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Some Properties of Trace of α Lower Level Partition of Fuzzy Square Matrices.

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Abstract:

A new definition of trace of fuzzy square matrix is introduced in accordance with newly defined α - lower level partition of fuzzy square matrices. Some properties of trace of α - lower level partition of fuzzy square matrices are established.

Keywords: α - lower level partitions, addition, multiplication, complementation, trace of fuzzy square matrices.

1.Introduction:

Matrices with entries in [0,1] and matrix operation defined by fuzzy logical operations are called fuzzy matrices. The concept of section of fuzzy matrices are introduced by Kim and Roush [17]. All fuzzy matrices are matrices but every matrix is not a fuzzy matrix. Fuzzy matrices play a fundamental role in fuzzy set theory. They provide us with a rich frame work within which many problems of practical applications of the theory can be formulated. Fuzzy matrices can be successfully used when fuzzy uncertainty occurs in a problem. These results are extensively used for cluster analysis and classification problem of static patterns under subjective measure of similarity.

On the otherhand, fuzzy matrices are generalized Boolean matrices which have been studied for fruitful results. And the theory of Boolean matrices can be backbone to the theory of matrices with non-negative contents, for which most famous classical results were obtained 1907 to 1912 by Parren and Frobenius. So the theory of fuzzy matrices is interesting in its own right.

Fuzzy matrix has been proposed to represent fuzzy relation in a system based on fuzzy set theory [1]. Fuzzy matrices were introduced first time by Thomson [2], who discussed the convergence of powers of fuzzy matrices.

Several authors have presented a number of results on the convergence of power sequence of fuzzy matrices [3,4,5]. Fuzzy Matrices play an important role in science and Technology. Kim [6,7] represented some important result on the determinant of a square matrix. He defined the determinant of a square fuzzy matrix and research works [6,7,8,9] has contributed a lot to the study of determinant theory of square fuzzy matrices. The adjoint of square fuzzy matrix was defined by Thomson [2] and Kim [6].

In case of fuzzy square matrices max and min operations are defined for addition and multiplication of two fuzzy square matrices, the resulting matrix is again a fuzzy matrix, Kandasamy [1].

In this paper trace of matrices for α -lower level partitions of fuzzy square matrices are introduced. Properties of trace of matrices under fuzzy α -lower level partitions

viz: commutative, associative, complementation and distributive properties are examined with counter examples.

2. Preliminaries:

Definition 2.1:

A fuzzy matrix is a matrix which has its elements from the interval [0,1] called the unit fuzzy interval. An mxn fuzzy matrix for which m = n ((ie) the number of rows is equal to the number of columns) and whose elements belong to the unit interval [0,1] is called a fuzzy square matrix of order n.

Example 1

$$\begin{pmatrix} 0.3 & 0.7 & 0.8 \\ 0.4 & 0.5 & 0.1 \\ 0.4 & 0.3 & 0.6 \end{pmatrix} \text{ is fuzzy square matrix of order 3.}$$

Definition 2.2

The α - lower level partition of a fuzzy matrix 'A' is a Boolean matrix denoted by

$$A_{(\alpha)} = \left(aij_{(\alpha)}\right) \text{ such that}$$
$$aij_{(\alpha)} = aij \;\; if \;\; aij \le \alpha$$
$$= 0 \;\; if \;\; aij > \alpha \quad \text{where } \alpha \in [0,1].$$

Addition of Fuzzy Square Matrices 2.3:

If $A_{(\alpha)}$ and $B_{(\alpha)}$ are the α - lower level partitions of fuzzy square matrices same order of A,B then addition is defined as,

$A_{(\alpha)} + B_{(\alpha)} = Max\left(aij_{(\alpha)}, bij_{(\alpha)}\right)$ where $aij_{(\alpha)}$ and $bij_{(\alpha)}$ membership function of the fuzzy matrices $A_{(\alpha)}$ and $B_{(\alpha)}$ for the i^{th} - row and j^{th} column and

$$aij_{(\alpha)} = \begin{cases} aij & if \quad aij \le \alpha \\ 0 & if \quad aij > \alpha \end{cases}$$

Example 2

$$A = \begin{bmatrix} 0.3 & 0.2 & 0.7 \\ 0.8 & 0.3 & 0.4 \\ 0.1 & 0.9 & 0.6 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 0.4 & 0.5 & 0.6 \\ 0.7 & 0.8 & 0.1 \\ 0.3 & 0.4 & 0.2 \end{bmatrix} \text{ be two fuzzy matrices of order 3.}$$

Take $\alpha = 0.3$

$$A_{(\alpha)} = \begin{bmatrix} 0.3 & 0.2 & 0 \\ 0 & 0.3 & 0 \\ 0.1 & 0 & 0 \end{bmatrix}$$

$$B_{(\alpha)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0.1 \\ 0.3 & 0 & 0.2 \end{bmatrix}$$

Then

$$A_{(\alpha)} + B_{(\alpha)} = \begin{bmatrix} 0.3 & 0.2 & 0 \\ 0 & 0.3 & 0.1 \\ 0.3 & 0 & 0.2 \end{bmatrix}$$

2.4 Multiplication of Fuzzy Square Matrices:

The product of two α- lower level partitions of fuzzy matrices $A_{(\alpha)}$ and $B_{(\alpha)}$ is defined as $A_{(\alpha)}B_{(\alpha)} = \min\{aij_{(\alpha)}, bij_{(\alpha)}\}$

Example 3

Take the matrix A and B of example 2.

$$A_{(\alpha)} = \begin{bmatrix} 0.3 & 0.2 & 0 \\ 0 & 0.3 & 0 \\ 0.1 & 0 & 0 \end{bmatrix}$$

$$B_{(\alpha)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0.1 \\ 0.3 & 0 & 0.2 \end{bmatrix}$$

then

$$A_{(\alpha)}B_{(\alpha)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.1 & 0 & 0 \end{bmatrix}$$

2.5 Complementation of Fuzzy Square matrices:

Complementation of fuzzy square matrix $A_{(\alpha)}$ is defined as

$$A^c_{(\alpha)} = \left(1 - A_{(\alpha)}\right)$$
$$= \left(1 - aij_{(\alpha)}\right)$$

Example 4

Take matrix A of Example 2.

$$A^c_{(\alpha)} = \begin{bmatrix} 0.7 & 0.8 & 1 \\ 1 & 0.7 & 1 \\ 0.9 & 0 & 0 \end{bmatrix}$$

3. Trace of a Fuzzy Square Matrix:

Definition 3.1

Let A be a fuzzy square matrix of order ‘n’. The trace of α-lower partition of fuzzy matrix $A_{(\alpha)}$ denoted by $tr\left(A_{(\alpha)}\right)$ is defined as

$$tr\left(A_{(\alpha)}\right) = \max\left(aii_{(\alpha)}\right)$$

where $aii_{(\alpha)}$ stands for the membership functions lying along the principal diagonal.

