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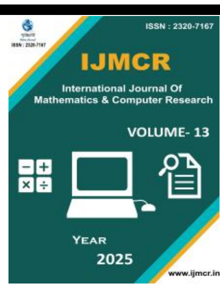

A Result on Fixed Points in Rectangular S-Metric Spaces

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ARTICLE INFO	ABSTRACT
Published Online: 15 May 2025	In this paper, the notion of rectangular S-metric which extends rectangular metric spaces introduced by Branciari. The results obtained expand and generalize several well-established findings in the existing literature.
Corresponding Author: A.S. Saluja	**MATHEMATICS SUBJECT CLASSIFICATION (2020) :** Primary: 54H25; Secondary:54E50, 47H10.

KEYWORDS: Fixed point, rectangular metric spaces, S-metric spaces, rectangular S metric spaces.

1. INTRODUCTION

Fixed point theory has been one of the most rapidly developing fields in analysis during the last few decades. In 1922,Banach proved his classical contraction principle. The investigation of existence and uniqueness of fixed point for a self mapping and common fixed points for two or more mappings has become a very active and natural subject of interest. Many researchers proved Banach contraction principle in multitude of generalized metric space. In 2000,Branciari [1] generalised the idea of metric space by replacing the triangular inequality with more general inequality, namely, quadrilateral inequality for introducing the notion of rectangular metric spaces and generalised branch contraction theorem. After eight years , George et al. [10] introduced rectangular b-metric spaces in order to generalised rectangular metric spaces. The notion of rectangular S-metric space which extends a rectangular metric space is presented. Motivated by these generalizations, in this paper, we stated and proved some fixed point theorems in these spaces. Prior to presenting our main result, the following definitions are required in the sequel.

2. PRELIMINARIES

Definition 2.1 [1] For a non empty set X and a function d: $X^2 \to [0, \infty)$satisfying the following properties:

(i) $d(\alpha, \beta) = 0$ iff $\alpha = \beta$ for all $\alpha, \beta \in X$;

(ii) $d(\alpha, \beta) = d(\beta, \alpha)$, $\forall \alpha, \beta \in X$;

(iii) $d(\alpha, \beta) \le d(\alpha, u) + d(u, v) + d(v, \beta)\ \ \forall\, \alpha, \beta \in X$ *and all distinct points* $u, v \in X - \{\alpha, \beta\}$

d is called a rectangular metric on X *and* $(X, d)$is called a rectangular metric space.

Definition 2.2 [12] let X be a non empty set and $\bar{S}$: $X^3 \to \mathbb{R}^+$, a function satisfying the following properties:

(i) $\bar{S}(\alpha, \beta, \gamma) = 0$ iff $\alpha = \beta = \gamma$

(ii) $\bar{S}(\alpha, \beta, \gamma) \le \bar{S}(\alpha, \alpha, a) + \bar{S}(\beta, \beta, a) + \bar{S}(\gamma, \gamma, a) \forall a, \alpha, \beta, \gamma \in X$ (rectangle inequality)

Then $(X, \bar{S})$ is called a S-metric space.

Definition2.3 [15] let X be a non empty set and $\bar{s}$: $X^3 \to \mathbb{R}^+$,a function with a strictly increasing continuous function , Ω : $[0, \infty) \to [0, \infty)$ such that $\Omega\,(t) \ge t$ for all $t > 0$ and $\Omega\,(0) = 0$, satisfying the following properties:

(i) $\bar{S}(\alpha, \beta, \gamma) = 0$ iff $\alpha = \beta = \gamma$

(ii) $\bar{S}(\alpha, \beta, \gamma) \le \Omega(\bar{S}(\alpha, \alpha, a) + \bar{S}(\beta, \beta, a) + \bar{S}(\gamma, \gamma, a)) \forall a, \alpha, \beta, \gamma \in X$ (rectangle inequality)

Then $(X, \bar{S})$ is called an S_p-metric space.

Remark 2.4

(i) If $\Omega(\gamma) = \gamma$, S_p-metric space reduces to S-metric space

(ii) If $\Omega(\gamma) = b\gamma$, S_p-metric space reduces to $S_b$metric space.

