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An Extension of G-Symbol of Three Variables

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Abstract:

The aim of this paper is to evaluate and extend the G-Symbole into three variables which is a special cases of Appell's Function.

1. Introduction:

Meijer in 1941 defined his G-Function by means of a Mellin-Barnes type of integral in the form

$$G_{p,q}^{m,n}\left(x\middle|\begin{matrix}a_1,a_2,\ldots..a_p\\ b_1,b_2,\ldots\ldots b_q\end{matrix}\right)=\frac{1}{2\pi i}\int\frac{\Gamma(b_1+s)\ldots\ldots\Gamma(b_m+s)\,\Gamma(1-a_1-s)\ldots\ldots\Gamma(1-a_n-s)}{\Gamma(a_{n+1}+s)\ldots\ldots\Gamma(a_p+s)\Gamma(1-b_{m+1}-s)\ldots..\Gamma(1-b_q-s)}\,x^s\,ds\,,$$

where an empty product is interpreted as i,o $\le$ m $\le$ q , o $\le$ n $\le$ p, and the parameters are such that no pole of Γ(bj-s) , j =1, 2, ………..m coincides with any pole of $\Gamma(1-a_k+s)$, k = 1, 2, ……..n. The path of integration L runs from -i∞ to i∞ so that all poles of Γ(bj-s), j = 1, 2, ……..m are to the right , and all the poles of $\Gamma(1-a_k+s)$, k = 1, 2, ……..n to the left of L. The integral converges for p+q<2(m+n) and $|\arg x| < \pi(m+n-½p-½q)$. The importance of the G-Function lies in the great many special functions that can be represented as its particular cases.

The Object of this paper is define a G-Function of three variables which not only includes the Meijer's G-Function as a particular cases but also most of the known function of three variables, e.g. the Appell's function F1, F2, F3, F4, the Whittaker function of three variables, etc. Besides including the known function of three variables as particular cases, it leaves the possibility of defining, through this new G-Symbole of three variables, a great many special function of three variables not hither to mentioned.

Let $(a_m) = a(a+1)(a+2)\ldots\ldots\ldots\ldots\ldots\ldots(a+m-1);\ (a)_0 = 1$

Also, the symbol (α_p) denotes the sequence of elements $\alpha_1, \alpha_2, \ldots\ldots\ldots\ldots\ldots\ldots \alpha_p$ and $(\alpha_{m,p})$ denotes the sequence $\alpha_m, \alpha_{m+1}, \ldots\ldots\ldots\ldots\ldots\ldots \alpha_p$.

Thus the triple hyper geometric function of higher order in three variables.

$$\left[\begin{array}{c|cccccc|c} p & \varepsilon_1 & ,\varepsilon_2 & ,\varepsilon_3 & ,\ldots\ldots\ldots\ldots\ldots & \varepsilon_p & & & \\ t & \Upsilon_1 & ,\Upsilon_1' & ,\Upsilon_1'' & ,\ldots\ldots\ldots\ldots\ldots & \Upsilon_t & ,\Upsilon_t' & ,\Upsilon_t'' & x \\ s & \delta_1 & ,\delta_2 & ,\delta_3 & ,\ldots\ldots\ldots\ldots\ldots & \delta s & & & \\ q & \beta_1 & ,\beta_2 & ,\beta_3 & ,\ldots\ldots\ldots\ldots\ldots & \beta_q & ,\beta_q' & ,\beta_q'' & Y \\ l & \rho_1 & ,\rho_2 & ,\rho_3 & ,\ldots\ldots\ldots\ldots.. & \rho_l & & & \\ \omega & \upsilon_1 & ,\upsilon_2 & ,\upsilon_3 & ,\ldots\ldots\ldots\ldots\ldots & \upsilon_\omega & ,\upsilon_\omega' & ,\upsilon_\omega'' & Z \end{array}\right]$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty}$$

$$\frac{[(\delta 1)_{m+n+k} \ldots\ldots(\delta_s)_{m+n+k}(\beta_1)_{m+n+k}(\beta_1')_{m+n+k}(\beta_1'')_{m+n+k} \ldots\ldots(\beta_q)_m(\beta_q')_n(\beta_q'')_k]}{[(\rho_1)_{m+n+k} \ldots\ldots(\rho_l)_{m+n+k} \ldots\ldots(\upsilon_1)_{m+n+k}(\upsilon_1')_{m+n+k}(V_1'')_{m+n+k} \ldots\ldots(\upsilon_\omega)_m(\upsilon_\omega')_n(\upsilon_\omega'')_k(1)_m(1_n)(1)_k}$$

Where p+t < s+q+l+ω+1 shall be abbreviated to

$$\left[\begin{array}{c|llll|c} p & (\varepsilon_p) & & & & \\ t & (Y_t) & ;(Y_t') & ;(Y_t'') & & x \\ s & (\delta_s) & & & & \\ q & (\beta_q) & ;(\beta_q') & ;(\beta_q'') & & y \\ l & (\rho_l) & & & & \\ \omega & (\upsilon_\omega) & ;(\upsilon_\omega') & ;(\upsilon_\omega'') & & z \end{array}\right]$$

The series for the F-Function converges absolutely for all complex values of x, y and z if p + t < s+q+l+w+1. In case p+t = s+q+l+w+1, it converges absolutely for all complex values of x, y and z.