Theorem 3.2 :

Let $A_{(\alpha)}$ and $B_{(\alpha)}$ be two α- lower level partitions of fuzzy square matrices each of order ‘n’ and λ be any scalar such that $0 \le \lambda \le 1$

then

(i) $tr\left(A_{(\alpha)} + B_{(\alpha)}\right) = tr\left(A_{(\alpha)}\right) + tr\left(B_{(\alpha)}\right)$

(ii) $tr(\lambda(A_{(\alpha)})) = \lambda\ tr(A_{(\alpha)})$

(iii) $tr(A_{(\alpha)}) = tr(A'_{(\alpha)})$ where $A'_{(\alpha)}$ is the transpose of $A_{(\alpha)}$.

Proof

By the proposed definition,

$$tr(A_{(\alpha)}) = \max(aii_{(\alpha)})$$

$$= \begin{cases} \max aii & if \quad aii \le \alpha \\ 0 & if \quad aii > \alpha \end{cases}$$

$$tr(B_{(\alpha)}) = \max(bii_{(\alpha)})$$

$$tr(A_{(\alpha)} + B_{(\alpha)}) = \max(\max(aii_{(\alpha)} bii_{(\alpha)}))$$

$$= \max(\max(aii_{(\alpha)}), \max(bii_{(\alpha)})).$$

Conversely,

$$trA_{\alpha} + trB_{\alpha} = \max(\max(aii_{(\alpha)}), \max(bii_{(\alpha)}))$$

$$= \max(\max(aii_{(\alpha)}, bii_{(\alpha)}))$$

$$= tr(A_{(\alpha)} + B_{(\alpha)})$$

Hence $tr(A_{(\alpha)} + B_{(\alpha)}) = tr(A_{(\alpha)}) + t_r(B_{(\alpha)})$.

Example 5

Let $A = \begin{bmatrix} 0.3 & 0.7 & 0.8 \\ 0.4 & 0.5 & 0.3 \\ 0.6 & 0.1 & 0.4 \end{bmatrix}$

and

$$B = \begin{bmatrix} 1 & 0.2 & 0.3 \\ 0.8 & 0.5 & 0.2 \\ 0.5 & 1 & 0.8 \end{bmatrix}$$

Take $\alpha = 0.3$

$$A_{(\alpha)} = \begin{bmatrix} 0.3 & 0 & 0 \\ 0 & 0 & 0.3 \\ 0 & 0.1 & 0 \end{bmatrix}$$

$$B_{(\alpha)} = \begin{bmatrix} 0 & 0.2 & 0.3 \\ 0 & 0 & 0.2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$tr(A_{(\alpha)}) = \max(0.3, 0)$$

$$= 0.3$$

$$tr(B_{(\alpha)}) = \max(0, 0)$$

$$= 0.$$

$$tr(A_{(\alpha)}) + tr(B_{(\alpha)}) = \max(0.3, 0)$$

$= 0.3$ .... (1)

$$A_{(\alpha)} + B_{(\alpha)} = \max\{aii_{(\alpha)}, bii_{(\alpha)}\}$$

$$= \begin{bmatrix} 0.3 & 0.2 & 0.3 \\ 0 & 0 & 0.3 \\ 0 & 0.1 & 0 \end{bmatrix}$$

$$\therefore tr\left(A_{(\alpha)} + B_{(\alpha)}\right) = \max(0.3, 0)$$

$$= 0.3 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

From (1) and (2),

$$tr\left(A_{(\alpha)} + B_{(\alpha)}\right) = tr\,A_{(\alpha)} + trB_{(\alpha)}$$

(ii) $tr\left(\lambda\, A_{(\alpha)}\right) = \lambda\, tr\left(A_{(\alpha)}\right)$

$$tr\left(\lambda\, A_{(\alpha)}\right) = \max\left(\lambda\, aii_{(\alpha)}\right)$$

$$= \lambda \max\left(aii_{(\alpha)}\right)$$

$$tr\left(\lambda\, A_{(\alpha)}\right) = \lambda\, tr\left(A_{(\alpha)}\right)$$

Example 6

Let $A = \begin{bmatrix} 0.3 & 0.5 & 0.7 \\ 0.4 & 0.5 & 0.3 \\ 0.6 & 0.1 & 0.4 \end{bmatrix}$

Take $\alpha = 0.3$ and $\lambda = 0.5$

$$A_{(\alpha)} = \begin{bmatrix} 0.3 & 0 & 0 \\ 0 & 0 & 0.3 \\ 0 & 0.1 & 0 \end{bmatrix}$$

$$\lambda\, A_{(\alpha)} = 0.5\, A_{0.3}$$

$$= 0.5 \begin{bmatrix} 0.3 & 0 & 0 \\ 0 & 0 & 0.3 \\ 0 & 0.1 & 0 \end{bmatrix}$$

$$tr\left(\lambda\, A_{(\alpha)}\right) = tr\left(0.5 A_{(0.3)}\right)$$

$$= 0.5\, tr\left(A_{(0.3)}\right)$$

$$= 0.5(\max(0.3, 0))$$

$$= (0.5)(0.3)$$

$$= \min(0.5)(0.3)$$

$$= 0.3$$

(iii) $tr\left(A_{(\alpha)}\right) = tr\left(A'_{(\alpha)}\right)$

proof

$$tr\left(A_{(\alpha)}\right) = \max\left(aii_{(\alpha)}\right)$$

$$\left(A'_{\alpha}\right) = \begin{bmatrix} a_{11_{(\alpha)}} & a_{12_{(\alpha)}} & a_{13_{(\alpha)}} \\ a_{21_{(\alpha)}} & a_{22_{(\alpha)}} & a_{23_{(\alpha)}} \\ a_{31_{(\alpha)}} & a_{32_{(\alpha)}} & a_{33_{(\alpha)}} \end{bmatrix}$$

$$tr\left(A'_{(\alpha)}\right) = \max\left(a_{11_{(\alpha)}}, a_{22_{(\alpha)}}, a_{33_{(\alpha)}}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

$$A'_{(\alpha)} = \begin{bmatrix} a_{11_{(\alpha)}} & a_{21_{(\alpha)}} & a_{31_{(\alpha)}} \\ a_{12_{(\alpha)}} & a_{22_{(\alpha)}} & a_{32_{(\alpha)}} \\ a_{13_{(\alpha)}} & a_{23_{(\alpha)}} & a_{33_{(\alpha)}} \end{bmatrix}$$

$$tr\left(A'_{(\alpha)}\right) = \max\left(a_{11_{(\alpha)}}, a_{22_{(\alpha)}}, a_{33_{(\alpha)}}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

