https://doi.org/10.2298/FIL2402487M
28. Mahmoud, A.M., and Bakhit, D. A. M. 2023. On the properties of solutions for nonautonomous third order stochastic differential equation with a constant delay, Turkish Journal of Mathematics 47(1) (2023) 135-158 https://doi.org/10.55730/1300-0098.3351
29. Mao, X. 2001. Some contributions to stochastic asymptotic stability and boundedness via multiple Lyapunov functions, Journal of Mathematical Analysis and Applications, 260, 325 –340 (2001) https://doi.org/10.1006/jmaa.2001.7451.
30. Oksendal, B. Stochastic Differential Equations: An introduction with applications, springer, 2000.
31. Oudjedi, L.D., Lekhmissi, B., and Remili, M. 2019. Asymptotic properties of solutions to third order neutral differential equations with delay, Proyecciones Journal of Mathematics, 38(1), (2019), 111 – 127 http://dx.doi.org/10.4067/S0716-09172019000100111
32. Raffoul, Y.N. 2007. Boundedness and exponential asymptotic stability in dynamical systems with applications to nonlinear differential equations with unbounded terms, Advances in Dynamical Systems and Applications, 2(1), (2007), 107-121 http://www.ripublication.com/adsa.htm
33. Remili, M., and Oudjedi, L.D. 2016. On asymptotic stability of solutions to third order nonlinear delay differential equation, Filomat, 30, 12, (2016), 3217–3226 https://www.jstor.org/stable/24899332
34. Remili, M., and Oudjedi, L.D. 2016. Stability of the solutions of nonlinear third order differential equations with multiple deviating arguments, Acta Univ. Sapientiae, Mathematica, 8, 1, (2016), 150- 165 http://dx.doi.org/0.1515/ausm-2016-0009
35. Remili, M., Oudjedi, L.D., and Beldjerd, D. 2016. On the qualitative behaviors of solutions to a kind of nonlinear third order differential equation with delay, Communications in Applied Analysis, 20, 53–64.
36. Rodkina, A., and Basin, M.2006. On delay-dependent stability for a class of nonlinear stochastic delay-differential equations, Mathematics of Control, Signals, and Systems, 18(2), (2006), 187–197. https://doi.org/10.1007/s00498-006-0163-1

37. Sadek, A.I. 2003. Stability and boundedness of a kind of third order delay differential system, Applied Mathematics Letters, 16, (2003), 657 - 662. https://doi.org/10.1016/S0893-9659(03)00063-6
38. Shaikihet, L. Lyapunov functional and stability of stochastic functional differential equations, Springer, International, 2013.
39. Tejumola H.O., and Tchegnani, B. 2000. Stability, boundedness and existence of periodic solutions of some third order and fourth-order nonlinear delay differential equations. Journal of the Nigerian Mathematical Society 2000; 19: 9-19.
40. Tunç, C. 2009. On the stability and boundedness of solutions to third order nonlinear differential equations with retarded argument. Nonlinear Dynamics 2009; 57 (1-2): 97-106. https://doi.org/10.1007/s11071-008-9423-6
41. Tunç C. 2010. Some stability and boundedness conditions for nonautonomous differential equations with deviating arguments. Electronic Journal Qualitative Theory of Differential Equations 2010; 1: 1-12. http://www.math.u-szeged.hu/ejqtde
42. Tunç C. 2015. Global stability and boundedness of solutions to differential equations of third order with multiple delays, Dynam. Systems Appl. 24, 467- 478.
43. Yoshizawa, T. Stability theory by Liapunov's second method, The Mathematical Society of Japan, 1966.