$|x|+|y|+|z| < \min(1, 2^{l+s-p+1})$

2. The $G_{p,q,s,t,l,w}^{m_1,m_2,m_3,n,v_1,v_2,v_3,}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ Functon

Consider the triple contour integral.

$$(1)\ \ I = \frac{-1}{4\pi 2}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty} \emptyset(\xi+\eta)\psi(\eta+\tau)\chi(\eta,\tau)x^{\xi}y^{\eta}z^{\tau}d\xi d\eta d\tau \quad \upsilon \quad \upsilon$$

Where

$$\chi(\eta,\tau) = \frac{\prod_{j=1}^{m_2}\Gamma(\upsilon_j-\eta)\prod_{j=1}^{\upsilon_2}\Gamma(\beta_j+\eta)\prod_{j=1}^{m_3}\Gamma(\upsilon_j''-\tau)\prod_{j=1}^{\upsilon_3}\Gamma(\beta_j''+\tau)}{\prod_{j=1}^{w}\Gamma(1-\upsilon_j+\eta)\prod_{j=\upsilon_2+1}^{q}\Gamma(1-\beta_j-\eta)\prod_{j=1+m_3}^{w}\Gamma(1-\upsilon_j''-\tau)\prod_{j=1+\upsilon_3}^{q}\Gamma(1-\beta_j''-\eta)}$$

$$\psi(\eta+\tau) = \frac{\prod_{j=1}^{n}\Gamma(1-\delta_j+\eta+\tau)}{\prod_{j=n+1}^{s}\Gamma(\delta_j-\eta-\tau)\prod_{j=1}^{l}\Gamma(\rho_j+\eta+\tau)}$$

and

$$\emptyset(\xi+\eta) = \frac{\prod_{j=1}^{n}\Gamma(1-\varepsilon_j+\xi+\eta)}{\prod_{j=n+1}^{p}\Gamma(\varepsilon_j-\xi-\eta)\prod_{j=1}^{s}\Gamma(\delta_j+\xi+\eta)}$$

and $0\le m_1 \le q$, $0\le m_2 \le w$, $0\le m_3 \le w$, $0\le v_1 \le t$, $0\le v_2 \le t$, $0\le v_3 \le t$, $0\le n \le p$,

The sequence of parameter (υ_{m1}) , $(\upsilon_m{'}_2)$, $(\upsilon_m{''}_3)$, $(\beta_{\upsilon 1})$, $(\beta_\upsilon{'}_2)$, $(\beta_\upsilon{''}_3)$ and (En) are s.t. none of the poles of the integrand coincide. The Paths of integration are indented , if necessary , in such a manner that all the poles of $\Gamma(\upsilon_{j-n})$, j = 1 , 2 , ………… m_2 and $\Gamma(\upsilon_{k-\tau})$, k = 1 , 2, …………. υ_3 lie to the right , and those of $\Gamma(\beta_{k+\eta})$, k = 1 , 2 , ………… υ_2 and $\Gamma(\beta_k{''}_{+\tau})$, k = 1 , 2 , ………… υ_3 and $\Gamma(1 - \varepsilon_k + \xi + \eta + \tau)$, k = 1 , 2 , ………… n lie to the left of the imaginary axis.

The integral (2.1) converges if

$$(2)\quad \begin{cases} p+q+l+w+s+t<3(m_1+v_1+n) \\ p+q+l+w+s+t<3(m_2+v_2+n) \\ p+q+l+w+s+t<3(m_3+v_3+n) \\ \qquad and \\ |\log x|<\pi(m_1+v_1+n-\frac{1}{2}(p+q+l+w+s+t) \\ |\log y|<\pi(m_2+v_2+n-\frac{1}{2}(p+q+l+w+s+t) \\ |\log z|<\pi(m_3+v_3+n-\frac{1}{2}(p+q+l+w+s+t) \end{cases}$$

Evaluating (2.1) by considering the residues at the poles of integrand that lie to the right of imaginary axis , we have

$$(3)\quad I=\sum_{h=1}^{m_1}\sum_{k=1}^{m_2}\sum_{d=1}^{m_3} x^{h}\, y^{\upsilon'_k} z^{\upsilon''_d}$$

$$\frac{\prod_{j=1}^{v_1}\Gamma(\gamma_j+\beta_j+\upsilon_h)\prod_{j=1}^{v_2}\Gamma(\gamma'_j+\beta'_j+\upsilon'_k)\prod_{j=1}^{v_3}\Gamma(\gamma''_j+\beta''_j+\upsilon''_d)\prod_{j=1}^{m_3}\Gamma(\upsilon''_j-\upsilon''_d)}{\prod_{j=\upsilon_1+1}^{t}\Gamma(1-\gamma_j-\beta_j-\upsilon_h)\prod_{j=1+v_2}^{t}\Gamma\left(1-\gamma'_j+\beta'_j+\upsilon'_k\right)\prod_{j=1+v_3}^{t}\Gamma(1-\gamma''_j+\beta''_j+\upsilon''_d)\prod_{j=1+m_1}^{w}\Gamma(1-\upsilon_j-\upsilon_h)}$$