From (1) &(2)

$$tr\left(A_{(\alpha)}\right) = tr\left(A'_{(\alpha)}\right)$$

Example 7

$$A = \begin{bmatrix} 0.3 & 0.5 & 0.6 \\ 0.4 & 0.7 & 0.8 \\ 0.9 & 0.4 & 0.5 \end{bmatrix}$$

Take $\alpha = 0.5$

$$A_{(0.5)} = \begin{bmatrix} 0.3 & 0.5 & 0 \\ 0.4 & 0 & 0 \\ 0 & 0.4 & 0.5 \end{bmatrix}$$

$$tr\left(A_{(0.5)}\right) = \max(0.3, 0, 0.5)$$

$$= 0.5$$

$$A'_{(0.5)} = \begin{bmatrix} 0.3 & 0.4 & 0 \\ 0.5 & 0 & 0.4 \\ 0 & 0 & 0.5 \end{bmatrix}$$

$$tr\left(A'_{(0.5)}\right) = \max(0.3, 0, 0.5)$$

$$= 0.5$$

3.3 Properties of Trace of Matrix

(i) $tr\left(A_{(\alpha)}\right) \neq tr\left(A^C_{(\alpha)}\right)$

Proof

Let $A_{(\alpha)} = \begin{bmatrix} a_{11_{(\alpha)}} & a_{12_{(\alpha)}} & a_{13_{(\alpha)}} \\ a_{21_{(\alpha)}} & a_{22_{(\alpha)}} & a_{23_{(\alpha)}} \\ a_{31_{(\alpha)}} & a_{32_{(\alpha)}} & a_{33_{(\alpha)}} \end{bmatrix}$

$$tr\left(A_{(\alpha)}\right) = \max\left(aii_{(\alpha)}\right)$$

$$A^C_{(\alpha)} = \begin{bmatrix} 1-a_{11_{(\alpha)}} & 1-a_{12_{(\alpha)}} & 1-a_{13_{(\alpha)}} \\ 1-a_{21_{(\alpha)}} & 1-a_{22_{(\alpha)}} & 1-a_{23_{(\alpha)}} \\ 1-a_{31_{(\alpha)}} & 1-a_{32_{(\alpha)}} & 1-a_{33_{(\alpha)}} \end{bmatrix}$$

$$tr\left(A^C_{(\alpha)}\right) = \max\left(1 - aii_{(\alpha)}\right)$$

Hence

$$tr\left(A_{(\alpha)}\right) \neq tr\left(A^C_{(\alpha)}\right)$$

Example 8

$$A = \begin{bmatrix} 0.3 & 0.5 & 0.7 \\ 0.6 & 0.5 & 0.8 \\ 0.4 & 0.1 & 0.6 \end{bmatrix}$$

Take $\alpha = 0.5$

$$A_{(0.5)} = \begin{bmatrix} 0.3 & 0.5 & 0 \\ 0 & 0.5 & 0 \\ 0.4 & 0.1 & 0 \end{bmatrix}$$

$$tr\left(A_{(0.5)}\right) = \max(0.3, 0.5, 0)$$

$$= 0.5 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (1)$$

$$A^C = \begin{bmatrix} 0.7 & 0.5 & 0.3 \\ 0.4 & 0.5 & 0.2 \\ 0.6 & 0.9 & 0.4 \end{bmatrix}$$

$$A^C_{(0.5)} = \begin{bmatrix} 0 & 0.5 & 0.3 \\ 0.4 & 0.5 & 0.2 \\ 0 & 0 & 0.6 \end{bmatrix}$$

$$tr\left(A^C_{(0.5)}\right) = \max(0.5, 0.6, 0)$$

$$= 0.6 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (2)$$

Hence

$$tr\left(A_{(0.5)}\right) \neq tr\left(A^C_{(0.5)}\right)$$

$$tr\left(A_{(\alpha)}\right) \neq tr\left(A^C_{(\alpha)}\right)$$

(ii) $$tr(A+B)^C_{(\alpha)} = tr\left(A^C_{(\alpha)}\right) + tr\left(B^C_{(\alpha)}\right)$$

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$A + B = \max(aii, bii)$$

$$(A+B)^C = (1 - \max(aii, bii))$$

$$(A+B)^C_{(\alpha)} = \left(1 - \max\left(aii_{(\alpha)}, bii_{(\alpha)}\right)\right) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (1)$$

$$A^C_{(\alpha)} = \begin{bmatrix} 1 - a_{11_{(\alpha)}} & 1 - a_{12_{(\alpha)}} & 1 - a_{13_{(\alpha)}} \\ 1 - a_{21_{(\alpha)}} & 1 - a_{22_{(\alpha)}} & 1 - a_{23_{(\alpha)}} \\ 1 - a_{31_{(\alpha)}} & 1 - a_{32_{(\alpha)}} & 1 - a_{33_{(\alpha)}} \end{bmatrix}$$

$$tr\left(A^C_{(\alpha)}\right) = \max\left(1 - aii_{(\alpha)}\right)$$

$$B_{(\alpha)}^{C} = \begin{bmatrix} 1-b_{11_{(\alpha)}} & 1-b_{12_{(\alpha)}} & 1-b_{13_{(\alpha)}} \\ 1-b_{21_{(\alpha)}} & 1-b_{22_{(\alpha)}} & 1-b_{23_{(\alpha)}} \\ 1-b_{31_{(\alpha)}} & 1-b_{32_{(\alpha)}} & 1-b_{33_{(\alpha)}} \end{bmatrix}$$

$$tr\left(B_{(\alpha)}^{C}\right) = \max\left(1-bii_{(\alpha)}\right)$$

$$tr\left(A_{(\alpha)}^{C}\right)+tr\left(B_{(\alpha)}^{C}\right) = \max\left(\left(\max\left(1-aii_{(\alpha)}\right)\right),\max\left(1-bii_{(\alpha)}\right)\right)$$

$$= \max\left(\max\left(\left(1-aii_{(\alpha)}\right),\left(1-bii_{(\alpha)}\right)\right)\right)$$

$$= \max\left(1-\max\left(aii_{(\alpha)},bii_{(\alpha)}\right)\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

From (1) and (2)

$$tr\left(A_{(\alpha)}+B_{(\alpha)}\right)^{C} = tr\left(A_{(\alpha)}^{C}\right)+tr\left(B_{(\alpha)}^{C}\right)$$

