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On Fixed Points in B-Rectangular Metric Spaces

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ABSTRACT

This paper presents fixed point results in b-rectangular metric spaces. The results obtained expand and generalize several well-established finding in the existing literature.

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KEYWORDS: Fixed point, b-metric space ,rectangular metric space, b-rectangular metric space.

1. INTRODUCTION

In 1922,Banach proved his classical contraction principle.The investiogation of existence and uniqueness of fixed point for a self mapping and common fixed point for two or more mappings has become a very active and natural subject of interest.many researchers proved Banach contraction principle in multitude of generalized metric space . In particular,b-metric spaces was introduced by Bakhtin[2] and Czerwik[6] as a generalization of metric spaces. They proved Banach contraction principle in b-metric spaces.

2. PRELIMINARIES

Definition 2.1([2],[6]) let X be a non empty set and $s \geq 1$ be a given real number .A function $d: X \times X \longrightarrow [0, \infty)$is a b-metric on X if , for all $x, y, z \in X$ the following conditions hold:

(b1) $d(x, y) = 0$ iff $x = y$

(b2)$d(x, y) = d(y, x)$

(b3)$d(x, z) \leq S[d(x, y) + d(y, z)]$ (b-tringular inequality)

In this case the pair $(X, d)$is called a b-metric space

Definition 2.2[1] let X be a non empty set , and let $d: X \times X \longrightarrow [0, \infty)$ be a mapping such that for all $x, y \in X$ and all distinct points $u, v \in X$,each distinct from x and y:

(r1) $d(x, y) = 0$ iff $x = y$

(r2) $d(x, y) = d(y, x)$

(r3)$d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$(rectangular inequality)

Then $(X, d)$is called rectangular or generalized metric space or Branciari's space.

Definition2.3[4] let X be a non empty set , $s \geq 1$ be a given real number and let $d: X \times X \longrightarrow [0, \infty)$ be a mapping such that for all $x, y \in X$and distinct $u, v \in X$,each distinct from x and y: (br1)$d(x, y) = 0$ iff $x = y$

(br2)$d(x, y) = d(y, x)$

(br3)$d(x, y) \leq S[d(x, u) + d(u, v) + d(v, y)]$ (b-rectangular inequality)

Then $(X, d)$is called a b-rectangular metric space or a b-generalized metric space (bg.m.s)

Note: that every metric space is a b-rectangular metric space and every rectangular metric space is a b-rectangular metric space (with coefficient s=1) However the converse is not necessarily true .

Also, every metric space is a b-metric space and every b-metric space is a b-rectangular metric space (not necessarily with the same coefficient

3. MAIN RESULT

Theorem 3.1 : let (X, d) be a complete b-rectangular metric space with $t > 2$and let $h, k: X \longrightarrow X$ be two self maps satisfying,

$d(hx, ky) \leq \delta P(x, y) + QR(x, y)$ (3.1.1)

For all $x, y \in X$ where $\delta \in [0, \frac{1}{t}]$ and $Q > 1$ and

$P(x, y) = max\{d(x, y), d(x, hx), d(y, ky)\}$,

$R(x, y) = min\{d(x, hx), d(y, ky), d(x, ky), d(y, hx)\}$.

Then h and k have a unique common fixed point.

Proof: let α_0 be an arbitrary point in X.

Define the sequence$\{\alpha_n\} in\ X\ as\ \alpha_{2n+1} = h\alpha_{2n} and\ \alpha_{2n+2} = k\alpha_{2n+1} for\ n \geq 1$

Suppose that there is some $n \geq 1$such that $\alpha_n = \alpha_{n+1}$

If $n = 2m\ then\ \alpha_{2m} = \alpha_{2m+1}$ and from (3.1.1)

$d(\alpha_2m+1, \alpha_2m+2 = d(h\alpha_2m, k\alpha_2m+1)$

$\leq \delta P(\alpha 2m, \alpha 2m{+}1) + QR(\alpha 2m, \alpha 2m{+}1)$

Where $P(\alpha 2m, \alpha 2m{+}1) = max\{d(\alpha 2m,\alpha 2m{+}1), d(\alpha 2m,h\alpha 2m), d(\alpha 2m{+}1, k\alpha 2m{+}1))\}$

$P(\alpha_2 m, \alpha_2 m{+}1) = max\{d(\alpha_2 m, \alpha_2 m{+}1), d(\alpha_2 m, \alpha_2 m{+}1), d(\alpha_2 m{+}1, \alpha_2 m{+}2)\}$