$$\times\frac{\prod_{j=1}^{n}\Gamma(1-\varepsilon_j+\upsilon_h+\upsilon'_k+\upsilon''_d)\ \ \prod_{j=1}^{m_1}\Gamma(\upsilon_j-\upsilon_h)}{\prod_{j=1+m_2}^{w}\Gamma(1+\upsilon'_j+\upsilon'_k)\prod_{j=1+m_3}^{w}\Gamma(1-\upsilon''_j-\upsilon''_k)\prod_{j=1+n}^{p}\Gamma(\varepsilon_j-\upsilon_n-\upsilon'_k-\upsilon''_d)\ \prod_{j=1}^{l}\Gamma\left(\rho_j+\upsilon_h+\upsilon'_k-\upsilon''_d\right)}$$

$$\times F\left[\begin{array}{c|l|l} p & (1-\varepsilon_p+\upsilon_h+\upsilon'_k+\ \upsilon''_d) & \\ t & (\gamma_t+\upsilon_h);\ (\gamma'_t+\upsilon'_k);\ (\gamma''_t+\upsilon''_d) & (-)^{m_1+p-n+t+q-v_1}x \\ s & (\delta_s+\upsilon_h+\upsilon'_k+\upsilon''_d) & \\ q & \left(\beta_q+\upsilon_h\right);\ (\beta'_q+\upsilon'_k);\ (\beta''_q+\upsilon''_d) & (-)^{m_2+p-n+t+q-v_2}y \\ l & (\rho_l+\upsilon_h+\upsilon'_k+\upsilon''_d) & \\ w-1 & (1-\upsilon_w+\upsilon_h)^*;(1-\upsilon'_w+\upsilon'_k)^*;(1-\upsilon''_w+\upsilon''_d)^* & (-)^{m_3+p-n+t+q-v_3}y \end{array}\right]$$

Where the prime in Π' indicates the omission of the factor of the type $\Gamma(\upsilon_j-\upsilon_n)$; the asterisk in the F denotes the omission of the parameter of the type $(1-\upsilon_n+\upsilon'_k+\upsilon''_d)$, (2.3) converges absolutely for $p+t<s+q+l+w$ or $p+t=s+q+l+w$, an $|x|+|y|+|z|<\min(1, 2(^{1+s-p+1})$. The right hand side of (2.3) shall, henceforth, be symbolically denoted by

$$G^{n,v_1,v_2,v_3,m_1,m_2,m_3}_{p,t,s,q,l,w}\left[\begin{array}{c|l} x & (\varepsilon_p) \\ & (\gamma_t);\ (\gamma'_t);\ (\gamma''_t) \\ & (\delta_s) \\ y & (\beta_t);\ (\beta'_t);\ (\beta''_t) \\ & (\varepsilon_p) \\ z & (\upsilon_t);\ (\upsilon'_t);\ (\upsilon''_t) \end{array}\right]$$

or $G^{m_1,m_2,m_3,n,v_1,v_2,v_3,}_{p,q,s,t,l,w}\begin{bmatrix}x\\y\\z\end{bmatrix}$ or simply $G\begin{bmatrix}x\\y\\z\end{bmatrix}$, whenever there is no chance of misunderstanding is the required extension of 1768Meijer's G-Function to three variables.

3. Certain Particular cases of $G\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$(1)\ G^{0,v_1,v_2,v_3,m_1,m_2,m_3}_{0,q,s,t,0,w}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = G^{m_1,v_1}_{t,s,q,w}\left(x\left|\begin{matrix}(1-\gamma_t)\\(\beta_q)\\(\upsilon_w)\end{matrix}\right.\right) G^{m_2,v_2}_{t,s,q,w}\left(y\left|\begin{matrix}(1-\gamma'_t)\\(\beta'_q)\\(\upsilon'_w)\end{matrix}\right.\right) G^{m_2,v_2}_{t,s,q,w}\left(z\left|\begin{matrix}(1-\gamma''_t)\\(\beta''_q)\\(\upsilon''_w)\end{matrix}\right.\right)$$