Example 9

Let $A = \begin{bmatrix} 0.3 & 0.5 & 0.6 \\ 0.4 & 0.7 & 0.8 \\ 0.9 & 0.4 & 0.5 \end{bmatrix}$

$$B = \begin{bmatrix} 0.1 & 0.3 & 0.5 \\ 0.2 & 0.6 & 0.4 \\ 0.7 & 0.8 & 0.3 \end{bmatrix}$$

$$(A+B) = \max(aij, bij)$$

$$= \begin{bmatrix} 0.3 & 0.5 & 0.6 \\ 0.4 & 0.7 & 0.8 \\ 0.9 & 0.8 & 0.5 \end{bmatrix}$$

$$(A+B)^{C} = \begin{bmatrix} 0.7 & 0.5 & 0.4 \\ 0.6 & 0.3 & 0.2 \\ 0.1 & 0.2 & 0.5 \end{bmatrix}$$

Take $\alpha = 0.3$

$$(A+B)_{(\alpha)}^{C} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.3 & 0.2 \\ 0.1 & 0.2 & 0 \end{bmatrix}$$

$$tr\left((A+B)_{(\alpha)}^{C}\right) = \max(0,0.3)$$

$$= 0.3$$

$$A^{C} = \begin{bmatrix} 0.7 & 0.5 & 0.4 \\ 0.6 & 0.3 & 0.2 \\ 0.1 & 0.6 & 0.5 \end{bmatrix}$$

$$A_{(\alpha)}^{C} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.3 & 0.2 \\ 0.1 & 0 & 0 \end{bmatrix}$$

$$tr\left(A^C_{(\alpha)}\right) = \max(0.3, 0)$$
$$= 0.3$$
$$B^C = \begin{bmatrix} 0.9 & 0.7 & 0.5 \\ 0.8 & 0.4 & 0.6 \\ 0.3 & 0.2 & 0.3 \end{bmatrix}$$
$$B^C_{(\alpha)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.3 & 0.2 & 0.3 \end{bmatrix}$$
$$tr\left(B^C_{(\alpha)}\right) = \max(0, 0.3)$$
$$= 0.3$$
$$tr\left(A^C_{(\alpha)}\right) + tr\left(B^C_{(\alpha)}\right) = \max(0.3, 0.3)$$
$$= 0.3$$

Hence

$$tr\left((A+B)^C_{(\alpha)}\right) = tr\left(A^C_{(\alpha)}\right) + tr\left(B^C_{(\alpha)}\right)$$

(iii) $tr\left(A^C_{(\alpha)} + B^C_{(\alpha)}\right) = tr\left(A^C_{(\alpha)}\right) + tr\left(B^C_{(\alpha)}\right)$

Proof

Let

$$A_{(\alpha)} = \begin{bmatrix} a_{11_{(\alpha)}} & a_{12_{(\alpha)}} & a_{13_{(\alpha)}} \\ a_{21_{(\alpha)}} & a_{22_{(\alpha)}} & a_{23_{(\alpha)}} \\ a_{31_{(\alpha)}} & a_{32_{(\alpha)}} & a_{33_{(\alpha)}} \end{bmatrix} \qquad B_{(\alpha)} = \begin{bmatrix} b_{11_{(\alpha)}} & b_{12_{(\alpha)}} & b_{13_{(\alpha)}} \\ b_{21_{(\alpha)}} & b_{22_{(\alpha)}} & b_{23_{(\alpha)}} \\ b_{31_{(\alpha)}} & b_{32_{(\alpha)}} & b_{33_{(\alpha)}} \end{bmatrix}$$

$$A^C_{(\alpha)} = \begin{bmatrix} 1-a_{11_{(\alpha)}} & 1-a_{12_{(\alpha)}} & 1-a_{13_{(\alpha)}} \\ 1-a_{21_{(\alpha)}} & 1-a_{22_{(\alpha)}} & 1-a_{23_{(\alpha)}} \\ 1-a_{31_{(\alpha)}} & 1-a_{32_{(\alpha)}} & 1-a_{33_{(\alpha)}} \end{bmatrix}$$
$$B^C_{(\alpha)} = \begin{bmatrix} 1-b_{11_{(\alpha)}} & 1-b_{12_{(\alpha)}} & 1-b_{13_{(\alpha)}} \\ 1-b_{21_{(\alpha)}} & 1-b_{22_{(\alpha)}} & 1-b_{23_{(\alpha)}} \\ 1-b_{31_{(\alpha)}} & 1-b_{32_{(\alpha)}} & 1-b_{33_{(\alpha)}} \end{bmatrix}$$
$$A^C_{(\alpha)} + B^C_{(\alpha)} = \max\left(1-aij_{(\alpha)}, 1-bij_{(\alpha)}\right)$$
$$tr\left(A^C_{(\alpha)} + B^C_{(\alpha)}\right) = \max\left(\max\left(\left(1-aii_{(\alpha)}\right), \left(1-bii_{(\alpha)}\right)\right)\right) \quad (1)$$
$$tr\left(A^C_{(\alpha)}\right) = \max\left(1-aii_{(\alpha)}\right)$$
$$tr\left(B^C_{(\alpha)}\right) = \max\left(1-bii_{(\alpha)}\right)$$
$$tr\left(A^C_{(\alpha)}\right) + tr\left(B^C_{(\alpha)}\right) = \max\left(\max\left(1-aii_{(\alpha)}\right), \max\left(1-bii_{(\alpha)}\right)\right)$$
$$= \max\left(\max\left(\left(1-aii_{(\alpha)}\right), \left(1-bii_{(\alpha)}\right)\right)\right) \quad (2)$$
$$tr\left(A^C_{(\alpha)} + B^C_{(\alpha)}\right) = tr\left(A^C_{(\alpha)}\right) + tr\left(B^C_{(\alpha)}\right)$$

Example 10

$$A = \begin{bmatrix} 0.7 & 0.5 & 0.6 \\ 0.4 & 0.8 & 0.7 \\ 0.9 & 0.8 & 0.6 \end{bmatrix} \qquad B = \begin{bmatrix} 0.5 & 0.6 & 0.5 \\ 0.3 & 0.9 & 0.7 \\ 0.8 & 0.3 & 0.8 \end{bmatrix}$$

$$A^C = \begin{bmatrix} 0.3 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.3 \\ 0.1 & 0.2 & 0.4 \end{bmatrix} \qquad B^C = \begin{bmatrix} 0.5 & 0.4 & 0.5 \\ 0.7 & 0.1 & 0.3 \\ 0.2 & 0.7 & 0.2 \end{bmatrix}$$