Definition 2.5 Let X be a non empty set and $\underline{S}$: $X^3 \to \mathbb{R}^+$ a function satisfying the following the following properties :

(i) $\underline{S}(\alpha, \beta, \gamma) = 0$ iff $\alpha = \beta = \gamma$

(ii) $\underline{S}(\alpha, \beta, \gamma) \le \underline{S}(\alpha, \alpha, a) + \underline{S}(\beta, \beta, a) + \underline{S}(\gamma, \gamma, a) \forall\ \alpha, \beta, \gamma \in X$ and all distinct points $a \in X - \{\alpha, \beta, \gamma\}$

Then $(X, \underline{S})$ is called a rectangular S-metric space.

Definition 2.6 Let $(X, \underline{S})$ be a rectangular S-metric space and $\{\alpha_n\}$ a sequence in X. Then $\{\alpha_n\}$ converges to α if and only if $\underline{S}(\alpha_n, \alpha, \alpha) \to 0$ as $n \to \infty$.

Definition 2.7 Let $(X, \underline{S})$ be a rectangular S-metric space and $\{\alpha_n\}$ a sequence in X. Then $\{\alpha_n\}$ is said to be a Cauchy sequence iff $\underline{S}(\alpha_n, \alpha_m, \alpha_l) \to 0$ as $n, m, l \to \infty$.

3. MAIN RESULT

Theorem 3.1: Let X be a complete rectangular S- metric space and $T: X \to X$ a map for which there exist the real number, p satisfying $0 \le p < 1$ such that for each pair $\alpha, \beta, \gamma \in X$

$\underline{S}(T\alpha, T\beta, T\gamma) \le pS(\alpha, \beta, \gamma)$ . . . (3.1.1)

Then T has a unique fixed point.

Proof: From (3.1.1)

$\underline{S}\ (T\alpha, T\beta, T\beta) \le p\ \underline{S}\ (\alpha, \beta, \beta)$ . . . (3.1.2)

Suppose T satisfies condition (3.1.2) and $\alpha_0 \in X$ be an arbitrary point

and define a sequence $\{\alpha_n\}$ by setting $\alpha_n = T^n\alpha_0$

and $\underline{S}(\alpha n, \alpha n, \alpha n{+}1) = \underline{S}(T\alpha n{-}1, T\alpha n{-}1, T\alpha n)$

Now $\underline{S}(T\alpha n{-}1, T\alpha n{-}1, T\alpha n) \le p\underline{S}(\alpha n{-}1, \alpha n{-}1, \alpha n)$

Therefore

$\underline{S}(\alpha n, \alpha n, \alpha n{+}1) \le p\underline{S}(\alpha n{-}1, \alpha n{-}1, \alpha n)$ Setting $s_n = \underline{S}(\alpha_n, \alpha_n, \alpha_{n+1})$ we have

$s_n \le ps_{n-1}$ . . . (3.1.3)

We deduce that

$s_n \le p^n s_0 \forall n \in N$ . . . (3.1.4)

Suppose there exist $n \in N$ such that $\alpha_0 = \alpha_n$

$\underline{S}\ (\alpha_0, \alpha_0, T\alpha_0) = \underline{S}\ (\alpha_n, \alpha_n, T\alpha_n)$

$\underline{S}\ (\alpha 0, \alpha 0, \alpha 1) = \underline{S}\ (\alpha n, \alpha n, \alpha n{+}1)$

$\alpha_0 = \alpha_n\ \alpha 0 \le pn\alpha 0$ Contradiction since $p < 1$.

Hence for all $n \in N$, $\alpha_0 \ne \alpha_n$

Repeating this argument , we have that $\forall\ n, m \in N$ with $n \ne m$, $\alpha_n \ne \alpha_m$.

Then the terms of a sequence $\{\alpha_n\}$ are distinct.