$$(2)\ G^{0,v_1,v_2,v_3,m_1,m_2,m_3}_{0,q,s,t,0,w}\left[\begin{matrix} x \\ \\ y \\ 0 \\ z \end{matrix}\left|\begin{matrix} \dots\dots\dots\dots\dots \\ (\gamma_t);\ (\gamma'_t);\ (\gamma''_t) \\ \dots\dots\dots\dots\dots \\ (\beta_t);\ (\beta'_t);\ (\beta''_t) \\ \dots\dots\dots\dots \\ (\upsilon_t);\ (\upsilon'_t);\ (\upsilon''_t) \end{matrix}\right.\right] = \frac{\prod_{j=1}^{t}\Gamma\left((\gamma''_j)\right)\quad \Gamma((\beta''_j)}{\prod_{j=2}^{q}\Gamma(1-\beta''_q)\prod_{j=3}^{w}\Gamma(1-(\upsilon''_w)} G^{v_1,m_1}_{t,q,w}\left(y\left|\begin{matrix}(1-\gamma'_t)\\(\beta_q)\\(1-\beta_q)\\(\upsilon_w)\end{matrix}\right.\right),$$

$$(q \geq t),$$
$$(w \geq t)$$

$$(3)\ G^{m,t,t,t,1,1,1}_{n,t,s,q,l,w}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = x^{\upsilon_1}y^{\upsilon'_1}z^{\upsilon''_1}\frac{\prod_{j=1}^{t}\{\Gamma(\gamma_j+\beta_1+\upsilon_1)\prod_{j=1}^{t}\Gamma(\gamma'_j+\beta'_1+\upsilon'_1)\prod_{j=1}^{t}\Gamma(\gamma'_j+\beta''_1+\upsilon''_1)\}}{\prod_{j=3}^{w}\{\Gamma(1-\upsilon_j+\upsilon_1)\ \Gamma\left(1-\upsilon'_j+\upsilon'_1\right)\Gamma(1-\upsilon''_j+\upsilon''_1)\}}$$

$$\times\ \frac{\prod_{j=1}^{n}\Gamma(1-\varepsilon_j+\upsilon_1+\upsilon'_1+\upsilon''_1)}{\prod_{j=1}^{l}\Gamma(\rho_j+\upsilon_1+\upsilon'_1+\upsilon''_1)}\ F\left[\begin{matrix} n \\ \\ t \\ s \\ q \\ l \\ w-1 \end{matrix}\left|\begin{matrix}\left(1-\varepsilon_n+\upsilon_1+\upsilon'_1+\upsilon''_1\right) \\ (\gamma_t+\upsilon_1);\ (\gamma'_t+\upsilon'_1);\ (\gamma''_t+\upsilon''_1) \\ (\delta_s+\upsilon_1+\upsilon'_1+\upsilon''_1) \\ \left(\beta_q+\upsilon_1\right);\ (\beta'_q+\upsilon'_1);\ (\beta''_q+\upsilon''_1) \\ (\rho_l+\upsilon_1+\upsilon'_1+\upsilon''_1) \\ (1-\upsilon_w+\upsilon_1)^*;(1-\upsilon'_w+\upsilon'_1)^*;\ (1-\upsilon''_w+\upsilon''_1)^* \end{matrix}\right|\begin{matrix} -x \\ \\ \\ -y \\ \\ -z \end{matrix}\right]$$

$$(4)\ \begin{matrix} Lt \\ y \to 0 \\ z \to 0 \end{matrix}\ G^{n,0,0,0,m,1,1,1}_{p,0,0,q,0,w}\left[\begin{matrix} x \\ \\ y \\ \\ z \end{matrix}\left|\begin{matrix} (\varepsilon_p) \\ \dots\dots\dots\dots\ . \\ (\beta_t),(0) \\ \dots\dots\dots\dots \\ (\upsilon_w),(0),(0) \end{matrix}\right.\right] = G^{m,n}_{p,q}\left(x\left|\begin{matrix}(\varepsilon_p)\\(\beta_q)\\(\upsilon_w)\end{matrix}\right.\right), \begin{pmatrix} p \leq q, \\ p \leq w \end{pmatrix}$$

$$(5)\ G^{1,1,1,1,1,1,1}_{1,1,1,1,1,1,1}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = x^{\upsilon_1}y^{\upsilon'_1}z^{\upsilon''_1}\frac{\Gamma(\gamma_1+\beta_1+\upsilon_1)\,\Gamma(\gamma'_1+\beta'_1+\upsilon'_1)\,\Gamma(\gamma''_1+\beta''_1+\upsilon''_1)}{\Gamma(\delta_1+\beta_1+\beta'_1+\beta''_1)\,\Gamma(\rho_1+\upsilon_1+\upsilon'_1+\upsilon''_1)}$$

$\times\ F_1\ [(1-\varepsilon_1+\beta_1+\beta'_1+\beta''_1);\ (1-\varepsilon_1+\upsilon_1+\upsilon'_1+\upsilon''_1);\ (\gamma_1+\beta_1+\upsilon_1),\ (\gamma'_1+\beta'_1+\upsilon'_1),\ (\gamma''_1+\beta''_1+\upsilon''_1);\ (\delta_1+\beta_1+\beta'_1+\beta''_1);$

