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Algebraic Structure of Nilpotent and Idempotent Matrices

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ABSTRACT

Group theory has significantly advanced the study of algebraic structures by examining examples that satisfy group axioms and by introducing modifications to define new types of groups. This research focuses on the algebraic structures formed by sets of nilpotent matrices and idempotent matrices. The first part investigates the set of nilpotent matrices under matrix multiplication, denoted as $(N,\cdot)$, analyzing its compliance with group properties. The second part explores the algebraic structure of idempotent matrices with multiplication as the operation, represented as $(A,\cdot)$. Through rigorous examination of closure, associativity, identity elements, and inverses within these sets, this study reveals that while $(N,\cdot)$ forms a semigroup due to lack of identity and inverses, $(A,\cdot)$ constitutes a monoid but not necessarily a group because some elements lack inverses. Additionally, intersections and unions between these sets are discussed to highlight their structural properties. These findings contribute to a deeper understanding of matrix-based algebraic systems and provide groundwork for further exploration in abstract algebra.

KEYWORDS: Semigroup, Monoid, Group, Matrix, Nilpotent, Idempotent, Abelian

I. INTRODUCTION

Algebraic structures play a fundamental role in modern mathematics, providing a framework to study sets equipped with operations that satisfy specific axioms [1], [2]. Among these structures, groups, semigroups, and monoids have been extensively studied due to their wide-ranging applications in various fields such as physics, computer science, and engineering [3], [4]. In particular, matrix theory offers rich examples of algebraic structures where the operation is typically matrix multiplication [5], [6].

Nilpotent and idempotent matrices are two important classes of matrices with distinctive algebraic properties [7], [8]. A nilpotent matrix is one that becomes the zero matrix when raised to some positive integer power, while an idempotent matrix remains unchanged when squared [9]. These special types of matrices not only appear naturally in linear algebra but also have significant implications in ring theory and module theory [10], [11].

This paper aims to explore the algebraic structures formed by sets of nilpotent and idempotent matrices under the operation of matrix multiplication. We investigate three main aspects: first, the structure of nilpotent matrices as an algebraic system; second, the structure formed by idempotent matrices; and third, the combined structure arising from their union. Additionally, we seek examples where these sets exhibit group properties—particularly focusing on cases where they form Abelian groups.

By understanding these structures more deeply, this research contributes to expanding knowledge on how classical concepts like nilpotency and idempotency interact within group theory frameworks and opens pathways for further exploration into new types of algebraic systems derived from matrix operations.

II. THEORETICAL REVIEW

The discussion of nilpotent and idempotent matrix groups naturally involves the binary operation "$\cdot$", which in this context refers to matrix multiplication. Specifically, for nilpotent and idempotent matrices, the focus is exclusively on square matrices of size $n \times n$.

A. *Definition of Nilpotent Matrices*

Definition 2.1 In linear algebra, a nilpotent matrix is defined as a square matrix N such that

$$N^k = \mathbf{0}$$

for some positive integer k. The smallest such integer k is called the index or degree of the nilpotent matrix N [12], [13]. Examples of Nilpotent Matrices:

1. The matrix $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is nilpotent with index 2, since $A^2 = 0$.
2. More generally, any $n \times n$ strictly upper (or lower) triangular matrix with zeros along its main diagonal is nilpotent with an index less than or equal to n. For example, consider the matrix

$$B = \begin{bmatrix} 0 & 2 & 1 & 6 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(where the full dimension and entries are specified accordingly)

B is a nilpotent matrix because its powers satisfy:

$$B^2 = \begin{bmatrix} 0 & 0 & 2 & 7 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B^3 = \begin{bmatrix} 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B^4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and thus the index of B is equal to 4.

Nilpotent matrices play a significant role in various fields of mathematics and their applications, such as spectral theory, Jordan decomposition, and the analysis of linear dynamical systems [14], [15].

Due to their unique property—namely, that repeated multiplication eventually results in the zero matrix—nilpotent matrices are often used to simplify complex problems and to understand the long-term behavior of systems [16].

B. Definition of Idempotent Matrices

Definition 2.2 An idempotent matrix is a matrix that, when multiplied by itself, yields itself. Formally, a matrix A is idempotent if and only if

$$A^2 = A$$

For the product A^2 to be defined, A must be a square matrix. Viewed in this way, an idempotent matrix is an idempotent element within the ring of matrices.