Take $\alpha = 0.5$

$$A^C_{(\alpha)} = \begin{bmatrix} 0.3 & 0.5 & 0.4 \\ 0 & 0.2 & 0.3 \\ 0.1 & 0.2 & 0.4 \end{bmatrix} \qquad B^C_{(\alpha)} = \begin{bmatrix} 0.5 & 0.4 & 0.5 \\ 0 & 0.1 & 0.3 \\ 0.2 & 0 & 0.2 \end{bmatrix}$$

$$A^C_{(\alpha)} + B^C_{(\alpha)} = \max\left(1 - aij_{(\alpha)}, 1 - bij_{(\alpha)}\right)$$

$$\therefore tr\left(A^C_{(\alpha)} + B^C_{(\alpha)}\right) = \max\left(\max\left(\left(1 - aii_{(\alpha)}\right), \left(1 - bii_{(\alpha)}\right)\right)\right)$$

$$= 0.5 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \; (1)$$

$$tr\left(A^C_{(\alpha)}\right) = \max\left(1 - aii_{(\alpha)}\right)$$

$$= \max(0.3, 0.2, 0.4)$$

$$= 0.4$$

$$tr\left(B^C_{(\alpha)}\right) = \max\left(1 - bii_{(\alpha)}\right)$$

$$= \max(0.5, 0.1, 0.2)$$

$$= 0.5$$

$$\therefore trA^C_{(\alpha)} + trB^C_{(\alpha)} = \max(0.4, 0.5)$$

$$= 0.5 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

From (1) and (2)

$$tr\left(\left(A^C_{(\alpha)}\right) + \left(B^C_{(\alpha)}\right)\right) = tr\left(A^C_{(\alpha)}\right) + tr\left(B^C_{(\alpha)}\right)$$

(iv) $tr\left(A_{(\alpha)} B_{(\alpha)}\right) \neq tr\left(A_{(\alpha)}\right) \cdot tr\left(B_{(\alpha)}\right)$

Proof

Let

$$A_{(\alpha)} = \begin{bmatrix} a_{11_{(\alpha)}} & a_{12_{(\alpha)}} & a_{13_{(\alpha)}} \\ a_{21_{(\alpha)}} & a_{22_{(\alpha)}} & a_{23_{(\alpha)}} \\ a_{31_{(\alpha)}} & a_{32_{(\alpha)}} & a_{33_{(\alpha)}} \end{bmatrix}$$

$$B_{(\alpha)} = \begin{bmatrix} b_{11_{(\alpha)}} & b_{12_{(\alpha)}} & b_{13_{(\alpha)}} \\ b_{21_{(\alpha)}} & b_{22_{(\alpha)}} & b_{23_{(\alpha)}} \\ b_{31_{(\alpha)}} & b_{32_{(\alpha)}} & b_{33_{(\alpha)}} \end{bmatrix}$$

$$A_{(\alpha)} B_{(\alpha)} = \min\left(aij_{(\alpha)}, bij_{(\alpha)}\right)$$

$$tr\left(A_{(\alpha)} B_{(\alpha)}\right) = \max\left(\min aii_{(\alpha)}, bii_{(\alpha)}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$trA_{(\alpha)} = \max\left(aii_{(\alpha)}\right)$$

$$trB_{(\alpha)} = \max\left(bii_{(\alpha)}\right)$$

$$tr(A_{(\alpha)})\cdot tr(B_{(\alpha)}) = \max(\max(aii_{(\alpha)}), \max(bii_{(\alpha)}))$$

$$= \max(\max(aii_{(\alpha)}), \max(bii_{(\alpha)})) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

From (1) and (2)

$$tr(A_{(\alpha)}\cdot B_{(\alpha)}) \neq trA_{(\alpha)}\cdot trB_{(\alpha)}$$

Example 11

$$A = \begin{bmatrix} 0.3 & 0.5 & 0.6 \\ 0.4 & 0.2 & 0.7 \\ 0.9 & 0.8 & 0.1 \end{bmatrix} \qquad B = \begin{bmatrix} 0.2 & 0.6 & 0.5 \\ 0.3 & 0.1 & 0.7 \\ 0.8 & 0.3 & 0.4 \end{bmatrix}$$

Take $\alpha = 0.5$

$$A_{(\alpha)} = \begin{bmatrix} 0.3 & 0.5 & 0 \\ 0.4 & 0.2 & 0 \\ 0 & 0 & 0.1 \end{bmatrix} \qquad B_{(\alpha)} = \begin{bmatrix} 0.2 & 0 & 0.5 \\ 0.3 & 0.1 & 0 \\ 0 & 0.3 & 0.4 \end{bmatrix}$$

$$A_{(\alpha)}B_{(\alpha)} = \min(aii_{(\alpha)}\, bii_{(\alpha)})$$

$$= \begin{bmatrix} 0.2 & 0 & 0 \\ 0.3 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}$$

$$tr(A_{(\alpha)}B_{(\alpha)}) = \max(0.2, 0.1, 0.1)$$

$$= 0.2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$tr(A_{(\alpha)}) = \max(0.3, 0.2, 0.4)$$

$$= 0.3$$

$$tr(B_{(\alpha)}) = \max(0.2, 0.1, 0.4)$$

$$= 0.4$$

$$tr(A_{(\alpha)})tr(B_{(\alpha)}) = \min(0.3, 0.4)$$

$$= 0.3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Hence

$$tr(A_{(\alpha)}\cdot B_{(\alpha)}) = tr(A_{(\alpha)})\cdot tr(B_{(\alpha)})$$

(v) Commutative Property:

$$tr(A_{(\alpha)}B_{(\alpha)}) = tr(B_{(\alpha)}A_{(\alpha)})$$

Proof

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$A_{(\alpha)} = \begin{bmatrix} a_{11_{(\alpha)}} & a_{12_{(\alpha)}} & a_{13_{(\alpha)}} \\ a_{21_{(\alpha)}} & a_{22_{(\alpha)}} & a_{23_{(\alpha)}} \\ a_{31_{(\alpha)}} & a_{32_{(\alpha)}} & a_{33_{(\alpha)}} \end{bmatrix} \qquad B_{(\alpha)} = \begin{bmatrix} b_{11_{(\alpha)}} & b_{12_{(\alpha)}} & b_{13_{(\alpha)}} \\ b_{21_{(\alpha)}} & b_{22_{(\alpha)}} & b_{23_{(\alpha)}} \\ b_{31_{(\alpha)}} & b_{32_{(\alpha)}} & b_{33_{(\alpha)}} \end{bmatrix}$$

$$A_{(\alpha)}B_{(\alpha)} = \min(aij_{(\alpha)}, bij_{(\alpha)})$$

$$tr\left(A_{(\alpha)}B_{(\alpha)}\right) = \max\left(\min aii_{(\alpha)}, bii_{(\alpha)}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$B_{(\alpha)}A_{(\alpha)} = \min\left(bij_{(\alpha)}\ aij_{(\alpha)}\right)$$

$$tr\left(B_{(\alpha)}A_{(\alpha)}\right) = \max\left(\min\left(bii_{(\alpha)}, aii_{(\alpha)}\right)\right)$$

$$= \max\left(\min\left(aii_{(\alpha)}, bii_{(\alpha)}\right)\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