$(\rho_1+\upsilon_1+\upsilon'_1+\upsilon''_1);\ -x,\ -y,\ -z]$

$$(6)\ G^{1,1,1,1,1,1,1}_{1,1,0,2,0,2}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = x^{\upsilon_1}y^{\upsilon'_1}z^{\upsilon''_1}\frac{\Gamma(\gamma_1+\beta_1+\upsilon_1)\,\Gamma(\gamma'_1+\beta'_1+\upsilon'_1)\,\Gamma(\gamma''_1+\beta''_1+\upsilon''_1)\Gamma(1-\varepsilon_1+\beta_1+\beta'_1+\beta''_1)\Gamma(1-\varepsilon_1+\upsilon_1+\upsilon'_1+\upsilon''_1)}{\left[\Gamma(1-\beta_2+\beta_1)\,\Gamma\left(1-\beta'_2+\beta'_1\right)\Gamma(1-\beta''_2+\beta''_1)\right][\Gamma(1-\upsilon_2+\upsilon_1)\Gamma(1-\upsilon'_2+\upsilon'_1)\Gamma(1-\upsilon''_2+\upsilon''_1)]}$$

$\times\ F_2\ [(1+\beta_1+\beta''_1-\varepsilon_1);\ (1+\upsilon_1+\upsilon''_1-\varepsilon_1);\ (\gamma_1+\beta_1+\upsilon_1),\ \left(\gamma'_1+\beta'_1+\upsilon'_1\right),(\ \gamma''_1+\beta''_1+\upsilon''_1);\ (1-\beta_2+\beta_1),$

$(1-\beta'_2+\beta'_1), (1-\beta''_2+\beta''_1); (1-\upsilon_2+\upsilon_1), (1-\upsilon'_2+\upsilon'_1), (1-\upsilon''_2+\upsilon''_1)$; -x, -y, z]

$$(7)\ G^{0,2,2,2,1,1,1}_{0,2,2,1,1,1}\begin{bmatrix}x\\y\\z\end{bmatrix} = x^{\upsilon_1}y^{\upsilon'_1}z^{\upsilon''_1}\ \frac{\Gamma(\gamma_1+\beta_1+\upsilon_1)\,\Gamma(\gamma_2+\beta_1+\upsilon_1)\,\Gamma(\gamma'_1+\beta'_1+\upsilon'_1)\,\Gamma(\gamma''_1+\beta''_1+\upsilon''_1)}{\Gamma(\delta_1+\beta_1+\beta'_1)\,\Gamma(\delta_1+\beta_1+\beta''_1)\,\Gamma(\rho_1+\upsilon_1+\upsilon'_1)\,\Gamma(\rho_1+\upsilon_1+\upsilon''_1)}$$

$$X\ \Gamma(\gamma'_2+\beta'_1)\,\Gamma(\gamma''_2+\beta''_1)\,\Gamma(\gamma'_2+\upsilon'_1)\,\Gamma(\gamma''_2+\upsilon''_1)\quad F_3\,[(\gamma_1+\beta_1+\upsilon_1), (\gamma'_1+\beta'_1+\upsilon'_1), (\gamma''_1+\beta''_1+\upsilon''_1), (\gamma_2+\beta_1), (\gamma'_2+\beta'_1),$$

$$(\gamma'_2+\upsilon'_1); (\delta_1+\beta_1+\beta'_1); (\delta_1+\beta_1+\beta''_1); (\rho_1+\upsilon_1+\upsilon'_1); (\rho_1+\upsilon_1+\upsilon''_1); -x, -y, z]$$

$$(8)\ G^{2,0,0,0,1,1,1}_{2,0,0,0,2,2}\begin{bmatrix}x\\y\\z\end{bmatrix} = x^{\upsilon_1}y^{\upsilon'_1}z^{\upsilon''_1}\ \frac{\Gamma(1-\varepsilon_1+\beta_1+\beta'_1+\beta''_1)\ \Gamma(1-\varepsilon_2+\beta_1+\beta'_1+\beta''_1)\ \Gamma(1-\varepsilon_1+\upsilon_1+\upsilon'_1+\upsilon''_1)\,\Gamma(1-\varepsilon_2+\upsilon_1+\upsilon'_1+\upsilon''_1)}{\Gamma(1-\beta_2+\beta_1)\,\Gamma(1-\beta'_2+\beta'_1)\,\Gamma(1-\beta''_2+\beta''_1)\,\Gamma(1-\upsilon_2+\upsilon_1)\,\Gamma(1-\upsilon'_2+\upsilon'_1)\,\Gamma(1-\upsilon''_2+\upsilon''_1)}$$

$$\times F_4\,\Gamma(1-\varepsilon_1+\beta_1+\beta'_1+\beta''_1), (1-\varepsilon_2+\beta_1+\beta'_1+\beta''_1);\ (1+\beta_1-\beta_2), (1+\beta'_1-\beta'_2), (1+\beta''_1-\beta''_2); (1-\varepsilon_1+\upsilon_1+\upsilon'_1+\upsilon''_1);$$

$$(1-\varepsilon_2+\upsilon_1+\upsilon'_1+\upsilon''_1); (1+\upsilon_1-\upsilon_2), (1+\upsilon'_1+\upsilon'_2), (1+\upsilon''_1+\upsilon''_2); -x, -y, z]$$