Examples of Idempotent Matrices:

- Examples of 2×2 idempotent matrices include:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix}$$

- Examples of 3×3 idempotent matrices include:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

Idempotent matrices play a significant role in various areas of mathematics and applied sciences [17], [18]. They are closely related to projection operators in linear algebra and functional analysis because they represent transformations that leave their image invariant after one application [14], [15], [19]. In ring theory, studying the properties and behavior of these elements helps understand decompositions and factorization within rings.

Moreover, understanding the algebraic structure formed by sets of idempotent matrices under multiplication provides insight into semigroup theory and monoid structures since these sets often satisfy closure properties but may lack invertibility required for group structures.

This paper further investigates how these unique properties influence combined algebraic systems involving both nilpotent and idempotent matrices.

C. Algebraic Structure of Groups

Definition 2.3 A group is defined as a set G combined with a binary operation $*$ that satisfies the following group axioms [1]:

1. **Closure Property**
   For every element a and b in G, the result of the operation $a * b$ must also be an element of G. If a structure $(G,*)$ only fulfills this closure property, it is called a *Groupoid*.
2. **Associativity**
   For all elements $a, b,$ and c in the set G, the equation $(a * b) * c = a * (b * c)$ holds true. When $(G,*)$ satisfies both closure and associativity properties exclusively, it forms what is known as a *Semigroup*.
3. **Existence of Identity Element**
   There exists an element $e \in G$, such that for any $a \in G$, performing $a * e = e * a = a$. It can be proven that within any group there is exactly one identity element. If $(G,*)$ meets closure, associativity, and has an identity element only, then it constitutes a *Monoid*.
4. **Existence of Inverse Elements**
   For each a in G, there exists an inverse element $a^{-1} \in G$, satisfying $a * a - 1 = a - 1 * a = e$, where e denotes the identity element. It can also be demonstrated that every member in G has exactly one inverse counterpart. When all four axioms above are fulfilled by $(G,*)$, this structure qualifies as a *Group*.
5. **Commutativity (Specific to Abelian Groups)**
   If for every pair of elements $a, b \in G$, the operation satisfies commutativity: $a * b = b * a$. Then if $(G,*)$ is commutative along with fulfilling previous axioms, it forms what is called an *Abelian Group*.

III. RESULTS AND DISSCUSSION

This section presents the detailed analysis and findings regarding the algebraic structures of specific matrix sets under multiplication. The focus is primarily on nilpotent matrices, exploring their behavior in relation to fundamental group axioms. By systematically examining properties such as closure, associativity, identity elements, and inverses within these sets, we aim to clarify the nature of their algebraic frameworks. The results provide insight into whether these matrix collections form groups, semigroups, or other algebraic structures.

A. Algebraic Structure of Nilpotent Matrices

Consider the pair $(N,\cdot)$, where N denotes the set of nilpotent matrices and " $\cdot$ " represents matrix multiplication. We will examine this structure against the four group axioms as follows:

1. **Closure**

   Take any matrices $A, B \in N$, where A has nilpotency index k_1 and B has index k_2. The product $A \cdot B$ is also a nilpotent matrix with an index equal to the least common multiple (LCM) of k_1 and k_2. This can be shown by:

   $$(A \cdot B)^{LCM(k_1,k_2)} = A^{LCM(k_1,k_2)} \cdot B^{LCM(k_1,k_2)}$$

   Since we can express:

   $$A^{LCM(k_1,k_2)} = A^{mk_1},$$

   $$B^{LCM(k_1,k_2)} = B^{nk_2}$$

   for some natural numbers $m, n \in \mathbb{N}$, it follows that:

   $$(A \cdot B)^{LCM(k_1,k_2)} = (A^{k_1})^m (B^{k_2})^n = \mathbf{0}^m \cdot \mathbf{0}^n = \mathbf{0}$$

   Therefore, it is confirmed that the product remains within the set of nilpotent matrices, i.e., $(A \cdot B) \in N$.

2. **Associativity**

   For arbitrary elements $A, B, C \in N$, associativity holds due to standard properties of matrix multiplication:

   $$A \cdot (B \cdot C) = (A \cdot B) \cdot C$$

   Hence, associativity is satisfied in $(N, \cdot)$.

3. **Existence of Identity Element**

   Since the identity matrix I_n does not belong to N because it is not nilpotent, there is no identity element in N. Consequently, there does not exist any element $A \in N$ such that for all $B \in N$ the relations hold $A \cdot B = B \cdot A = A$.

4. **Existence of Inverse Elements**

   Because an identity element does not exist in this structure, it follows that no inverse elements can be defined for members within N.

From points one through four above, we conclude that N forms a Semigroup.