From (1) and (2)

$$tr\left(A_{(\alpha)}B_{(\alpha)}\right) = tr\left(B_{(\alpha)}A_{(\alpha)}\right)$$

Example 12

Let $A = \begin{bmatrix} 0.5 & 0.3 & 0.6 \\ 0.7 & 08 & 0.4 \\ 0.3 & 0.5 & 0.9 \end{bmatrix}$ and $B = \begin{bmatrix} 0.4 & 0.3 & 0.6 \\ 0.8 & 0.5 & 0.1 \\ 0.7 & 0.4 & 0.2 \end{bmatrix}$

Take $\alpha = 0.5$

$$A_{(\alpha)} = \begin{bmatrix} 0.5 & 0.3 & 0 \\ 0 & 0 & 0.4 \\ 0.3 & 0.5 & 0 \end{bmatrix} \qquad B_{(\alpha)} = \begin{bmatrix} 0.4 & 0.3 & 0 \\ 0 & 0.5 & 0.1 \\ 0 & 0.4 & 0.2 \end{bmatrix}$$

$$A_{(\alpha)}\ B_{(\alpha)} = \min\left(aij_{(\alpha)}\ bij_{(\alpha)}\right)$$

$$= \begin{bmatrix} 0.4 & 0.3 & 0 \\ 0 & 0 & 0.1 \\ 0 & 0.4 & 0 \end{bmatrix}$$

$$tr\left(A_{(\alpha)}B_{(\alpha)}\right) = \max(0.4,0)$$

$$= 0.4$$

$$tr\left(B_{(\alpha)}A_{(\alpha)}\right) = \max(0,0.4)$$

$$= 0.4$$

$$\therefore tr\left(A_{(\alpha)}B_{(\alpha)}\right) = tr\left(B_{(\alpha)}A_{(\alpha)}\right)$$

(vi) $tr\left(A^C_{(\alpha)}B^C_{(\alpha)}\right) = tr\left(B^C_{(\alpha)}A^C_{(\alpha)}\right)$

Proof

Let $A_{(\alpha)} = \begin{bmatrix} a_{11_{(\alpha)}} & a_{12_{(\alpha)}} & a_{13_{(\alpha)}} \\ a_{21_{(\alpha)}} & a_{22_{(\alpha)}} & a_{23_{(\alpha)}} \\ a_{31_{(\alpha)}} & a_{32_{(\alpha)}} & a_{33_{(\alpha)}} \end{bmatrix}$ $B_{(\alpha)} = \begin{bmatrix} b_{11_{(\alpha)}} & b_{12_{(\alpha)}} & b_{13_{(\alpha)}} \\ b_{21_{(\alpha)}} & b_{22_{(\alpha)}} & b_{23_{(\alpha)}} \\ b_{31_{(\alpha)}} & b_{32_{(\alpha)}} & b_{33_{(\alpha)}} \end{bmatrix}$

$$A^C_{(\alpha)} = \begin{bmatrix} 1-a_{11_{(\alpha)}} & 1-a_{12_{(\alpha)}} & 1-a_{13_{(\alpha)}} \\ 1-a_{21_{(\alpha)}} & 1-a_{22_{(\alpha)}} & 1-a_{23_{(\alpha)}} \\ 1-a_{31_{(\alpha)}} & 1-a_{32_{(\alpha)}} & 1-a_{33_{(\alpha)}} \end{bmatrix}$$

$$B^C_{(\alpha)} = \begin{bmatrix} 1-b_{11_{(\alpha)}} & 1-b_{12_{(\alpha)}} & 1-b_{13_{(\alpha)}} \\ 1-b_{21_{(\alpha)}} & 1-b_{22_{(\alpha)}} & 1-b_{23_{(\alpha)}} \\ 1-b_{31_{(\alpha)}} & 1-b_{32_{(\alpha)}} & 1-b_{33_{(\alpha)}} \end{bmatrix}$$

$$A^C_{(\alpha)}\ B^C_{(\alpha)} = \min\left(\left(1-aij_{(\alpha)}\right),\left(1-bij_{(\alpha)}\right)\right)$$

$$tr\left(A^C_{(\alpha)}\ B^C_{(\alpha)}\right) = \max\left(\min\left(\left(1-aii_{(\alpha)}\right),\left(1-bii_{(\alpha)}\right)\right)\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (1)$$

$$B^C_{(\alpha)}\ A^C_{(\alpha)} = \min\left(\left(1-bij_{(\alpha)}\right),\left(1-aij_{(\alpha)}\right)\right)$$

$$tr\left(B^C_{(\alpha)}\ A^C_{(\alpha)}\right) = \max\left(\min\left(\left(1-bii_{(\alpha)}\right),\left(1-aii_{(\alpha)}\right)\right)\right)$$

$$= \max\left(\min\left(\left(1-aii_{(\alpha)}\right),\left(1-bii_{(\alpha)}\right)\right)\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (2)$$

From (1) and (2)

$$tr\left(A^C_{(\alpha)} B^C_{(\alpha)}\right) = tr\left(B^C_{(\alpha)} A^C_{(\alpha)}\right)$$

Example13

$$A = \begin{bmatrix} 0.5 & 0.6 & 0.7 \\ 0.8 & 0.9 & 0.3 \\ 0.6 & 0.8 & 0.2 \end{bmatrix} \qquad B = \begin{bmatrix} 0.8 & 0.9 & 0.3 \\ 0.6 & 0.7 & 0.2 \\ 0.5 & 0.8 & 0.1 \end{bmatrix}$$

$$A^C = \begin{bmatrix} 0.5 & 0.4 & 0.3 \\ 0.2 & 0.1 & 0.7 \\ 0.4 & 0.2 & 0.8 \end{bmatrix} \qquad B^C = \begin{bmatrix} 0.2 & 0.1 & 0.7 \\ 0.4 & 0.3 & 0.8 \\ 0.5 & 0.2 & 0.9 \end{bmatrix}$$