$$(9)\ )\ G^{2,0,0,0,1,1,1}_{2,0,0,0,2,2}\begin{bmatrix}x\\y\\z\end{bmatrix} = x^{\upsilon_1}y^{\upsilon'_1}z^{\upsilon''_1}\ \frac{\prod_{j=1}^{n}\Gamma(1-\varepsilon_j+\beta_1+\beta'_1+\beta''_1)\ \prod_{j=1}^{n}\Gamma(1-\varepsilon_j+\upsilon_1+\upsilon'_1+\upsilon''_1)}{\prod_{j=1}^{s}\Gamma(\delta_j+\beta_1+\beta'_1+\beta''_1)\ \prod_{j=n+1}^{p}\Gamma(\varepsilon_j-\beta_1-\beta'_1-\beta''_1)\ \prod_{j=1}^{l}\Gamma(\rho_j+\upsilon_1+\upsilon'_1+\upsilon''_1)\ \prod_{j=n+1}^{p}\Gamma(\varepsilon_j-\upsilon_1-\upsilon'_1-\upsilon''_1)}$$

$$\times\ {}_pF_s\begin{bmatrix}(1-\varepsilon_p+\beta_1+\beta'_1+\beta''_1);\ (-)^{n-p}(x+y)\\(\delta_s+\beta_1+\beta'_1+\beta''_1)\end{bmatrix}\times pF_l\begin{bmatrix}(1-\varepsilon_p+\upsilon_1+\upsilon'_1+\upsilon''_1);\ (-)^{n-p}(x+z)\\(\rho_l+\upsilon_1+\upsilon'_1+\upsilon''_1)\end{bmatrix},$$

$$p\le s+1,\quad p\le l+1$$

$$(10)\ )\ G^{0,1,v_2,v_3,l,m_2,m_3}_{0,t,s,q,l,w}\begin{bmatrix}x\\y\\z\end{bmatrix}=\sum_{n=1}^{l}\sum_{k=1}^{m2}\sum_{d=1}x^{\upsilon_h}\,y^{\upsilon'_k}z^{\upsilon''_d}\ [\gamma_1+\beta_1+\upsilon_h]$$

$$\times\frac{\prod_{j=1}^{v_2}\Gamma(\gamma'_j+\beta'_j+\upsilon'_k)\ \prod_{j=1}^{v_3}\Gamma(\gamma''_j+\beta''_j+\upsilon''_d)\ \prod_{j=1}^{l}\Gamma(\upsilon_j-\upsilon_h)\ \prod_{j=1}^{m_2}\Gamma(\upsilon_j-\upsilon'_k)\prod_{j=1}^{m_3}\Gamma(\upsilon''_j-\upsilon''_k)}{\prod_{j=1}^{l}\Gamma(\rho_j+\upsilon_h+\upsilon'_k-\upsilon''_d)\ \prod_{j=2}^{s}\Gamma(1-\delta_j-\upsilon_h)\ \prod_{j=1+v_2}^{s}\Gamma(1-\delta'_j-\upsilon'_k)\prod_{j=1+v_3}^{s}\Gamma(1-\delta''_j-\upsilon''_d)\prod_{j=1+m_2}^{w}\Gamma(1+\upsilon'_k+\upsilon'_j)\prod_{j=1+m_3}^{w}\Gamma(1+\upsilon''_k-\upsilon''_j)}$$

$$\times\left[\left|\begin{matrix}\dots\dots\dots\dots\dots\\(\gamma_t+\upsilon_h);\ (\gamma'_t+\upsilon'_k);\ (\gamma''_t+\upsilon''_d)\\(\delta_s+\upsilon_h+\upsilon'_k+\upsilon''_d)\\(\beta_q+\upsilon_h);\ (\beta'_q+\upsilon'_k);\ (\beta''_q+\upsilon''_d)\\(\rho_l+\upsilon_h+\upsilon'_k+\upsilon''_d)\\(1-\upsilon_w+\upsilon_h); (1-\upsilon'_w+\upsilon'_k); (1-\upsilon''_w+\upsilon''_d)\end{matrix}\right|\begin{matrix}(-)^{w+q+t+1}x\\ \\(-)^{m_2+t-v_2}y\\ \\(-)^{m_3+t-v_3}y\end{matrix}\right]$$

4. Some Simple Properties And Recurrence Relations for $G\begin{bmatrix}x\\y\\z\end{bmatrix}$

In this section certain elementary properties of $G\begin{bmatrix}x\\y\\z\end{bmatrix}$- functions are enumerated. The proofs are fairly simple and follow by easy change in variable in the integral (2.1) and hence are omitted.