$P(\alpha 2m, \alpha 2m{+}1) = max\{0,0, d(\alpha 2m{+}1, \alpha 2m{+}2) \}$

$R(\alpha_2 m, \alpha_2 m{+}1) = min\{d(\alpha_2 m, h\alpha_2 m), d(\alpha_2 m, \alpha_2 m{+}1), d(\alpha_2 m, k\alpha_2 m{+}1), d(\alpha_2 m{+}1, h\alpha_2 m)\}$

$= min\{d(\alpha_2 m, \alpha_2 m{+}1), d(\alpha_2 m{+}1, \alpha_2 m{+}2), d(\alpha_2 m, \alpha_2 m{+}2), d(\alpha_2 m{+}1, \alpha_2 m{+}1)\}$

$= 0$

Thus we have,

$d(\alpha_2 m{+}1, \alpha_2 m{+}2) \leq \delta d(\alpha_2 m{+}1, \alpha_2 m{+}2)$

Which is a contradiction with $\delta \in [\, 0, \frac{1}{t}]$

Therefore $\alpha_{2m+1} = \alpha_{2m+2}$ Thus we have $\alpha 2m = \alpha 2m{+}1 = \alpha 2m{+}2$

It means that $\alpha_{2m} = h\alpha_{2m} = k\alpha_{2m}$

That is α_{2m} is a common fixed point of f and g

If $n = 2m + 1$ then using same arguments, it can be shown that α_{2m+1} is a common fixed point h and k

Now suppose that $\alpha_n \neq \alpha_{n+1}$ for all $n \geq 1$

$d(\alpha_2 n{+}1, \alpha_2 n{+}2) = d(h\alpha_2 n, k\alpha_2 n{+}1)$

$\leq \delta P(\alpha_{2n}, \alpha_{2n+1}) + QR(\alpha_{2n}, \alpha_{2n+1})$ …(3.1.2)

Where

$P(\alpha 2n, \alpha 2n{+}1) = max\{d(\alpha 2n, \alpha 2n{+}1), d(\alpha 2n, h\alpha 2n), d(\alpha 2m{+}1, k\alpha 2n{+}1)\}$

$= max\{d(\alpha 2n, \alpha 2n{+}1), d(\alpha 2n, \alpha 2n{+}1), d(\alpha 2n{+}1, \alpha 2n{+}2)\}$

$= max\{d(\alpha 2n, \alpha 2n{+}1), d(\alpha 2n{+}1, \alpha 2n{+}2)\}$

And

$R(\alpha 2n, \alpha 2n{+}1) = min\{d(\alpha 2n, h\alpha 2n), d(\alpha 2n{+}1, k\alpha 2n{+}1), d(\alpha 2n, k\alpha 2n{+}1), d(\alpha 2n{+}1, h\alpha 2n)\}$

$= min\{d(\alpha 2n, \alpha 2n{+}1), d(\alpha 2n{+}1, \alpha 2n{+}2), d(\alpha 2n, \alpha 2n{+}2), 0\}$

$= 0$

If $P(\alpha_{2n}, \alpha_{2n+1}) = d(\alpha_{2n+1}, \alpha_{2n+1})$ *then by* (3.1.2)

$d(\alpha_2 n{+}1, \alpha_2 n{+}2) \leq \delta d(\alpha_2 n{+}1, \alpha_2 n{+}2)$

Which is a contradiction.

Thus $P(\alpha 2n, \alpha 2n{+}1) = d(\alpha 2n, \alpha 2n{+}1)$

And from (3.1.2)

$d(\alpha_2 n{+}1, \alpha_2 n{+}2) \leq \delta d(\alpha_2 n, \alpha_2 n{+}1)$

Similarly it can be proved that

$d(\alpha_2 n{+}3, \alpha_2 n{+}2) \leq \delta d(\alpha_2 n{+}2, \alpha_2 n{+}1)$

So,

$d(\alpha_{n+1}, \alpha_n) \leq \delta d(\alpha_n, \alpha_{n-1}) \leq \delta^n d(\alpha_1, \alpha_0)$ for all $n \geq 1$.