Take $\alpha = 0.5$

$$A^C_{(\alpha)} = \begin{bmatrix} 0.5 & 0.4 & 0 \\ 0.2 & 0.1 & 0 \\ 0.4 & 0.2 & 0 \end{bmatrix} \quad B^C_{(\alpha)} = \begin{bmatrix} 0.2 & 0.1 & 0 \\ 0.4 & 0.1 & 0 \\ 0.5 & 0.2 & 0 \end{bmatrix}$$

$$\left(A^C_{(\alpha)} B^C_{(\alpha)}\right) = \min\left(\left(1-aij_{(\alpha)}\right)\left(1-bij_{(\alpha)}\right)\right)$$

$$= \begin{bmatrix} 0.2 & 0.1 & 0 \\ 0.2 & 0.1 & 0 \\ 0.4 & 0.2 & 0 \end{bmatrix}$$

$$tr\left(A^C_{(\alpha)} B^C_{(\alpha)}\right) = \max\left(\min\left(\left(1-aii_{(\alpha)}\right)\left(1-bii_{(\alpha)}\right)\right)\right)$$

$$= \max(0.2, 0.1, 0)$$

$$= 0.2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$B^C_{(\alpha)} A^C_{(\alpha)} = \begin{bmatrix} 0.2 & 0.1 & 0 \\ 0.2 & 0.1 & 0 \\ 0.4 & 0.2 & 0 \end{bmatrix}$$

$$tr\left(B^C_{(\alpha)} A^C_{(\alpha)}\right) = \max(0.2, 0.1, 0)$$

$$= 0.2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

From (1) and (2)

$$tr\left(A^C_{(\alpha)} B^C_{(\alpha)}\right) = tr\left(B^C_{(\alpha)} A^C_{(\alpha)}\right)$$

(vii) Distributive Property

$$tr\left(A^C_{(\alpha)}\left(B^C_{(\alpha)} + C^C_{(\alpha)}\right)\right) = tr\left(A^C_{(\alpha)} B^C_{(\alpha)}\right) + tr\left(A^C_{(\alpha)} C^C_{(\alpha)}\right)$$

Proof

$$A_{(\alpha)} = \begin{bmatrix} a_{11_{(\alpha)}} & a_{12_{(\alpha)}} & a_{13_{(\alpha)}} \\ a_{21_{(\alpha)}} & a_{22_{(\alpha)}} & a_{23_{(\alpha)}} \\ a_{31_{(\alpha)}} & a_{32_{(\alpha)}} & a_{33_{(\alpha)}} \end{bmatrix} \qquad B_{(\alpha)} = \begin{bmatrix} b_{11_{(\alpha)}} & b_{12_{(\alpha)}} & b_{13_{(\alpha)}} \\ b_{21_{(\alpha)}} & b_{22_{(\alpha)}} & b_{23_{(\alpha)}} \\ b_{31_{(\alpha)}} & b_{32_{(\alpha)}} & b_{33_{(\alpha)}} \end{bmatrix}$$

$$C_{(\alpha)} = \begin{bmatrix} c_{11_{(\alpha)}} & c_{12_{(\alpha)}} & c_{13_{(\alpha)}} \\ c_{21_{(\alpha)}} & c_{22_{(\alpha)}} & c_{23_{(\alpha)}} \\ c_{31_{(\alpha)}} & c_{32_{(\alpha)}} & c_{33_{(\alpha)}} \end{bmatrix}$$

$$A^C_{(\alpha)} = \begin{bmatrix} 1-a_{11_\alpha} & 1-a_{12_\alpha} & 1-a_{13_\alpha} \\ 1-a_{21_\alpha} & 1-a_{22_\alpha} & 1-a_{23_\alpha} \\ 1-a_{31_\alpha} & 1-a_{32_\alpha} & 1-a_{33_\alpha} \end{bmatrix}$$

$$B^C_{(\alpha)} = \begin{bmatrix} 1-b_{11_{(\alpha)}} & 1-b_{12_{(\alpha)}} & 1-b_{13_{(\alpha)}} \\ 1-b_{21_{(\alpha)}} & 1-b_{22_{(\alpha)}} & 1-b_{23_{(\alpha)}} \\ 1-b_{31_{(\alpha)}} & 1-b_{32_{(\alpha)}} & 1-b_{33_{(\alpha)}} \end{bmatrix}$$

$$C^C_{(\alpha)} = \begin{bmatrix} 1-c_{11_{(\alpha)}} & 1-c_{12_{(\alpha)}} & 1-c_{13_{(\alpha)}} \\ 1-c_{21_{(\alpha)}} & 1-c_{22_{(\alpha)}} & 1-c_{23_{(\alpha)}} \\ 1-c_{31_{(\alpha)}} & 1-c_{32_{(\alpha)}} & 1-c_{33_{(\alpha)}} \end{bmatrix}$$

$$\left(B^C_{(\alpha)} + C^C_{(\alpha)}\right) = \max\left(\left(1-bij_{(\alpha)}\right),\left(1-cij_{(\alpha)}\right)\right)$$

$$A^C_{(\alpha)}\left(B^C_{(\alpha)} + C^C_{(\alpha)}\right) = \min\left(\max\left(\left(1-bij_{(\alpha)}\right),\left(1-cij_{(\alpha)}\right)\right),\left(1-aij_{(\alpha)}\right)\right)$$

$$= \min\left(\max\left(\left(1-bij_{(\alpha)}\right),\left(1-aij_{(\alpha)}\right)\right),\max\left(\left(1-cij_{(\alpha)}\right),\left(1-aij_{(\alpha)}\right)\right)\right)$$

$$tr\left(A^C_{(\alpha)}\left(B^C_{(\alpha)} + C^C_{(\alpha)}\right)\right) = \max\left(\min\left(\max\left(\left(1-bii_{(\alpha)}\right),\left(1-aii_{(\alpha)}\right)\right),\max\left(1-cii_{(\alpha)}\right),\left(1-aii_{(\alpha)}\right)\right)\right) \quad (1)$$

$$tr\left(A^C_{(\alpha)}B^C_{(\alpha)}\right) = \max\left(\min\left(1-aii_{(\alpha)}\right)\left(1-bii_{(\alpha)}\right)\right)$$

$$tr\left(A^C_{(\alpha)}C^C_{(\alpha)}\right) = \max\left(\min\left(1-aii_{(\alpha)}\right)\left(1-cii_{(\alpha)}\right)\right)$$

$$\therefore tr\left(A^C_{(\alpha)}B^C_{(\alpha)}\right) + tr\left(A^C_{(\alpha)}C^C_{(\alpha)}\right) = \max\left(\min\left(\min\left(1-aii_{(\alpha)}\right)\left(1-bii_{(\alpha)}\right)\right),\max\left(\min\left(1-aii_{(\alpha)}\right)\left(1-cii_{(\alpha)}\right)\right)\right) \quad \mathbf{(2)}$$