$$\text{(A)}\, x^{\sigma}y^{\mu}z^{\zeta}\, G\begin{bmatrix}x\\y\\z\end{bmatrix} = \left[\begin{array}{l|l} x & (\varepsilon_p+\sigma+\mu+\zeta) \\ & (\gamma_t-\sigma);\ (\gamma_t'-\mu);\ (\gamma_t''-\zeta) \\ & (\delta_s-\sigma-\mu-\zeta) \\ y & (\beta_q+\sigma);\ (\beta_q'+\mu);\ (\beta_q''+\zeta) \\ z & (\rho_l+\sigma+\mu+\zeta) \\ & (\upsilon_w-\sigma);\ (\upsilon_w'-\mu);\ (\upsilon_w''-\zeta) \end{array}\right]$$

$$\text{(B)}\, G^{n,v_1-1,v_{2-1},v_{3-1},m_1,m_2,m_3}_{p,t-1,s,q-1,l,w-1}\left[\begin{array}{l|l} x & (\varepsilon_p) \\ & (\gamma_3,t);\ (\gamma_3',t);\ (\gamma_3'',t) \\ & (\delta_s) \\ y & (\beta_3,q);\ (\beta_3',q);\ (\beta_3'',q) \\ & (\rho_l) \\ z & (\upsilon_{w-1});\ (\upsilon_{w-1}');\ (\upsilon_{w-1}'') \end{array}\right]$$

$$= \left[\begin{array}{l|l} x & (\varepsilon_p) \\ & (1-\beta_q),(\gamma_2,t),(\gamma_3,t);(1-\beta_q'),(\gamma_2',t),((\gamma_3',t);(1-\beta_q''),(\gamma_2'',t),(\gamma_3'',t) \\ & (\delta_s) \\ y & (1-\upsilon_w),(\beta_2,s),(\beta_3,s);(1-\upsilon_w'),(\beta_2',s),(\beta_3',s);(1-\upsilon_w''),(\beta_2'',s),(\beta_2'',s) \\ & (\rho_l) \\ z & (\upsilon_w);\ (\upsilon_w');\ (\upsilon_w'') \end{array}\right]$$

$$\text{(C)}\, x\frac{\partial}{\partial x}\, G\begin{bmatrix}x\\y\\z\end{bmatrix} = G\left[\begin{array}{l|l} x & (\varepsilon_p) \\ & (1+\gamma_1),(\gamma_2,t),(\gamma_2,t);\ (\gamma_t');(\gamma_t'') \\ & (\delta_s) \\ y & (\beta_q);\ (\beta_q');\ (\beta_q'') \\ & (\rho_l) \\ z & (\upsilon_w);\ (\upsilon_w');\ (\upsilon_w'') \end{array}\right] - \gamma_1.G\begin{bmatrix}x\\y\\z\end{bmatrix}$$

A Similar Formula holds for $y\frac{\partial G}{\partial y}$ and $z\frac{\partial G}{\partial Z}$.

$$\text{(D)}\,(\gamma_1+\beta_1+\upsilon_1)\, G\begin{bmatrix}x\\y\\z\end{bmatrix} = G\left[\begin{array}{l|l} x & (\varepsilon_p) \\ & (1+\gamma_1),(\gamma_2,t),(\gamma_2,t);\ (\gamma_t');(\gamma_t'') \\ & (\delta_s) \\ y & (\beta_q);\ (\beta_q');\ (\beta_q'') \\ & (\rho_l) \\ z & (\upsilon_w);\ (\upsilon_w');\ (\upsilon_w'') \end{array}\right]$$

$$+G\left[\begin{array}{c|l} x & (\varepsilon_p) \\ & (\gamma_1);\ (\gamma_t');(\gamma_t'') \\ & (\delta_s) \\ y & (1+\beta_1),(\beta_2,q),(\beta_3,q);\ (\beta_q')(\beta_q'') \\ & (\rho_l) \\ z & (\upsilon_w);\ (\upsilon_w');\ (\upsilon_w'') \end{array}\right] +G\left[\begin{array}{c|l} x & (\varepsilon_p) \\ & (\gamma_1);\ (\gamma_t');(\gamma_t'') \\ & (\delta_s) \\ y & (\beta_q);\ (\beta_q');\ (\beta_q'') \\ & (\rho_l) \\ z & (1+\upsilon_1),(\upsilon_2,w),(\upsilon_2,w);(\upsilon_w');\ (\upsilon_w'') \end{array}\right]$$

A Similar relation holds for $(\gamma_1' + \beta_1' + \upsilon_1')G$ and $(\gamma_1'' + \beta_1'' + \upsilon_1'')G$.