By repeated use of (ii) in definition (2.4) and all distinct points $\alpha n{+}1, \alpha n{+}2, \ldots \alpha m{-}1$ with $m > n$

$\underline{S}\ (\alpha_n, \alpha_m, \alpha_m)$

$\le \underline{S}\ (\alpha_n, \alpha_n, \alpha_{n+1}) + \underline{S}\ (\alpha_m, \alpha_m, \alpha_{n+1}) + \underline{S}\ (\alpha_m, \alpha_m, \alpha_{n+1})$

$= \underline{S}\ (\alpha_n, \alpha_n, \alpha_{n+1}) + 2\underline{S}(\alpha_m, \alpha_m, \alpha_{n+1})$

$= s_n + 2\underline{S}\ (\alpha_m, \alpha_m, \alpha_{n+1})$

$\le s_n + 2s_{n+1} + 2^2\underline{S}(\alpha_m, \alpha_m, \alpha_{n+2})$

$\le s_n + 2s_{n+1} + 2^2s_{n+2} + 2^3\underline{S}\ (\alpha_m, \alpha_m, \alpha_{n+3})$

$\le s_n + 2s_{n+1} + 2^2s_{n+2} + \cdots + 2^{m-1}s_m$

$\le s_n + 2s_{n+1} + 2^2s_{n+2} + 2^3s_{n+3} + \cdots$ . . . (3.1.5)

From (3.1.4) and (3.1.5), we have

$\underline{S}\ (\alpha n, \alpha m, \alpha m)$

$\le p^n s0 + 2p^{n+1}s0 + 2^2p^{n+2}s0 + \cdots + 2^{m-1}p^{m-1}s0$

$\le [p^n + 2p^{n+1} + 2^2p^{n+2} + \cdots + 2^{m-1}p^{m-1}]s0$

$\le p^n[1 + 2p + (2p)^2 + \cdots + (2p)^{m-n-1}]s_0$

$\le p^n[1 + 2p + (2p)^2 + \cdots]s_0$

$\le p^n(1 - 2p)^{-1}s_0$ . . . (3.1.6)

Taking the limit of $\underline{S}\ (\alpha_n, \alpha_m, \alpha_m)\ as\ n, m \to \infty, we\ have$

$$\lim_{m,n\to\infty} \underline{S}\ (\alpha n, \alpha m, \alpha m)$$

$$= \lim_{m,n\to\infty} [p^n(1 - 2p)^{-1}]\ \underline{S}(x_0, x_0, x_1) = 0$$

. . . (3.1.7)

For $n, m, l \in N\ with\ n > m > l$

$\underline{S}\ (\alpha_n, \alpha_n, \alpha_l) \le \underline{S}\ (\alpha_n, \alpha_n, \alpha_{n-1}) + \underline{S}\ (\alpha_m, \alpha_m, \alpha_{n-1}) + \underline{S}\ (\alpha_l, \alpha_l, \alpha_{n-1})$ . . . (3.1.8)

On taking the limit of $\underline{S}\ (\alpha_n, \alpha_m, \alpha_l)\ as\ n, m, l \to \infty$, we obtain

We have $\lim_{m,n,l\to\infty} \underline{S}\ (\alpha_n, \alpha_m, \alpha_l) = 0$

So, $\{\alpha_n\}$,is a $\underline{S}$ −cauchy sequence.

By completeness of $(x, \underline{S})$, there exist $u \in X$ such that $\{\alpha_n\}$ is $\underline{S}$- convergent to u.

Suppose $Tu \ne u$, then

$\underline{S}\ (\alpha_n, Tu, Tu) \le p\underline{S}\ (\alpha_{n-1}, u, u)$

Taking the limit as $n \to \infty$ and using the fact that function is $\underline{S}$- continuous in its variables,

We get, $\underline{S}\ (u, Tu, Tu) \le p\underline{S}\ (u, u, u)$

Hence, $\underline{S}\ (u, Tu, Tu) \le 0$

Which is a contradiction.

So that, $Tu = u$

To prove uniqueness, suppose $v \ne u$ is such that $Tv = v, then$ from (3.1.1)

$\underline{S}\ (Tu, Tv, Tv) \le p\underline{S}\ (u, v, v)$

Since $Tu = u\ and\ \ Tv = v$,we have

$\underline{S}\ (u, v, v) \le 0$ Which implies that $v = u$.

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