Similarly we can show

$d(\alpha_{n+2}, \alpha_n) \leq \delta^n d(\alpha_2, \alpha_0)$

We can show that $\{\alpha_n\}$ is a b- rectangular - Cauchy sequence. using b-rectangular inequality and

$\alpha_n \neq \alpha_{n+1}$ for all $n \geq 1$ and $d_n = d(\alpha_n, \alpha_{n+1})$,

$$d_n^* = d(\alpha_n, \alpha_{n+2})$$

$d(\alpha_n, \alpha_n{+}2m{+}1) \leq t[d(\alpha_n, \alpha_n{+}1) + d(\alpha_n{+}1, \alpha_n{+}2) + d(\alpha_n{+}2, \alpha_n{+}2m{+}1)]$

$\leq t[d_n + d_n{+}1] + t^2[d_n{+}2 + d_n{+}3] + t^3[d_n{+}4 + d_n{+}5] + \cdots + t^{m+1}d_n{+}2m$

$\leq t[\delta n d_0 + \delta n{+}1 d_0] + t2[\delta n{+}2 d_0 + \delta n{+}3 d_0]{+}\ldots{+}tm\delta n{+}2m d_0$

$= \frac{1+\delta}{1-t\delta^2} t\delta^n d_0 \qquad (t\delta^2 < 1)$

Hence, $d(\alpha n, \alpha n{+}2m{+}1) \leq \frac{1+\delta}{1-t\delta^2} t \quad \delta^n d_0$ …(3.1.3)

Similarly we can show that

$d(\alpha n, \alpha n{+}2m) \leq \frac{1+\delta}{1-t\delta^2} t \quad \delta^n d_0 + \delta^{n-2} d^*_0$ …(3.1.4)

Thus from (3.1.3) and (3.1.4) we obtain that

$\lim d(\alpha_n, \alpha_{n+p}) = 0 \quad for\ all \quad p = 1,2,3 \ldots n{\to}\infty$

Hence $\{\alpha_n\}$ is a b-rectangular -cauchy sequence in (X, d).

By completeness of (X, d), $\exists\ r \in X$ such that $\alpha_n = h\alpha_{n-1} \to r$ as $n \to \infty$

Now we prove that $hr = r$ (by b-rectangular-inequality)

$\frac{1}{t} d(hr, r) \leq d(hr, k\alpha n) + d(k\alpha_n, \alpha_n) + d(\alpha_n, r)$

$\leq \delta P(r, \alpha_n) + QR(r, \alpha_n) + d(\alpha_{n+1}, \alpha_n) + d(\alpha_n, r)$

Where $P(r, \alpha_n) = \max\{d(r, \alpha_n), d(r, hr), d(\alpha_n, k\alpha_n)\} \to d(r, hr)$ as $n \to \infty$

$R(r, \alpha_n) = \min\{d(r, hr), d(\alpha_n, k\alpha_n), d(r, g\alpha_n), d(\alpha_n, hr)\} \to 0$ as $n \to \infty$

Hence taking the limit as $n \to \infty$, *we obtain*

$\frac{1}{t} d(hr, r) \leq \delta d(r, hr) + Q.\ 0 + 0 + 0$

That is $hr = r$.

Hence r is a fixed point of h.

Now we show that $kr = r$. Suppose that $r \neq kr$ by (3.1)

$d(r, kr) = d(hr, kr) \leq \delta P(r, r) + QR(r, r)$

Where $P(r, r) = \max\{(r, r), d(r, hr), d(r, kr)\} = \max\{0,0, d(r, kr)\} = d(r, kr)$

and $R(r, r) = \min\{d(r, hr), d(r, kr), d(r, kr), d(r, hr)\} = 0$

By (3.1.1)

$d(r, kr) \leq \delta d(r, kr)$

Which is a contradiction.

Thus $kr = r$.

Now we show that uniqueness

Suppose r and s are different common fixed points of h and k

$d(r, s) = d(hr, ks) \leq \delta P(r, s) + QR(r, s)$

where $P(r, s) = max.\ \{d(r, s), d(r, hr), d(s, ks)\} = d(r, s)$

and $R(r, s) = \min\{d(r, hr), d(s, ks), d(r, ks), d(s, hr)\} = 0$

Thus from (3.5)

$d(r, s) \leq \delta d(r, s)$

So $d(r, s) = 0$

i. e. $r = s$

4. CONCLUSION

As the rectangular b-metric space is relatively new addition two the existing literature. Therefore we endeavour to further enrich this notion by introducing the idea of exetended rectangular b- metric spaces.

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