$$\text{(E) } (\varepsilon_P + \delta_S + \rho_l - 3)G\begin{bmatrix} x \\ y \\ z \end{bmatrix} = G\left[\begin{array}{c|l} x & (\varepsilon_{p-1}),(\varepsilon_p - 1) \\ & (\gamma_t);\ (\gamma_t');\ (\gamma_t'') \\ & (\delta_s) \\ y & (\beta_q);\ (\beta_q');\ (\beta_q'') \\ & (\rho_l) \\ z & (\upsilon_w);\ (\upsilon_w');\ (\upsilon_w'') \end{array}\right] +G\left[\begin{array}{c|l} x & (\varepsilon_p) \\ & (\gamma_t);\ (\gamma_t');\ (\gamma_t'') \\ & (\delta_{s-1}),(\delta_s - 1) \\ y & (\beta_q);\ (\beta_q');\ (\beta_q'') \\ & (\rho_l) \\ z & (\upsilon_w);\ (\upsilon_w');\ (\upsilon_w'') \end{array}\right]$$

$$+G\left[\begin{array}{c|l} x & (\varepsilon_p) \\ & (\gamma_t);\ (\gamma_t');\ (\gamma_t'') \\ & (\delta_s) \\ y & (\beta_q);\ (\beta_q');\ (\beta_q'') \\ & (\rho_{l-1}),(\rho_l - 1) \\ z & (\upsilon_w);\ (\upsilon_w');\ (\upsilon_w'') \end{array}\right]$$

$$\text{(F) } \left(\varepsilon_P + \gamma_1 + \gamma_1' + \gamma_1'' - 1\right)G\begin{bmatrix} x \\ y \\ z \end{bmatrix} = G\left[\begin{array}{c|l} x & (\varepsilon_{p-1}),(\varepsilon_p - 1) \\ & (\gamma_t);\ (\gamma_t');\ (\gamma_t'') \\ & (\delta_s) \\ y & (\beta_q);\ (\beta_q');\ (\beta_q'') \\ & (\rho_l) \\ z & (\upsilon_w);\ (\upsilon_w');\ (\upsilon_w'') \end{array}\right] +$$

$$G\left[\begin{array}{c|l} x & (\varepsilon_p) \\ & (\gamma_1+1),(\gamma_2,t),(\gamma_3,t);\ (\gamma_t');(\gamma_t'') \\ & (\delta_s) \\ y & (\beta_q);\ (\beta_q');\ (\beta_q'') \\ & (\rho_l) \\ z & (\upsilon_w);\ (\upsilon_w');\ (\upsilon_w'') \end{array}\right]$$

$$+G\left[\begin{array}{c|l} x & (\varepsilon_p) \\ & (\gamma_t);(1+\gamma_1'),(\gamma_2',t),(\gamma_3',t) \\ & (\delta_s) \\ y & (\beta_q);\ (\beta_q');\ (\beta_q'') \\ & (\rho_l) \\ z & (\upsilon_w);\ (\upsilon_w');\ (\upsilon_w'') \end{array}\right] +G\left[\begin{array}{c|l} x & (\varepsilon_p) \\ & (\gamma_t);((\gamma_t');\left(1+\gamma_1''\right),(\gamma_2'',t),\left(\gamma_3'',t\right) \\ & (\delta_s) \\ y & (\beta_q);\ (\beta_q');\ (\beta_q'') \\ & (\rho_l) \\ z & (\upsilon_w);\ (\upsilon_w');\ (\upsilon_w'') \end{array}\right]$$

$$(G)\ (\upsilon_1+\upsilon_1'+\upsilon_1''+\rho_l-1)\ G\begin{bmatrix}x\\y\\z\end{bmatrix} = G\left[\begin{array}{l|l} x & (\varepsilon_p) \\ & (\gamma_t);\ (\gamma_t');\ (\gamma_t'') \\ & (\delta_s) \\ y & (\beta_q);\ (\beta_q');\ (\beta_q''\ \ ) \\ & (\rho_l) \\ z & (\upsilon_1+1),(\upsilon_2,w),(\upsilon_3,w);(\upsilon_w');\ (\upsilon_w'') \end{array}\right]$$

$$+G\left[\begin{array}{l|l} x & (\varepsilon_p) \\ & (\gamma_t);\ (\gamma_t');\ (\gamma_t'') \\ & (\delta_s) \\ y & (\beta_q);\ (\beta_q');\ (\beta_q''\ \ ) \\ & (\rho_l) \\ z & (\upsilon_w);(\upsilon_1'+1),(\upsilon_2',w),(\upsilon_3'',w);\ (\upsilon_w'') \end{array}\right] + G\left[\begin{array}{l|l} x & (\varepsilon_p) \\ & (\gamma_t);\ (\gamma_t');\ (\gamma_t'') \\ & (\delta_s) \\ y & (\beta_q);\ (\beta_q');\ (\beta_q''\ \ ) \\ & (\rho_l) \\ z & (\upsilon_w);(\upsilon_w');(\upsilon_1''+1),(\upsilon_2',w),(\upsilon_3'',w) \end{array}\right]$$

$$+G\left[\begin{array}{l|l} x & (\varepsilon_p) \\ & (\gamma_t);\ (\gamma_t');\ (\gamma_t'') \\ & (\delta_s) \\ y & (\beta_q);\ (\beta_q');\ (\beta_q'') \\ & (\rho_l-1)\ ,(\rho_2,l),(\rho_3,l) \\ z & (\upsilon_w);\ (\upsilon_w');\ (\upsilon_w'') \end{array}\right],\ l\geq 1$$

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