B. Algebraic Structure of Idempotent Matrices

Consider the pair $(\boldsymbol{A},\cdot)$, where $\boldsymbol{A}$ is the set of idempotent matrices and " $\cdot$ " denotes matrix multiplication. We will examine this structure based on the four group axioms as follows:

1. **Closure**

   Take any $A, B \in \boldsymbol{A}$. The product $A \cdot B$ is also an idempotent matrix, which can be shown by:

   $$(A \cdot B)^2 = A^2 \cdot B^2$$

   Since A and B are idempotent, meaning $A^2 = A$ and $B^2 = B$, it follows that:

   $$(A \cdot B)^2 = A \cdot B$$

   Therefore, it is confirmed that the product remains within the set of idempotent matrices; hence, closure holds.

2. **Associativity**

   For any matrices $A, B, C \in \boldsymbol{A}$, associativity naturally holds due to standard properties of matrix multiplication:

   $$A \cdot (B \cdot C) = (A \cdot B) \cdot C$$

   Thus, associativity applies in this context.

3. **Existence of Identity Element**

   The identity matrix I_n belongs to the set $\boldsymbol{A}$ such that for every element $A \in \boldsymbol{A}$ the following holds true:

   $$I_n \cdot A = A \cdot I_n = A$$

4. **Existence of Inverse Elements**

   Consider a matrix $A = \begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix} \in \boldsymbol{A}$, determinant is calculated as: $det(A) = 3(-2) - 1(-6) = 0$, which means it is singular and therefore does not have an inverse. This shows there exists at least one element in $\boldsymbol{A}$ without an inverse.

From points one through four above—closure, associativity, identity exist but inverses do not—it can be concluded that $(\boldsymbol{A},\cdot)$ forms a Monoid.

C. Example of Nilpotent and Idempotent Matrix Groups

A detailed investigation reveals that no matrix can simultaneously exhibit both nilpotent and idempotent characteristics, except for a very special case. Nilpotent matrices are defined by the property that some positive power of the matrix equals the zero matrix, while idempotent matrices satisfy the condition that squaring them yields themselves. These two properties are fundamentally different and generally incompatible in non-trivial cases.

However, by considering the nilpotency condition $N^k = \mathbf{0}$ where k is a non-negative integer, it becomes evident that the zero matrix lies at the intersection of these two sets: $0 \in \boldsymbol{N} \cap \boldsymbol{A}$. The zero matrix is unique in this regard because it trivially satisfies both definitions — raising it to any power results in itself (zero), fulfilling nilpotency as well as idempotency.

Based on this premise, we can consider the algebraic structure formed solely by this zero element under multiplication, denoted as $(0,\cdot)$. This structure meets all group axioms: closure (the product of zero with itself is zero), associativity (inherited from standard multiplication), existence of an identity element (which coincides with zero here), and existence of inverses (trivially satisfied since there is only one element). Moreover, this trivial group is commutative because multiplication involving only one element cannot violate commutativity.

This example serves as an important illustration within algebraic structures: although more complex matrices cannot be both nilpotent and idempotent simultaneously, their intersection contains at least one fundamental object — the zero matrix. It highlights how certain extreme or degenerate cases can satisfy multiple algebraic properties concurrently.

In summary, while exploring groups formed by nilpotent or idempotent matrices individually leads to rich structures with diverse behaviors, their overlap reduces to a minimal but mathematically significant entity. The trivial group consisting solely of the zero matrix exemplifies how foundational elements underpin broader algebraic concepts.

D. Example of an Idempotent Matrix Group

An illustrative example of an idempotent matrix group is the set consisting solely of identity matrices, denoted as I_n. The structure $(I_n,\cdot)$, where the operation is standard matrix multiplication, forms a group that not only satisfies all group axioms but also exhibits commutativity. This means it qualifies as an Abelian group.

The defining property of idempotent matrices is that squaring the matrix yields the same matrix. In this case, for every identity matrix In, it holds true that: $I_n^2 = I_n$. This confirms that each element in this set inherently possesses the idempotent property.

Because $(I_n,\cdot)$ contains only identity elements and respects commutative multiplication, it represents a simple yet fundamental example of a commutative (Abelian) idempotent group. This trivial structure serves as a foundational building block in linear algebra and abstract algebra alike.

Beyond its simplicity, this example highlights important concepts about how certain sets with very restrictive membership can still form rich algebraic structures. The identity matrix plays a crucial role in many mathematical contexts due to its neutral behavior under multiplication and its unique position as both an idempotent and invertible element.

Moreover, understanding such elementary groups helps build intuition for more complex systems where multiple elements interact under various operations but may not always satisfy properties like commutativity or idempotency simultaneously.