From (1) and (2)

$$tr\left(A^C_{(\alpha)}\left(B^C_{(\alpha)} + C^C_{(\alpha)}\right)\right) = tr\left(A^C_{(\alpha)}\,B^C_{(\alpha)}\right) + tr\left(A^C_{(\alpha)}\,C^C_{(\alpha)}\right)$$

Example 14

Let

$$A = \begin{bmatrix} 0.5 & 0.6 & 0.7 \\ 0.8 & 0.9 & 0.3 \\ 0.6 & 0.8 & 0.2 \end{bmatrix} \qquad B = \begin{bmatrix} 0.8 & 0.9 & 0.3 \\ 0.6 & 0.7 & 0.2 \\ 0.5 & 0.8 & 0.1 \end{bmatrix} \qquad C = \begin{bmatrix} 0.4 & 0.6 & 0.8 \\ 0.3 & 0.2 & 0.7 \\ 0.8 & 0.1 & 0.5 \end{bmatrix}$$

$$A^C = \begin{bmatrix} 0.5 & 0.4 & 0.3 \\ 0.2 & 0.1 & 0.7 \\ 0.4 & 0.2 & 0.8 \end{bmatrix} B^C = \begin{bmatrix} 0.2 & 0.1 & 0.7 \\ 0.4 & 0.3 & 0.8 \\ 0.5 & 0.2 & 0.9 \end{bmatrix} C^C = \begin{bmatrix} 0.6 & 0.4 & 0.2 \\ 0.7 & 0.8 & 0.3 \\ 0.2 & 0.9 & 0.5 \end{bmatrix}$$

Take $\alpha = 0.5$

$$A^C_{(\alpha)} = \begin{bmatrix} 0.5 & 0.4 & 0.3 \\ 0.2 & 0.1 & 0 \\ 0.4 & 0.2 & 0 \end{bmatrix} B^C_{(\alpha)} = \begin{bmatrix} 0.2 & 0.1 & 0 \\ 0.4 & 0.3 & 0 \\ 0.5 & 0.2 & 0 \end{bmatrix} C^C_{(\alpha)} = \begin{bmatrix} 0 & 0.4 & 0.2 \\ 0 & 0 & 0.3 \\ 0.2 & 0 & 0.5 \end{bmatrix}$$

$$\left(B^C_{(\alpha)} + C^C_{(\alpha)}\right) = \begin{bmatrix} 0.2 & 0.4 & 0.2 \\ 0.4 & 0.3 & 0.3 \\ 0.5 & 0.2 & 0.5 \end{bmatrix}$$

$$A^C_{(\alpha)}\left(B^C_{(\alpha)} + C^C_{(\alpha)}\right) = \begin{bmatrix} 0.2 & 0.4 & 0.2 \\ 0.2 & 0.1 & 0 \\ 0.4 & 0.2 & 0 \end{bmatrix}$$

$$= 0.2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$\left(A^C_{(\alpha)} B^C_{(\alpha)}\right) = \begin{bmatrix} 0.2 & 0.1 & 0 \\ 0.2 & 0.1 & 0 \\ 0.4 & 0.2 & 0 \end{bmatrix}$$

$$tr\left(A^C_{(\alpha)} B^C_{(\alpha)}\right) = \max(0.2, 0.1, 0)$$

$$= 0.2$$

$$\left(A^C_{(\alpha)} C^C_{(\alpha)}\right) = \begin{bmatrix} 0 & 0.4 & 0.2 \\ 0 & 0 & 0 \\ 0.2 & 0 & 0 \end{bmatrix}$$

$$tr\left(A^C_{(\alpha)} C^C_{(\alpha)}\right) = \max(0)$$

$$= 0$$

$$tr\left(A^C_{(\alpha)} + B^C_{(\alpha)}\right) + tr\left(A^C_{(\alpha)} C^C_{(\alpha)}\right) = \max(0.2, 0)$$

$$= 0.2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

From (1) and (2)

$$tr\left(A^C_{(\alpha)}\left(B^C_{(\alpha)} + C^C_{(\alpha)}\right)\right) = tr\left(A^C_{(\alpha)} B^C_{(\alpha)}\right) + tr\left(A^C_{(\alpha)} C^C_{(\alpha)}\right)$$

Theorem 3.4:

Let A be a fuzzy matrix and M_α be the set of al lower level partitions of A for each $\alpha \in [0,1]$. Then the collection of all traces of matrices in M_n forms a partial ordered set under the relation $\leq$ defined by,

$tr\left(A_{(\alpha_1)}\right) \leq tr\left(A_{(\alpha_2)}\right)$ if and only if $aii_{(\alpha_1)} \leq aii_{(\alpha_2)}$ for some i.

Proof

(i) $tr\left(A_{(\alpha_1)}\right) \leq tr\left(A_{(\alpha_2)}\right)$ for all each $\alpha \in [0,1]$ if and only if,

$aii_{(\alpha_1)} \leq aii_{(\alpha_2)}$ for every i.

Hence reflexivity is true.

(ii) If $tr\left(A_{(\alpha_1)}\right) \leq tr\left(A_{(\alpha_2)}\right)$ then $aii_{(\alpha_1)} \leq aii_{(\alpha_2)}$

If $tr\left(A_{(\alpha_2)}\right) \leq tr\left(A_{(\alpha_1)}\right)$ then $aii_{(\alpha_2)} \leq aii_{(\alpha_1)}$

Hence $aii_{(\alpha_1)} = aii_{(\alpha_2)}$

$\Rightarrow tr\left(A_{(\alpha_1)}\right) = tr\left(A_{(\alpha_2)}\right)$

Anti symmetry is not true.

(iii) If $tr\left(A_{(\alpha_1)}\right) \leq tr\left(A_{(\alpha_2)}\right)$ where $A_{(\alpha_1)}, A_{(\alpha_2)} \in M_n$ implies $aii_{(\alpha_1)} \leq aii_{(\alpha_2)}$

$$tr\left(A_{(\alpha_2)}\right) \le tr\left(A_{(\alpha_3)}\right) \text{ implies } aii_{(\alpha_2)} \le aii_{(\alpha_3)}$$

Hence

$$aii_{(\alpha_1)} \le aii_{(\alpha_2)} \text{ and } aii_{(\alpha_2)} \le aii_{(\alpha_3)} \text{ implies } aii_{(\alpha_1)} \le aii_{(\alpha_3)}$$

$$\therefore tr\left(A_{(\alpha_2)}\right) \le tr\left(A_{(\alpha_3)}\right)$$

Hence transitivity is true.

Thus the collection of all traces of matrices in M_n forms a partial ordered set.

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