In conclusion, while more intricate examples exist within matrix theory and algebraic structures at large, the set containing just identity matrices remains one of the clearest demonstrations of an Abelian idempotent group — simple yet mathematically significant.

E. Example of a Nilpotent Matrix Group

When considering groups formed by nilpotent matrices that are not idempotent, it becomes apparent that such groups do not exist in the conventional sense. This is primarily because the identity matrix I_n, which serves as the multiplicative identity in matrix groups, does not belong to the set of nilpotent matrices N.

As a result, these structures fail to satisfy all group axioms and can only be classified as semigroups. Nilpotent matrices inherently lack an identity element within their set since raising them to some power yields zero rather than an invertible element. Without this identity matrix present, it is impossible for $(N,\cdot)$ to form a full group under multiplication.

Examples of such algebraic structures are therefore limited to semigroups unless additional assumptions or modifications are introduced—similar to those discussed in previous sections regarding nilpotent and idempotent matrix groups. These assumptions often involve restricting attention to trivial cases or extending the set with additional elements.

This limitation highlights an important aspect of algebraic structures involving nilpotency: while they exhibit interesting properties individually, their inability to include an identity element restricts their classification within group theory. Consequently, researchers often study these sets under weaker algebraic frameworks like semigroups or monoids instead.

F. Summary of Algebraic Properties of Matrix Sets

In order to provide a clear overview of the algebraic characteristics discussed in this study, the following table summarizes the key properties—closure, associativity, identity element existence, and invertibility—of each matrix set under multiplication. These sets include nilpotent matrices ($\boldsymbol{N}$), idempotent matrices ($\boldsymbol{A}$), their union ($\boldsymbol{N} \cup \boldsymbol{A}$), and their intersection ($\boldsymbol{N} \cap \boldsymbol{A}$).

Tabel 1. Summary of Algebraic Properties

Algebraic Property	$(N,\cdot)$	$(\boldsymbol{A},\cdot)$	$(\boldsymbol{N} \cup \boldsymbol{A},\cdot)$	$(N \cap A,\cdot)$
Closure	Yes	Yes	Depends on elements; generally no	Yes (contains zero matrix only
Associativity	Yes	Yes	Yes	Yes
Identity Element	No	Exists (identity matrix I_n)	No	Zero matrix acts as identity-like element
Inverse Element	No	Not always (some singular matrices exist)	No	Trivial inverse due to single element

This summary highlights that while both nilpotent and idempotent sets individually satisfy closure and associativity under multiplication, only the idempotent set contains an identity element. The union does not generally form a group or monoid due to lack of closure or identity. The intersection is minimal — essentially containing only the zero matrix — which trivially satisfies closure and associativity but lacks a true identity or inverses.

This tabular presentation aids in understanding how these algebraic structures relate to one another within the context of matrix multiplication.

IV. RECOMMENDATION AND CONCLUSION

This study has explored the algebraic structures formed by sets of nilpotent and idempotent matrices under matrix multiplication. The investigation focused on verifying group axioms within these sets and identifying the nature of their algebraic systems.

It is recommended that future research expand upon this work by incorporating a wider variety of examples, including non-trivial cases or larger classes of matrices. Additionally, exploring modifications or generalizations to existing definitions could provide deeper insights into how these matrix sets behave under different operations or constraints. Such extensions may reveal new types of algebraic structures beyond semigroups and monoids.

The findings confirm that the set of nilpotent matrices $(N,\cdot)$, with multiplication as the operation, forms a semigroup but does not satisfy all group axioms due to the absence of an identity element and inverses within N. On the other hand, the set of idempotent matrices $(A,\cdot)$ constitutes a monoid, since it includes an identity element but generally lacks inverses for all elements.

Furthermore, trivial yet important examples were identified:

- The zero matrix serves as both nilpotent and idempotent simultaneously, forming a minimal commutative group structure under multiplication.
- The set containing only identity matrices forms an Abelian idempotent group due to its closure properties, existence of identity elements, invertibility, and commutativity.

These results highlight fundamental limitations in constructing full groups from nilpotent or idempotent matrices alone but also emphasize how certain special cases can satisfy stronger algebraic properties. Understanding these foundational structures provides valuable insight into more complex algebraic systems encountered in linear algebra and abstract algebra.

In conclusion, this research lays groundwork for further exploration into generalized matrix groups involving nilpotency and idempotency concepts. Future studies might investigate additional operations or hybrid structures combining multiple properties to enrich theoretical understanding as well as practical applications in mathematics and related fields.

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